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THE METHOD OF FINDING POINTS OF INTERSECTION OF TWO CUBIC BÉZIER CURVES USING THE SYLVESTER MATRIX

Abstract. Sylvester matrix is used to create a 9th degree polynomial from coefficients of two cubic Bézier curves. The real roots of this polynomial allow to compute points of intersection of aforementioned curves. An additional constraint is used to indicate valid points. The "reverse-inverse law" is presented in order to reduce the cost of calculation in this particular case. Also some limitations of the method as well as the ways to avoid them, if possible, are pointed out.

1. Introduction

Bézier curves have been known since 1912, the year they were invented by S.N. Bernstein (1880–1968), Russian then Soviet mathematician. They were first put to practical use in the early 1960s, when P.É. Bézier and P. de Casteljau, French engineers, started using them in the automobile industry. They came to spotlight once again when graphic capabilities of personal computers had become advanced enough. Today Bézier curves are being frequently used in computer graphics, animation, CAD, and many other related fields [3].

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Despite their popularity, some questions still have not been answered satisfactorily in terms of intuitiveness or computational cost. The problem of intersection of two cubic Bézier curves is one of them.

In paper [10] the method called Bezier clipping is described. In this method, Bézier curves are substituted by their convex hulls, bounded by four line segments. As a typical "divide and conquer" algorithm, it is quite simple to adopt. However, the drawback of this method is associated with quadruple recursion, see the live code snippet from [2]:

```
static void intersectBeziers(&intersections,
                             const Bezier &a, const Bezier &b){
  if (accuracy) {  ...  } else {
    Bezier leftA, rightA, leftB, rightB;
    a.subdivide(leftA, rightA);
    b.subdivide(leftB, rightB);
    intersectBeziers(intersections, leftA, leftB);
    intersectBeziers(intersections, rightA, leftB);
    intersectBeziers(intersections, leftA, rightB);
    intersectBeziers(intersections, rightA, rightB);
  }
}
```

Therefore this method for today users has a big disadvantage, forcing a low accuracy under threat of slowdown of an application or even a stack-overflow crash.

Another approach is presented in [5]. In that case points of intersection emerges as eigenvalues of a properly created matrix. Those matrix-oriented transformations absorb most computational time. Provided example shows highly accurate results. However, these computations depend on libraries available on specific mainframes, so they are hard to reproduce and verify.

This paper presents a new approach to the problem of intersection of two cubic Bézier curves. The solution is based on the shortening or lengthening one of them. Proper length change is possible through use of a Sylvester matrix.

2. Sylvester matrix

Sylvester matrices are named after J.J. Sylvester, a mathematician [9] who lived in England 1814–1897. He made fundamental contributions to matrix theory, invariant theory, number theory, partition theory, and combinatorics.

A Sylvester matrix is a matrix associated to two univariate polynomials. For purpose of this article, let us limit these polynomials to the 3rd degree:

$$\begin{aligned} f(x) =& \quad a_3x^3 + a_2x^2 + a_1x + a_0, \qquad a_3 \neq 0,\\ g(x) =& \quad b_3x^3 + b_2x^2 + b_1x + b_0, \qquad b_3 \neq 0. \end{aligned}$$

The entries of the Sylvester matrix are coefficients of these polynomials:

$$S_{f,g} = \begin{bmatrix} a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & b_3 & b_2 & b_1 & b_0 \end{bmatrix}.$$

The determinant of the Sylvester matrix of two polynomials is their resultant [4]:

$$R(f,g) = a_3^3\, b_3^3 \prod_{i=1}^{3}\prod_{j=1}^{3}(\alpha_i - \beta_j).$$

where: α_i and β_j $(i, j = 1, 2, 3)$ are roots of $f(x)$ and $g(x)$, respectively.

We will use the following theorem [6]:

Theorem 1. *Two polynomials have a common root if and only if the resultant is equal to zero:*

$$\exists i,\ \exists j: \quad \alpha_i = \beta_j \qquad \Longleftrightarrow \qquad R(f,g) = \det(S_{f,g}) = 0.$$

3. Introductory example

To present a general idea, a simple example with two line segments will be used.

