ECONTECHMOD. AN INTERNATIONAL QUARTERLY JOURNAL – 2013. Vol. 2. No. 4. 29–37

Some features of application the delayed feedback control method to Cournot-Puu duopoly model

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Received Oktober 8.2013: accepted November 10.2013

Abstract. The Cournot-Puu duopoly model is considered. Delayed feedback control method (DFC-method) is applied to this model. The dependence of rate coming of the system at Cournot equilibrium on the feedback coefficient K choice is shown. The optimal value of this coefficient is defined. The dependence of rate coming of the system at Cournot equilibrium on parameter c_r (the ratio of marginal cost firms) is set. The application of DFC-method with two control laws to duopoly model is considered.

Key words: Cournot-Puu duopoly model, Cournot equilibrium, chaos, stability, delayed feedback control method (DFC-method).

INTRODUCTION

Review of the literature of recent years (see review in [15, 27]) shows the great interest of researchers to study of the oligopoly models and nonlinear dynamics, which is typical for them. The simplest but at the same time the most investigated among them is the Cournot oligopoly model [12]. One of the modifications of the model was proposed by T. Puu. He introduced the assumption that demand in oligopoly must be isoelastic and competitors must have constant, but different marginal costs [24].

Recent studies also indicate the existence of the chaotic dynamics in oligopoly models [1-3, 5, 6, 8-11, 13, 19, 21-23, 25, 26, 29, 30]. Among them, much attention is paid to the duopoly model, including the model of Cournot-Puu.

Our recent papers have been devoted to the construction of the generalized Cournot-Puu oligopoly model and study the stability of its equilibrium point [14, 17, 18, 20]. And in [18, 20] we described in detail duopoly model and defined such parameters of the system (the marginal cost of the firms), at which equilibrium point is stable, and there is chaos in the system (more precisely, there is a cascade of period doubling, which leads to chaos).

In this regard, there is a need to control this chaotic dynamic because unstable oscillations are undesirable for any economic system or process. Some methods for chaos control, such as OGY chaos control method [4,7], the adaptive control method and pole placement method [21] were applied to the Cournot-Puu duopoly model. In particular, in [11], the authors proposed the delayed feedback control method (DFC-method) to control the chaos that occurs in the Cournot-Puu duopoly model. This method is based on a feedback of the difference between the current state and the delayed state of the system. It requires relatively little information about the system and, therefore, is easy to use. We have implemented a generalization of this method for the case of presence n firms in the market (situation of oligopoly) for the generalized Cournot-Puu model, with the possibility of the application k $(1 \le k \le n)$ control laws to stabilize the system [16].

However, there are several important aspects that require detailed consideration and study. The first question is: what should be the optimal value of the feedback coefficient K at which the system will come to the Cournot equilibrium on the minimum number of steps (minimum time)? The second question is: far as will increase or decrease the number of time steps which must be executed to reach a state of equilibrium, depending on admissible values of the system parameters (marginal costs of the firms c_i). And the third, how effective the use of more than one control law to control the chaos is effective? Let us consider these questions in detail, for example for duopoly.

INFLUENCE OF CHOICE OF THE FEEDBACK COEFFICIENT K TO SPEED OF SETTING THE SYSTEM AT COURNOT EQUILIBRIUM

In the case of duopoly there are only two firms F_1 and F_2 on the market in the same industry, with output q_1 and q_2 respectively. Firms have constant but different marginal costs c_1 and c_2, respectively.

According to the generalized model [18], Cournot-Puu duopoly model is as follows (see also [11]):

$$q_1(t+1)=\sqrt{\frac{q_2(t)}{c_1}}-q_2(t),$$
$$q_2(t+1)=\sqrt{\frac{q_1(t)}{c_2}}-q_1(t). \tag{1}$$

Functions $q_1(t+1)$ and $q_2(t+1)$ with parameter values $c_1=1, c_2=6{,}25$ and initial conditions $q_1(0)=q_2(0)=0{,}01$ have the form as it is shown in Fig. 1.

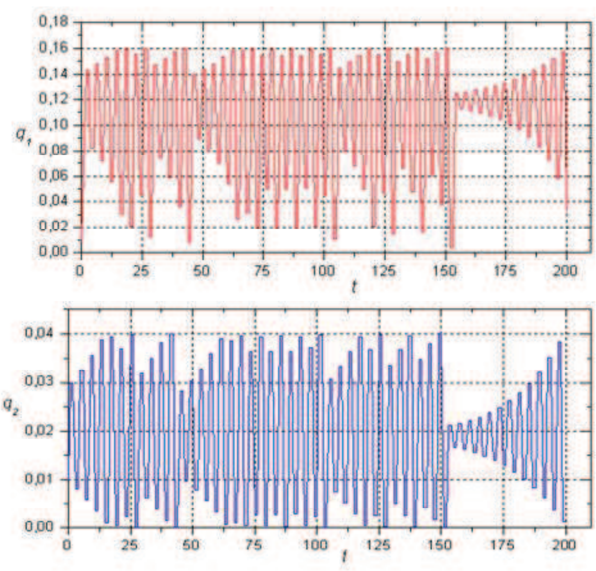

Fig. 1. The reaction functions of the firms F_1 and F_2

Nontrivial equilibrium point of the system (1) – Cournot equilibrium (Nash equilibrium) – is a point of intersection the reaction curves and has a value:

$$q_1^*=\frac{c_2}{(c_1+c_2)^2},\quad q_2^*=\frac{c_1}{(c_1+c_2)^2}. \tag{2}$$

Profit of the duopolists at the Cournot equilibrium is respectively:

$$U_1^*=\frac{c_2^2}{(c_1+c_2)^2},\quad U_2^*=\frac{c_1^2}{(c_1+c_2)^2}. \tag{3}$$