Let us examine a possible intersection of two line segments, given in a parametric form, as presented in Figure 1. The first will be:

$$\mathbf{l_1} = \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \begin{bmatrix} 1+9t \\ 2+3t \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix} = \mathbf{C_1 T}, \quad t \in \langle 0, 1 \rangle. \qquad (1)$$

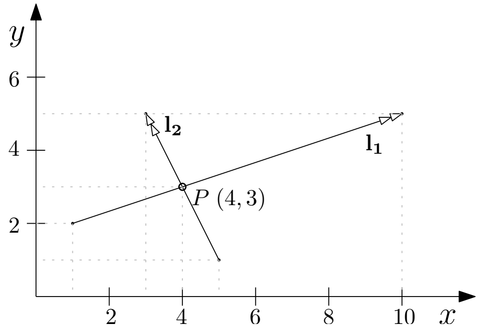

Fig. 1. Intersection of line segments

and the second one:

$$\mathbf{l_2} = \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 5-2t \\ 1+4t \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix} = \mathbf{C_2 T}, \quad t \in (0, 1\rangle. \tag{2}$$

Double arrows indicate an increase in the parameter t. The intersection occurs at the point $P(4,3)$, for parameters $t_1 = 1/3$ and $t_2 = 1/2$, respectively:

$$\mathbf{l_1}(t_1 = \frac{1}{3}) = \mathbf{l_2}(t_2 = \frac{1}{2}) = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

These values of t_1 and t_2 were calculated using elementary methods of an analytical geometry.

Now let us compute the subtracted curve $\mathbf{l_{1-2}}$:

$$\mathbf{l_{1-2}} = \mathbf{l_1} - \mathbf{l_2} = (\mathbf{C_1} - \mathbf{C_2})\mathbf{T} = \begin{bmatrix} -4 & 11 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix}, \tag{3}$$

thus:

$$\mathbf{l_{1-2}} = \begin{bmatrix} x_{1-2}(t) \\ y_{1-2}(t) \end{bmatrix} = \begin{bmatrix} -4 & + & 11\,t \\ 1 & - & 1\,t \end{bmatrix}.$$

An associated Sylvester matrix [8] to x_{1-2} and y_{1-2} polynomials is:

$$\mathbf{S_{1-2}} = \begin{bmatrix} 11 & -4 \\ -1 & 1 \end{bmatrix}.$$

Because $t_1 \neq t_2$, therefore $\det[S_{1-2}] = 7 \neq 0$. Can Sylvester matrix be useful in this case? The answer is presented below:

If two line segments* $\mathbf{l_1}$ *and* $\mathbf{l_2}$*:

- ***intersects at*** t_1 ***and*** t_2***, respectively,***
- $0 < t_1 < t_2 \leq 1$

then $\mathbf{l_1}$ ***can be shortened by factor*** $k < 1$ ***in such manner that the new shortened line segment*** $\mathbf{l_{1K}}$ ***will intersect*** $\mathbf{l_2}$ ***at the same parameter*** $t = t_2$***. Effectively, the determinant of associated Sylvester matrix*** $\det[\mathbf{S_{1K-2}}] = 0$***.***

If $0 < t_2 < t_1 \leq 1$, then $\mathbf{l_2}$ should be shortened.

In the discussed example, it happens when $\mathbf{l_1}$ turns into $\mathbf{l_{1K}}$ after shortening by factor $k_1 = \frac{2}{3}$. The new endpoint will have coordinates $Q(7, 4)$. In this case, $t_1 = t_2 = \frac{1}{2}$. In Figure 2 all these changes are shown.