Stability of the equilibrium point (2) of system (1) is investigated in detail in [18,20], and in [24].

Denote the ratio of marginal costs $\frac{c_2}{c_1}=c_r$, and without loss of generality we will assume that $c_2 \ge c_1$ (i.e., $c_r \ge 1$). The equilibrium point (2) is stable if the ratio of marginal costs is in the range:

$$1 \le c_r < 3+\sqrt{8}. \tag{4}$$

Then equilibrium point is unstable if:

$$3+\sqrt{8} \le c_r \le 25/4. \tag{5}$$

Limit cycles and chaos exist in the system at these values c_r. Bifurcation diagram for firms F_2 with output q_2 with respect to the ratio c_r of marginal costs is presented in Fig. 2.

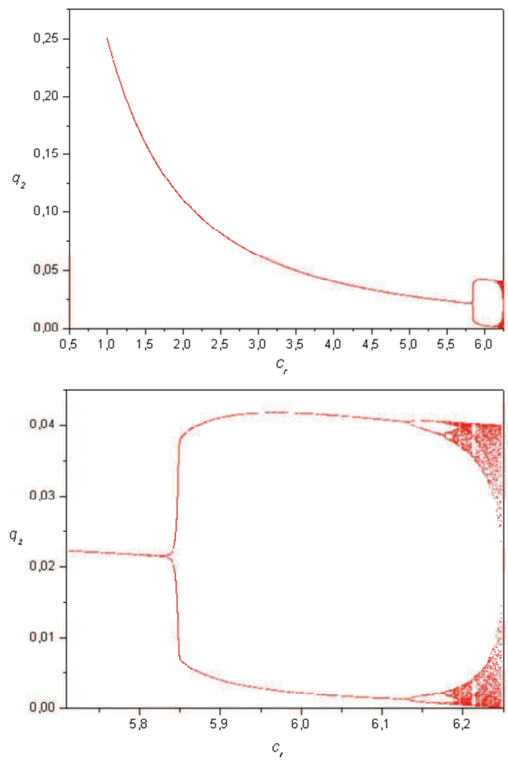

Fig. 2. Bifurcation diagram of the firm F_2 with the production q_2

Consider the following control form of the duopoly model (1):

$$\begin{cases} q_1(t+1)=\sqrt{\dfrac{q_2(t)}{c_1}}-q_2(t)+u(t), \\ q_2(t+1)=\sqrt{\dfrac{q_1(t)}{c_2}}-q_1(t). \end{cases} \tag{6}$$

$u(t)$ represents such DFC-law:

$$u(t)=K\big(q_1(t)-q_1(t-1)\big),\quad t\ge 1, \tag{7}$$

where: K is feedback coefficient.

In this paper DFC-law is applied to the structure (state) (output of the firm) of duopoly model. We can consider the application of DFC-law to the parameters (the marginal cost of firm) of the model.

As it is shown in [11], and also in detail in [20], Cournot equilibrium $\left(q_1^*, q_2^*\right)$ (2) is locally asymptotically stable if and only if:

$$-\frac{1}{2}-\frac{(c_r-1)^2}{8c_r}<K<1-\frac{(c_r-1)^2}{4c_r}. \quad (8)$$

A graphical depiction of the region of asymptotic stability of the equilibrium point $\left(q_1^*, q_2^*\right)$ in the space of parameters $\{c_r, K\}$ is shown in Fig. 3.

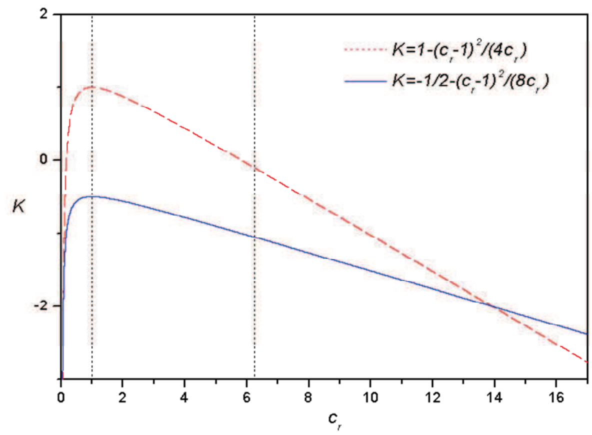

Fig. 3. The region of asymptotic stability of the equilibrium point $\left(q_1^*, q_2^*\right)$ when DFC-method is applied to the state of duopoly model