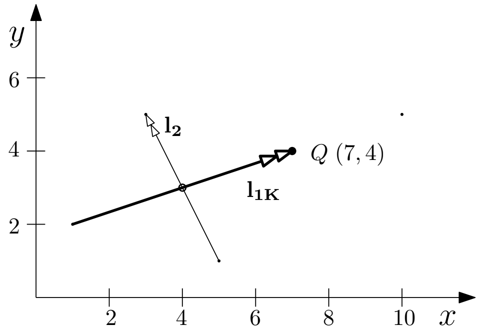

Fig. 2. Line segment $\mathbf{l_1}$ shortened to $\mathbf{l_{1K}}$

As will be explained later, equivalently $\mathbf{l_2}$ can be lengthened into $\mathbf{l_{2K}}$ by factor $k_2 = \frac{3}{2}$. The new endpoint of $\mathbf{l_{2K}}$ will have coordinates $R(2, 7)$. In this case, $t_1 = t_2 = \frac{1}{3}$. These changes are shown in Figure 3.

To achieve these results, (1) has to be modified

$$\mathbf{l_{1K}} = \mathbf{C_1KT} = \begin{bmatrix} 1 & 9 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix} = \begin{bmatrix} 1 + 9kt \\ 2 + 3kt \end{bmatrix}, \quad t \in (0, 1), \tag{4}$$

where $\mathbf{K}$ is a shortening matrix [1]:

$$\mathbf{K} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}, \qquad 0 < k \leq 1. \tag{5}$$

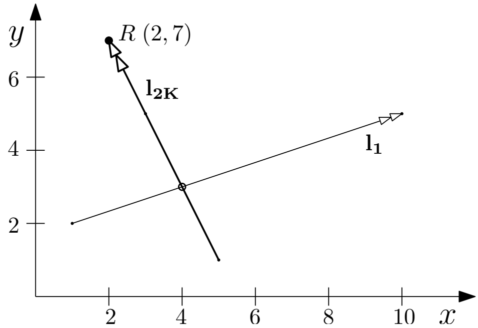

Fig. 3. Line segment $\mathbf{l_2}$ lengthened to $\mathbf{l_{2K}}$

Then (3) takes the form:

$$\mathbf{l_{1K-2}} = \mathbf{l_{1K}} - \mathbf{l_2} = (\mathbf{C_{1K}} - \mathbf{C_2})\mathbf{T} =$$
$$= \begin{bmatrix} x_{1K-2}(t,k) \\ y_{1K-2}(t,k) \end{bmatrix} = \begin{bmatrix} -4 & + & (9k+2)\,t \\ 1 & + & (3k-4)\,t \end{bmatrix}. \tag{6}$$

Now, Sylvester matrix to x_{1K-2} and y_{1K-2} polynomials is:

$$\mathbf{S_{1K-2}} = \begin{bmatrix} (9k+2) & -4 \\ (3k-4) & 1 \end{bmatrix},$$

and its determinant has form:

$$\det[\mathbf{S_{1K-2}}] = 21k - 14.$$

The important point to note here is ***the search for the intersection has been substituted by a root-finding issue for a simple polynomial.***

It is clear, which value of k ensures zero-value of determinant of Sylvester matrix:

$$k = k_1 = \frac{2}{3} \;\Rightarrow\; \det[\mathbf{S_{1K-2}}] = 0, \qquad 0 < k \le 1.$$

Intersection of given line segment is confirmed.

Choosing the parameter k_1 is always possible for given $0 < t_1 \le t_2 \le 1$. Due to symmetry also for k_2 is always possible for $0 < t_2 \le t_1 \le 1$. Both cases are shown in Figure 4.

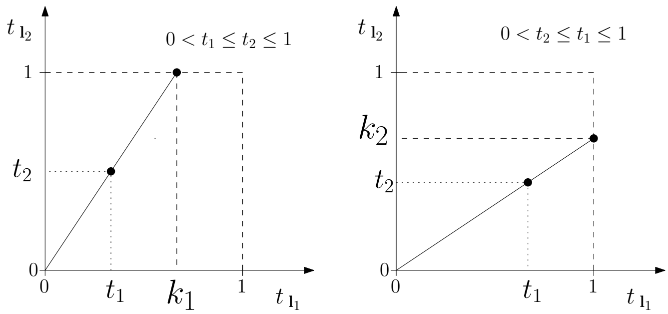

Fig. 4. Dependencies of k_1 and k_2 for given t_1, t_2

If we know value of k then it is possible to compute:

- value of t, where intersection occurs,
- coordinates x, y of the point of intersection.

Because at the intersection point:

$$\mathbf{l_{1K-2}} = \begin{bmatrix} x_{1K-2}(t,k) \\ y_{1K-2}(t,k) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

then putting $k = k_1 = \frac{2}{3}$ into any equation (the first one, for instance) of (6) we get:

$$\begin{aligned} 0 &= -4 + (9k+2)\,t = \\ &= -4 + (9 \cdot \tfrac{2}{3} + 2)\,t = \\ &= -4 + 8t, \end{aligned}$$

hence $t = \frac{1}{2}$. Knowing t after insertion into unchanged equation (2), we get coordinates of the point of intersection P:

$$P = \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 5-2t \\ 1+4t \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

One could have doubts, because there is no a priori knowledge about which line segment intersects at lower value of t. To explain this, let us replace line segments. After that, we get:

$$\mathbf{l_{2K}} = \mathbf{C_2KT} = \begin{bmatrix} 5 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix} = \begin{bmatrix} 1-2kt \\ 1+4kt \end{bmatrix}, \quad t \in (0,1).$$

The modified (3) will be:

$$\mathbf{l_{2K-1}} = \begin{bmatrix} x_{2K-1}(t,k) \\ y_{2K-1}(t,k) \end{bmatrix} = \begin{bmatrix} 4 & + & (-2k-9)\,t \\ -1 & + & (4k-3)\,t \end{bmatrix}.$$

Sylvester matrix:

$$\mathbf{S_{2K-1}} = \begin{bmatrix} (-2k-9) & 4 \\ (4k-3) & -1 \end{bmatrix},$$

and his determinant:

$$\det[\mathbf{S_{2K-1}}] = -14k + 21.$$

Hence

$$k = k_2 = \frac{3}{2} \;\Rightarrow\; \det[\mathbf{S_{2K-1}}] = 0.$$

In fact we can stop here, because from the shortening matrix (5) we require $0 < k \leq 1$, *but for reasons explained later, let us continue.*

Now $k > 1$ and the $\mathbf{l_2}$ is extended beyond its endpoint, so it could be $t \notin \langle 0, 1\rangle$. To check this, let take into account the determinant of $\mathbf{l_{2K-1}}$ matrix, which will have zero values in the intersection point:

$$\mathbf{l_{2K-1}} = \begin{bmatrix} x_{2K-1}(t,k) \\ y_{2K-1}(t,k) \end{bmatrix} = \begin{bmatrix} 4 & + & (-2k-9)\,t \\ -1 & - & (4k-3)\,t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

what gives two equations:

$$\begin{aligned} 4 + (-2k-9)\,t &= 0, \\ -1 - (4k-3)\,t &= 0. \end{aligned}$$

Putting $k = \frac{3}{2}$ and solving any of them, we get $t = \frac{1}{3}$, so t belongs to the required range $t \in \langle 0, 1\rangle$, where the intersection occurs.

As before, this value of t can be inserted into unchanged equation (1), to get the coordinates of point of intersection P:

$$P = \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \begin{bmatrix} 1 + 9t \\ 2 + 3t \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

Summary:

$$\begin{cases} k \in \langle 0, 1\rangle & \text{– intersection occurs;} \\ k > 1 & \text{– additional check is required; most simple one is: } kt \leq 1; \\ k < 0 & \text{– no intersection (self-evident).} \end{cases}$$

Finally, let us notice some kind of similarity between:

$$\det[\mathbf{S_{1K-2}}] = 21k - 14,$$

and

$$\det[\mathbf{S_{2K-1}}] = -14k + 21.$$

It is not a coincidence, but an evidence of ***reverse-inverse law***, introduced later.