This region is determined by the conditions (8) and is bounded by the lines:

$$\mathrm{K}=1-\frac{(\mathrm{c_r}-1)^2}{4\mathrm{c_r}},$$

$$\mathrm{K}=-\frac{1}{2}-\frac{(\mathrm{c_r}-1)^2}{8\mathrm{c_r}}, \quad (9)$$

$$\mathrm{c_r}=1, \quad \mathrm{c_r}=\frac{25}{4}.$$

Numerical experiments with using DFC-method to the state of Cournot-Puu duopoly model are carried out in [11]. The parameters are fixed as $c_1=1$, , $c_2=6{,}25$ initial conditions $q_1(0)=q_2(0)=0{,}01$, and the feedback coefficient $K=-0{,}5$. Chaotic trajectory is stabilized at Cournot equilibrium and control DFC-law $u(t)$, acting since time $t=50$, tends to zero. But some question arises here: how effective is the choice of such coefficient K? Maybe, there is some value of this coefficient from the allowable interval (8), for a given value of the ratio of marginal costs, which faster leads the unstable system to equilibrium point. The answer to this question we'll get in this section.

Let us consider the range of the parameter c_r at which Cournot equilibrium is unstable, i.e., $3+\sqrt{8}\le c_r\le\frac{25}{4}$. Consider the left end of this interval, i.e., $c_r=3+\sqrt{8}$ ($c_1=1, c_2=3+\sqrt{8}$). (We consider the ends of the interval of unstability, as they are the most interesting to study). Then, coefficient K, according to the system of inequalities (8), can be selected from the interval:

$$-1<K<0. \quad (10)$$

For the right end of the interval, i.e., values $c_r=25/4$ ($c_1=1, c_2=25/4$), coefficient K, according to the system of inequalities (8), can be selected from the interval:

$$-\frac{841}{800}<K<-\frac{82}{800},$$

namely:

$$-1{,}05125<K<-0{,}1025. \quad (11)$$

As we can see from (10) and (11), the left and right ends of the interval of values of the coefficient K, are displaced to the left (from (-1) to $(-1{,}05125)$ – for the left end, and from 0 to $(-0{,}1025)$ – for the right end) with increasing c_r from value $c_r=3+\sqrt{8}$ to the value $c_r=25/4$. That is, common to all values c_r from the interval $3+\sqrt{8}\le c_r\le\frac{25}{4}$ are values of K from the interval:

$$-1<K<-0{,}1025. \quad (12)$$

Cournot equilibrium at selected values of the parameter c_r, according to (2), has the values that are presented in the Table 1.

Table 1. The equilibrium point $\left(q_1^*, q_2^*\right)$ at different values c_r

q^* \ c_r	$3+\sqrt{8}$	$25/4$
q_1^*	$\frac{1}{8}=0{,}125$	$\frac{100}{841}\approx 0{,}118906$
q_2^*	$\frac{1}{8(3+\sqrt{8})}\approx 0{,}021447$	$\frac{16}{841}\approx 0{,}019025$

We have conducted numerous studies to model (6) with control functions (7) for value $c_r=3+\sqrt{8}$ and initial conditions $q_1(0)=q_2(0)=0{,}01$, selecting the coefficient K from the allowable range (10). We received a number of time steps (intervals) needed the each of the firms F_1 and F_2 with output q_1 and q_2 to come to the Cournot equilibrium $\left(q_1^*, q_2^*\right)$, for each selected value K (DFC-law $u(t)$ acts since time $t = 50$). Graphically this dependence is shown in Fig. 4.

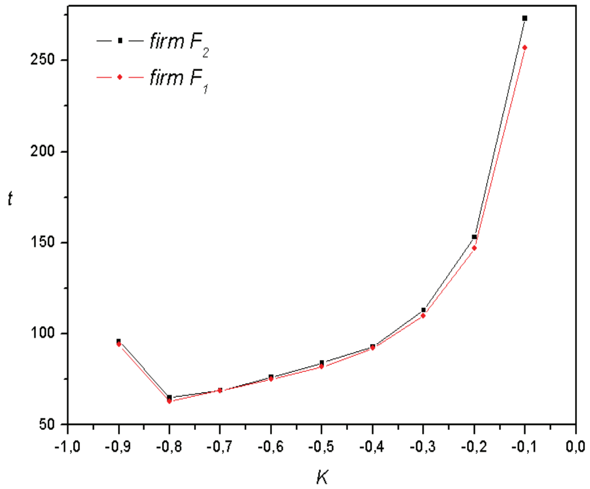

Fig. 4. Dependence of the number of time steps t on the selection of the coefficient K ($c_r = 3+\sqrt{8}$)

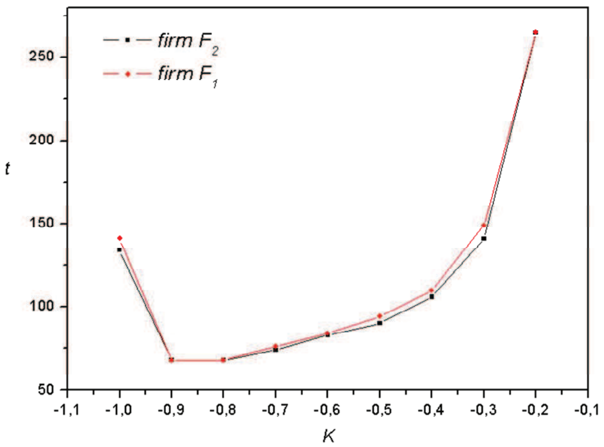

Fig. 5. Dependence of the number of time steps t on the selection of the coefficient K ($c_r = 25/4$)

The more we are moving away from the optimal value in the direction of the right end of the range of K, the system needs more time to come to the equilibrium point.