4. Intersection of two cubic Bézier curves

Moving on from linear to a cubic Bézier curves, patterns become more complicated. The first cubic Bézier curve we denote as:

$$\mathbf{B_1} = \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \begin{bmatrix} c_{0x} & c_{1x} & c_{2x} & c_{3x} \\ c_{0y} & c_{1y} & c_{2y} & c_{3y} \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} = \mathbf{CT}, \qquad \begin{matrix} t \in \langle 0, 1\rangle, \\ c_{3x}^2 + c_{3y}^2 > 0, \end{matrix} \tag{7}$$

and the second one:

$$\mathbf{B_2} = \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} d_{0x} & d_{1x} & d_{2x} & d_{3x} \\ d_{0y} & d_{1y} & d_{2y} & d_{3y} \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} = \mathbf{DT}, \qquad \begin{matrix} t \in \langle 0, 1\rangle, \\ d_{3x}^2 + d_{3y}^2 > 0. \end{matrix}$$

The shortening matrix $\mathbf{K}$ [1]:

$$\mathbf{K} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & k^2 & 0 \\ 0 & 0 & 0 & k^3 \end{bmatrix}, \qquad 0 < k < 1.$$

For instance, let us shorten $\mathbf{B_2}$ curve. Afterwards it will look like $\mathbf{B_{2K}}$:

$$\mathbf{B_{2K}} = \mathbf{DKT} = \begin{bmatrix} x_{2K}(t) \\ y_{2K}(t) \end{bmatrix} = \begin{bmatrix} d_{0x} & d_{1x}k & d_{2x}k^2 & d_{3x}k^3 \\ d_{0y} & d_{1y}k & d_{2y}k^2 & d_{3y}k^3 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}.$$

The subtracted curve:

$$\mathbf{l_{2K-1}} = \mathbf{B_{2K}} - \mathbf{B_1} = \mathbf{CT} - \mathbf{DKT} = \begin{bmatrix} x_{1,2K}(t) \\ y_{1,2K}(t) \end{bmatrix} =$$

$$= \begin{bmatrix} d_{0x} - c_{0x} & d_{1x} - c_{1x}k & d_{2x} - c_{2x}k^2 & d_{3x} - c_{3x}k^3 \\ d_{0y} - c_{0y} & d_{1y} - c_{1y}k & d_{2y} - c_{2y}k^2 & d_{3y} - c_{3y}k^3 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} =$$

$$= \begin{bmatrix} d_{0x} - c_{0x} & (d_{1x} - c_{1x}k)t & (d_{2x} - c_{2x}k^2)t^2 & (d_{3x} - c_{3x}k^3)t^3 \\ d_{0y} - c_{0y} & (d_{1y} - c_{1y}k)t & (d_{2y} - c_{2y}k^2)t^2 & (d_{3y} - c_{3y}k^3)t^3 \end{bmatrix}. \tag{8}$$

After loading this into Sylvester matrix, it gets a following form:

$$\mathbf{S_{2K-1}} = \begin{bmatrix} a - bk^3 & c - dk^2 & e - fk & \Delta_x & 0 & 0 \\ 0 & a - bk^3 & c - dk^2 & e - fk & \Delta_x & 0 \\ 0 & 0 & a - bk^3 & c - dk^2 & e - fk & \Delta_x \\ m - nk^3 & p - qk^2 & r - sk & \Delta_y & 0 & 0 \\ 0 & m - nk^3 & p - qk^2 & r - sk & \Delta_y & 0 \\ 0 & 0 & m - nk^3 & p - qk^2 & r - sk & \Delta_y \end{bmatrix},$$

where:

$$\begin{aligned} &a = d_{3x}, \quad b = c_{3x}, \\ &c = d_{2x}, \quad d = c_{2x}, \\ &e = d_{1x}, \quad f = c_{1x}, \quad \Delta_x = d_{0x} - c_{0x}, \\ &m = d_{3y}, \quad n = c_{3y}, \\ &p = d_{2y}, \quad q = c_{2y}, \\ &r = d_{1y}, \quad s = c_{1y}, \quad \Delta_y = d_{0y} - c_{0y}. \end{aligned}$$

Calculating a determinant of this matrix requires considerable effort[1], which can be significantly reduced[2]. It is still too large to be shown here, though. After some factorizing it rises the sum:

$$\det[\mathbf{S_{1,2K}}] = \sum_{i=0}^{9} \beta_i k^i = 0, \tag{9}$$

where:

$$\beta_i = \Delta_x \cdot \beta_{ix}(a, b, c, d, e, f, \Delta_x, m, n, p, q, r, s, \Delta_y) + \\ + \Delta_y \cdot \beta_{iy}(a, b, c, d, e, f, \Delta_x, m, n, p, q, r, s, \Delta_y). \tag{10}$$

This is a polynomial of 9th degree of variable k, which can have up to 9 real roots. In fact, it is easy to show two cubic Bézier curves which intersect 9 times, as shown in Figure 5.

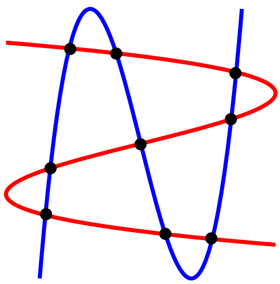

Fig. 5. Two cubic Bézier curves with 9 points of intersection

According to Abel-Ruffini theorem, it is possible to find the roots of this polynomial most likely by numerical approximation[3]. Now, it is required to:

- solve (9) to find all real roots k_i,
- select those from $0 < k_i \leq 1$ range, each of them is responsible for one point of intersection,

[1]The work consist of adding up 1728 ($= 54 \cdot 32$) elements, each containing a product of at least 6 multipliers, because the determinant of sparse Sylvester matrix consist of 54 non-zero permutations.

[2]It is possible to rotate the curves in such way that Δ_x (or Δ_y) is reduced to zero. As a result, the count of non-zero permutations drops from 54 to 17.

[3]The negligible exception are symmetrical equations [6], mentioned later due to other reasons.

- optionally: calculate[4] all real roots of $x_{1,2K}(t)$ and $y_{1,2K}(t)$ from equation (8), corresponding to known k_i values, then select the identical t_i from these two groups of roots (existence of such identical roots warrants theorem [1]),
- optionally: calculate coordinates of intersection points of $P_i(x_i, y_i)$ from equation (7) corresponding to t_i.

It might seem like there is a need to replace the curves and repeat the process. So that was, if not ***the reverse-inverse law***.

5. Reverse-inverse law

As one might have noticed already, a certain pattern emerges. It is called:

Theorem 2 (The reverse-inverse law). *Let $p(z)$ is a polynomial of n-th degree with real coefficients a_i:*

$$p(z) = \sum_{i=0}^{n} a_i z^i = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$

where:

$$\begin{aligned} &a_i \in \mathbb{R}, \quad i = 0, 1, 2, \ldots, n, \\ &a_n \neq 0, \\ &a_0 \neq 0. \end{aligned}$$

If polynomial $q(z)$ of n-th degree

$$q(z) = \sum_{i=0}^{n} b_i z^i$$

has coefficients b_i associated with a_i of $p(z)$ ***in reverse order****:*

$$b_i = a_{n-i}, \qquad i = 0, 1, 2, \ldots, n,$$

then the roots ζ_i of the equation $q(z) = 0$, both real and complex, are associated with the roots z_i of the equation $p(z) = 0$ ***in inverse manner****:*

$$\zeta_i = \frac{1}{z_i}.$$

[4]Cardano method is required to solve 3-rd degree polynomial.