As it is shown in Fig. 4, both firms (with the same value of K) need almost the same number of steps to come to the equilibrium point. Also we see that the fastest system stabilizes at $K = -0{,}8$ ($t_1 = 63, t_2 = 65$, that is, by 13 steps to the first firm and 15 steps for the second, control law stabilizes the chaotic situation). Therefore, in order to stabilize the system as quickly as possible, it is better to choose K that close to this optimal value.

Similarly, we have conducted numerous studies for values $c_r = 25/4$ and initial conditions $q_1(0) = q_2(0) = 0{,}01$, selecting the coefficient K from the allowable range (11). Graphically the results of research are presented in Fig. 5.

As it is shown in Fig. 5, both firms (with the same value of K) need almost the same number of steps to come to the equilibrium point. Also we see that the fastest system stabilizes at $K = -0{,}9$ and $K = -0{,}8$ (the second value $K = -0{,}8$ is optimal for the previous case too) ($t_1 = 68, t_2 = 68$ for both values $K = -0{,}9$ and $K = -0{,}8$). So, in order to stabilize the system as soon as possible, it is better to choose those values K that are between the above optimal value and close to them. As in the previous case, the more we are moving away the optimal values to the right end of the range of values K, the system needs more time to come into equilibrium.

Application of the DFC-method to duopoly model with parameter value $c_r = 25/4$, initial conditions $q_1(0) = q_2(0) = 0{,}01$ and the value of the coefficient $K = -0{,}9$ is shown graphically in Fig. 6.

Control law begins to act from the moment of time $t = 50$ and stabilizes the system to the equilibrium point.

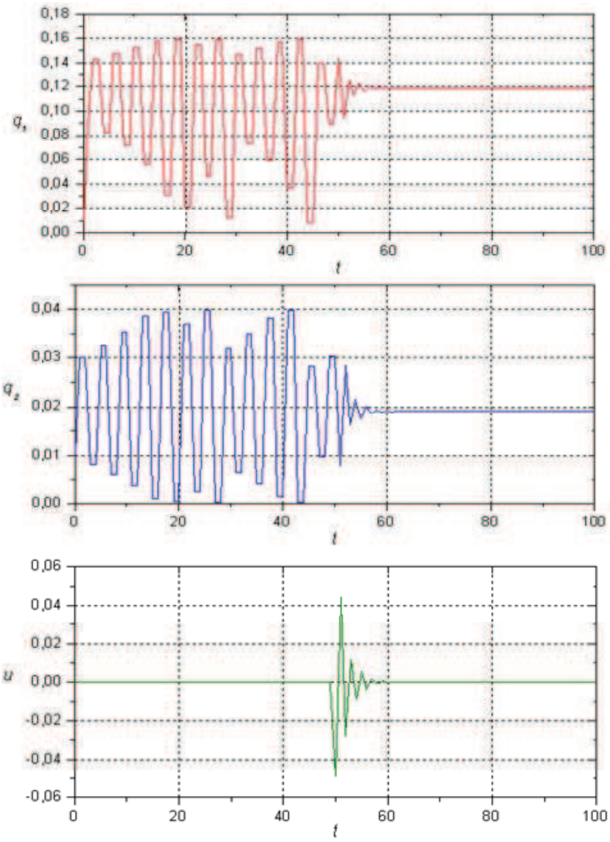

Fig. 6. Application of the DFC-method to duopoly model

DEPENDENCE OF THE RATE SETTING SYSTEM AT COURNOT EQUILIBRIUM ON THE VALUE OF THE c_r

In the study of duopoly model with applying DFC-method to it, the question arises: how will be change the number of time steps required the system to come to the Cournot equilibrium (increase or decrease) for different values of the ratio of marginal costs c_r ?

We have conducted two numerical study for the model (6), with the control function (7), for two values of the coefficient K from the interval (12) with the initial conditions $q_1(0) = q_2(0) = 0{,}01$. In the first case we have the optimal value of the coefficient $K = -0{,}8$, and in the second case, for comparison, value $K = -0{,}5$. Choosing the values of c_r in the interval of unstability $3+\sqrt{8} \le c_r \le \frac{25}{4}$, we got a number of time steps necessary to each of the firms F_1 and F_2 with the production q_1 and q_2, to come to the Cournot equilibrium. Graphically this dependence is shown in Fig. 7.

As it is shown in Fig. 7, in the first case (at $K = -0{,}8$) the system requires significantly less steps to come to the equilibrium point (Fig.7, left) than in the second case, i.e., at $K = -0{,}5$ (Fig. 7, right). Therefore, the results obtained in the preceding section are confirmed.

At the beginning of the interval of unstability (at $c_r = 3+\sqrt{8}$) the system needs less time to come to equilibrium. However, with the increasing c_r the number of steps also increases, but at the end of the interval (at $c_r = 25/4$) the number of steps decreases. If you look at the bifurcation diagram (Fig. 2), it is possible to explain this result. At the beginning of the unstable interval the equilibrium point oscillates between two values (beginning of period doubling tree) that are close to one another. With increasing c_r, two branches of the period doubling tree divergent more (the difference between the values, between which equilibrium point oscillates, increases) and the number of steps increases. Decreasing the number of time steps, for some values c_r, may indicate to a smaller difference of values between which equilibrium point oscillates.

APPLICATION of the DFC-METHOD WITH TWO control LAWS

In this section we'll consider application of the two control laws to the state of the duopoly model (1).

Consider the system:

$$\begin{cases} q_1(t+1) = \sqrt{\dfrac{q_2(t)}{c_1}} - q_2(t) + u_1(t), \\ q_2(t+1) = \sqrt{\dfrac{q_1(t)}{c_2}} - q_1(t) + u_2(t). \end{cases} \tag{13}$$

$u_1(t)$, $u_2(t)$ are the DFC-laws:

$$\begin{aligned} u_1(t) &= K_1\left(q_1(t) - q_1(t-1)\right), \quad t \ge 1, \\ u_2(t) &= K_2\left(q_2(t) - q_2(t-1)\right), \quad t \ge 1, \end{aligned} \tag{14}$$

where: K_1, K_2 are the feedback coefficients.

According to the generalized DFC-method [16, 20], for $n = 2$ and $k = 2$ (k is the number of the control laws), the Jacobi matrix of the linearized system of model (13) looks as follows:

$$J = \begin{pmatrix} 0 & p_1 & K_1 & 0 \\ p_2 & 0 & 0 & K_2 \\ -1 & p_1 & K_1 & 0 \\ p_2 & -1 & 0 & K_2 \end{pmatrix}, \tag{15}$$

where:

$$p_1 = \frac{c_2 - c_1}{2c_1}, \quad p_2 = \frac{c_1 - c_2}{2c_2}. \tag{16}$$

Next, define the conditions imposed on the choice of coefficients K_1 and K_2 to control laws (14) stabilize the system to the Cournot equilibrium. We apply the Routh-Hurwitz procedure for $n + k = 4$ [28].

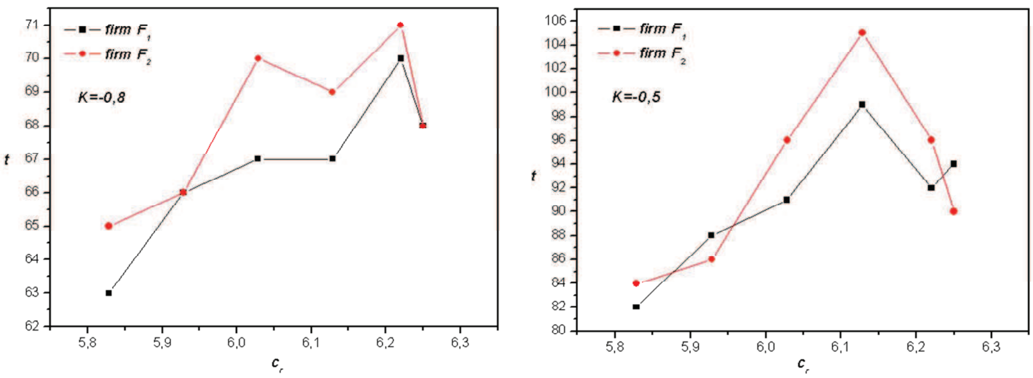

Fig. 7. Dependence the number of time steps t on the parameter c_r.

Stability of the linearized system of model (13) is determined by the characteristic equation:

$$\det\begin{pmatrix} 0 & p_1 & K_1 & 0 \\ p_2 & 0 & 0 & K_2 \\ -1 & p_1 & K_1 & 0 \\ p_2 & -1 & 0 & K_2 \end{pmatrix} = 0,$$

or

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0. \qquad (17)$$

The coefficients of equation (17) look as follows:

$$\begin{aligned} a_1 &= -(K_1 + K_2), \\ a_2 &= K_1 + K_2 + K_1K_2 - p_1p_2, \\ a_3 &= -2K_1K_2, \\ a_4 &= K_1K_2. \end{aligned} \qquad (18)$$

Equilibrium point (q_1^*, q_2^*) is locally asymptotically stable if for all eigenvalues λ of the Jacobi matrix J condition holds [28]:

$$|\lambda| < 1. \qquad (19)$$

According to the classical Routh-Hurwitz procedure, all eigenvalues satisfy condition (19) if the conditions hold:

$$\begin{aligned} &b_0 > 0, \quad b_1 > 0, \quad b_2 > 0, \quad b_3 > 0, \quad b_4 > 0, \\ &b_1b_2 - b_0b_3 > 0, \quad b_3(b_1b_2 - b_0b_3) - b_1^2b_4 > 0, \end{aligned} \qquad (20)$$

where:

$$\begin{aligned} b_0 &= 1 + a_1 + a_2 + a_3 + a_4, \\ b_1 &= 2 + a_1 - a_3 - 2a_4, \\ b_2 &= 3 - a_2 + 3a_4, \\ b_3 &= 2 - a_1 + a_3 - 2a_4, \\ b_4 &= 1 - a_1 + a_2 - a_3 + a_4. \end{aligned} \qquad (21)$$

According to the parameters $b_i, i = \overline{0,4}$ (21), the coefficients $a_i, i = \overline{1,4}$ (18) and the elements $p_i, i = 1,2$ (16), conditions (20) can be rewritten as:

$$\begin{aligned}
&(0)\quad 1 + \frac{(c_r - 1)^2}{4c_r} > 0, \\
&(1)\quad 2 - (K_1 + K_2) > 0, \\
&(2)\quad 3 - (K_1 + K_2) + 2K_1K_2 - \frac{(c_r - 1)^2}{4c_r} > 0, \\
&(3)\quad 2 + (K_1 + K_2) - 4K_1K_2 > 0, \\
&(4)\quad 1 + 2(K_1 + K_2) + 4K_1K_2 + \frac{(c_r - 1)^2}{4c_r} > 0, \\
&(5)\quad \left(2 - (K_1 + K_2)\right)\left(3 - (K_1 + K_2) + 2K_1K_2 - \frac{(c_r - 1)^2}{4c_r}\right) - \\
&\quad -\left(1 + \frac{(c_r - 1)^2}{4c_r}\right)\left(2 + (K_1 + K_2) - 4K_1K_2\right) > 0, \\
&(6)\quad \left(2 + (K_1 + K_2) - 4K_1K_2\right)\cdot \\
&\quad \cdot\left(\left(2 - (K_1 + K_2)\right)\left(3 - (K_1 + K_2) + 2K_1K_2 - \frac{(c_r - 1)^2}{4c_r}\right) - \right. \\
&\quad \left. -\left(1 + \frac{(c_r - 1)^2}{4c_r}\right)\left(2 + (K_1 + K_2) - 4K_1K_2\right)\right) - \\
&\quad -\left(2 - (K_1 + K_2)\right)^2 \cdot \\
&\quad \cdot\left(1 + 2(K_1 + K_2) + 4K_1K_2 + \frac{(c_r - 1)^2}{4c_r}\right) > 0,
\end{aligned} \qquad (22)$$

where: $c_r = \dfrac{c_2}{c_1}$.

Obviously, the first condition $1 + \frac{(c_r - 1)^2}{4c_r} > 0$ is always satisfied, since $c_r > 0$. You must choose the following coefficient K_1 and K_2 that satisfy the remaining six inequalities. This region is graphically represented in Fig. 8 (in the plane $\{K_1 + K_2, K_1K_2\}$), which satisfies the inequalities (22.1)-(22.6) at the value of the parameter $c_r = 3 + \sqrt{8}$. This region is bounded by curves constructed using the inequalities (22) by replacing the sign of inequality to equality, i.e.,

$$\begin{aligned}
&(1)\quad 2 - (K_1 + K_2) = 0, \\
&(2)\quad 3 - (K_1 + K_2) + 2K_1K_2 - \frac{(c_r - 1)^2}{4c_r} = 0, \\
&(3)\quad 2 + (K_1 + K_2) - 4K_1K_2 = 0, \\
&(4)\quad 1 + 2(K_1 + K_2) + 4K_1K_2 + \frac{(c_r - 1)^2}{4c_r} = 0, \\
&(5)\quad \left(2 - (K_1 + K_2)\right)\left(3 - (K_1 + K_2) + 2K_1K_2 - \frac{(c_r - 1)^2}{4c_r}\right) - \\
&\quad -\left(1 + \frac{(c_r - 1)^2}{4c_r}\right)\left(2 + (K_1 + K_2) - 4K_1K_2\right) = 0, \\
&(6)\quad \left(2 + (K_1 + K_2) - 4K_1K_2\right)\cdot \\
&\quad \cdot\left(\left(2 - (K_1 + K_2)\right)\left(3 - (K_1 + K_2) + 2K_1K_2 - \frac{(c_r - 1)^2}{4c_r}\right) - \right. \\
&\quad \left. -\left(1 + \frac{(c_r - 1)^2}{4c_r}\right)\left(2 + (K_1 + K_2) - 4K_1K_2\right)\right) - \\
&\quad -\left(2 - (K_1 + K_2)\right)^2\left(1 + 2(K_1 + K_2) + 4K_1K_2 + \frac{(c_r - 1)^2}{4c_r}\right) = 0.
\end{aligned} \qquad (23)$$

As it is shown on the right of Fig. 8, the range of permissible values of the coefficients K_1 and K_2 is determined by the fourth and sixth equalities.

We have conducted numerous studies that answer the question: whether is it appropriate to use two control laws in the duopoly model? Choosing values $K_1 + K_2$ and K_1K_2 from the permissible region, we have the number of steps required for the system to come to equilibrium point. The result of the study is shown graphically in Fig. 9.

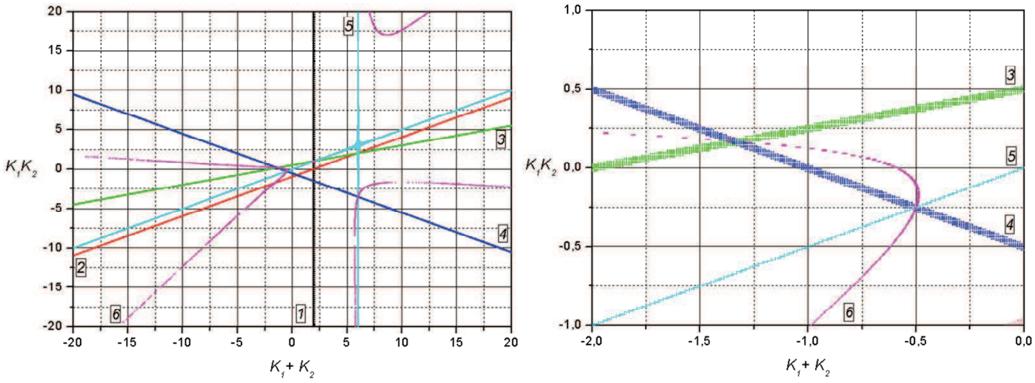

Fig. 8. The region of admissible values of the coefficients K_1 and K_2 ($c_r = 3+\sqrt{8}$)

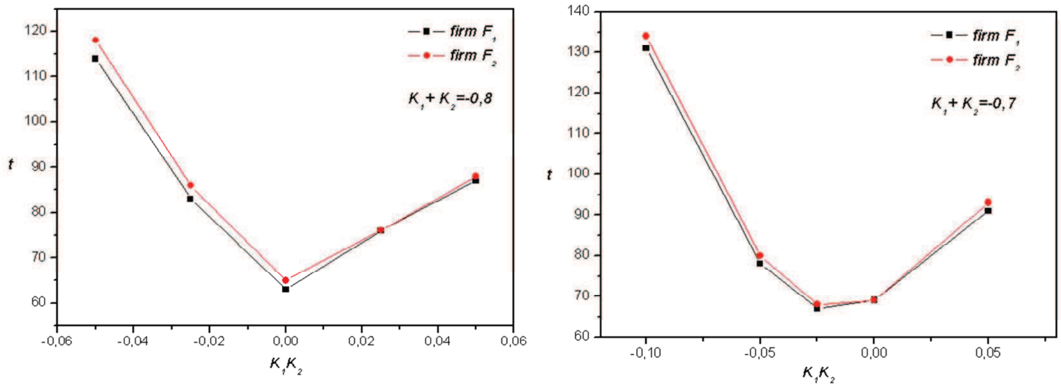

Fig. 9. Dependence the number of time steps t on K_1K_2 ($c_r = 3+\sqrt{8}$)

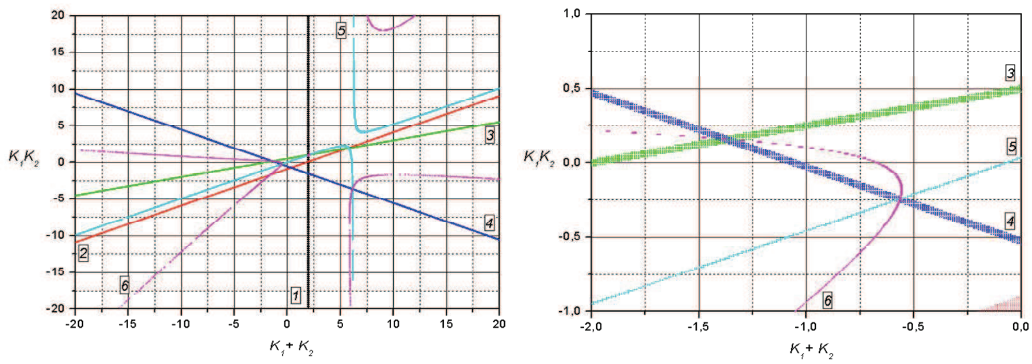

Fig. 10. The region of admissible values of coefficients K_1 and K_2 ($c_r = 25/4$)

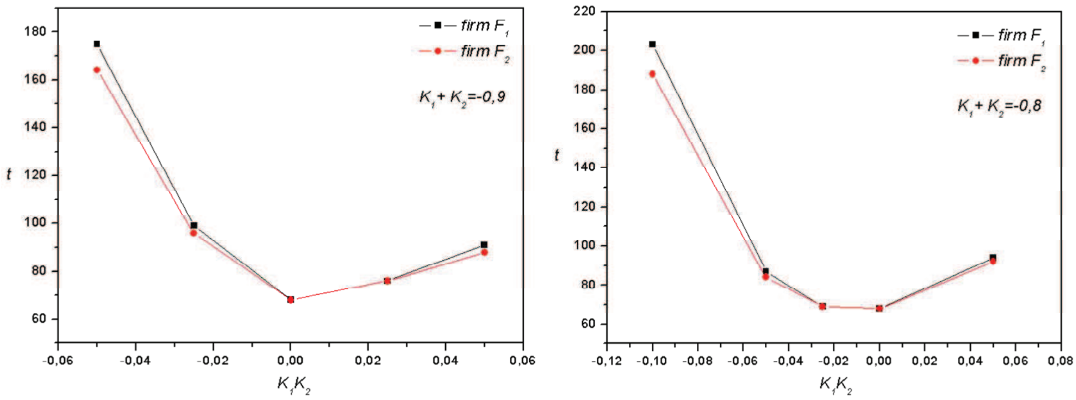

Fig. 11. Dependence the number of time steps t on K_1K_2 ($c_r = 25/4$)

As seen in Fig. 9 (left) at the value $K_1 + K_2 = -0,8$ and $K_1K_2 = 0$ it needs the least time for the system to come to the Cournot equilibrium. The product $K_1K_2 = 0$ means that either $K_1 = 0$ or $K_2 = 0$. Without loss of generality suppose that $K_2 = 0$. Then we have a situation when only one control law is applied to the model with the value of the coefficient $K_1 = -0,8$. This situation we have described in the preceding section when the value of coefficient $K = -0,8$ is the best value for DFC-method with the one control law. But the more we move to the left and to the right of the value $K_1K_2 = 0$, the more time it takes to stabilize the system. The same situation we have for $K_1 + K_2 = -0,7$ (Fig. 9, on right).

Similar studies we have done for the right end of the unstability interval, that is for $c_r = 25/4$. This region is graphically represented in Fig. 10 (in the plane $\{K_1 + K_2, K_1K_2\}$), which satisfies the inequalities (22.1)–(22.6) at the value of the parameter $c_r = 25/4$.

As in the previous case, the range of permissible values of the coefficients K_1 and K_2 is determined by the fourth and sixth equalities. Similarly, we have conducted numerous studies, choosing values $K_1 + K_2$ and K_1K_2 from the permissible region. We received a number of steps necessary for the system to come to equilibrium point. The result of the study is shown graphically in Fig. 11.

As seen in Fig. 11 on left, at the value $K_1 + K_2 = -0,9$ and $K_1K_2 = 0$ it needs the least time for the system to come to the Cournot equilibrium. But the more we move to the left or right of the value $K_1K_2 = 0$, the more time it takes to stabilize the system. Similar conclusions we can do for $K_1 + K_2 = -0,8$ and $K_1K_2 = 0$ (Fig. 11, right). But the product $K_1K_2 = 0$ means that one of the coefficients is zero, i.e., we use DFC-method with only one control law for the model. And in this case, the values $K = -0,9$ and $K = -0,8$ are optimal.

So, summing up the results of research of this section, we have shown that the use of DFC-method with the two control laws to duopoly model is not effective.

Application of DFC-method to the state of the model is the method of individual control on a chaotic market when one of the firms can examine the market situation and change their course of action, observing the volume of production in the current and past periods. According to our research, controlling of unstable fluctuations by only one firm of the oligopolistic industry is more effective. If the two firms simultaneously will wish to do this, it'll need more time to stabilize the market situation (so that both firms have come to an equilibrium value of output).

CONCLUSIONS

Oligopoly is the predominant form of the market structure. Automobile industry, steelmaking industry, petrochemical industry, electrical industry, energy industry, computer industry and others are the oligopolistic industries. That is why it is important to study the processes occurring in such organizations of market relations.

In the paper we considered Cournot-Puu duopoly model (two firms in oligopolistic industries). Chaotic behavior at certain values of the ratio of marginal costs of firms c_r ($3+\sqrt{8} \le c_r \le 25/4$) is observed in the model. We studied some features of the application of the DFC-method to controlling the chaos that arises in this model.

First, we have shown that there is such value of feedback coefficient K at which firms come to the Cournot equilibrium fastest, namely $K = -0,8$. This value is optimal for almost all values of parameter c_r from the unstability interval.

We also examined how the value of parameter c_r influences on the rate of the system setting at the equilibrium point. At the beginning of the unstability interval (at $c_r = 3+\sqrt{8}$) the system needs less time to come to equilibrium. However, with the increasing c_r the number of steps also increases, but at the end of the interval (at $c_r = 25/4$) the number of steps decreases. This can be explained by the fact that at the beginning of the instability interval the equilibrium point oscillates between two values that are close to one another. With increasing of c_r, two branches of the period doubling tree divergent more (the difference between the values, between which equilibrium point oscillates, increases) and the number of steps increases too.

In this paper we have shown that the use of DFC-method with the two control laws (the ability of the two firms to control unstable fluctuations) is ineffective. We have shown that there is a certain range of values of the coefficients K_1 and K_2 at which firms come to Cournot equilibrium, but these values are not optimal. Because for any K_1 or K_2 from permissible range of values it needs more time to stabilize the system than with one control law.

Conducting such studies for the triopoly model is an alternative for further study.

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