Proof. Let us divide $p(z)$ by z^n:

$$p(z)/z^n = \sum_{i=0}^{n} a_i z^{(i-n)} = \frac{a_0}{z^n} + \frac{a_1}{z^{(n-1)}} + \frac{a_2}{z^{(n-2)}} + \cdots + \frac{a_{n-1}}{z} + \frac{a_n}{1}.$$

This transformation does not change the roots of the new polynomial. Next, let us substitute z by $\zeta = \frac{1}{z}$:

$$\begin{aligned} p(z)/z^n = p(\frac{1}{\zeta})\, \zeta^n &= a_0\zeta^n + a_1\zeta^{(n-1)} + a_2\zeta^{(n-2)} + \cdots + a_{n-1}\zeta + a_n = \\ &= a_n + a_{n-1}\zeta + a_{n-2}\zeta^2 + \cdots + a_0\zeta^n = \\ &= b_0 + b_1\zeta + b_2\zeta^2 + \cdots + b_n\zeta^n = \\ &= \sum_{i=0}^{n} b_i\zeta^i = \\ &= q(\zeta). \end{aligned}$$

Finally:

$$p(z) = z^n q(\zeta).$$

Hence, if $p(z)$ has roots at z_i: $p(z_i) = 0$, then $q(\zeta_i = \frac{1}{z_i}) = 0$. □

Remark 3. This law works also for less restrictive assumptions, namely when a few of leading coefficients of polynomial $p(z)$ have value equal to zero:

$$p(z) = \sum_{i=k}^{n} a_i z^i = a_k z^k + a_{k+1} z^{k+1} + \cdots + a_n z^n,$$

where:

$$\begin{aligned} &0 < k < n, \\ &a_i \in \mathbb{R}, \quad i = k, k+1, \ldots, n, \\ &a_n \neq 0, \\ &a_k \neq 0. \end{aligned}$$

In this case, $p(z)$ changes into:

$$p(z) = z^k\, \breve{p}(z),$$

where:

$$\breve{p}(z) = \sum_{i=0}^{n-k} a_{i+k} z^i$$

meets all assumptions of the law.

Example 4. Let:

$$\begin{aligned} p(x) = \quad & x^5 - 5x^4 + 6x^3 = x^3(x^2 - 5x + 6) \quad \Rightarrow \quad & \breve{p}(x) = x^2 - 5x + 6 \\ & & \text{(reverted coefficients)} \\ & & q(x) = 6x^2 - 5x + 1 \end{aligned}$$

then:

$$\begin{array}{ccc} p(2) = p(3) = p(0) = 0 & & \\ \Downarrow & & \text{(inverted roots)} \\ \breve{p}(2) = \breve{p}(3) = 0 & \Longleftrightarrow & q(\frac{1}{2}) = q(\frac{1}{3}) = 0. \end{array}$$

Symmetric equations

Symmetric equations [6] are equations, where a polynomial has symmetric coefficients, e.g.:

$$\forall i: \ a_i = a_{n-i}.$$

Example 5. Following equations:

$$\begin{aligned} &0 = 1 - 10x + 100x^2 - 10x^3 + x^4, \\ &0 = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 4x^5 + 3x^6 + 2x^7 + x^8, \\ &0 = 1 + 5x^4 + x^8, \end{aligned}$$

all are symmetric equations.

For these equations, we provide two laws, because they naturally result from the reverse-inverse law [2].

Theorem 6. *If z is a root of a symmetric equation, then $\frac{1}{z}$ is also a root of this equation.*

Theorem 7. *All symmetric equations of degree* $2n$ *(even ones) can be created by this formula:*

$$f(x) = a \prod_{i=1}^{n} (x - z_i)\left(x - \frac{1}{z_i}\right)$$

and symmetric equations of degree $2n + 1$ *(odd ones):*

$$f(x) = a\,(x + 1) \prod_{i=1}^{n} (x - z_i)\left(x - \frac{1}{z_i}\right).$$

Importance of reverse-inverse law for this method of finding intersections

First thought was the process of shortening should be performed twice, one time for each of two curves. This involves calculating a determinant of Sylvester matrix from equation (9):

$$\det[\mathbf{S_{1K-2}}],$$

then repeating the same steps for:

$$\det[\mathbf{S_{2K-1}}].$$

However, the polynomials resulting from these two Sylvester matrices are always reversed each other. So there is no need to solve two polynomial equations, because roots of both polynomials are governed by reverse-inverse law.

Finally, instead of:

- solving two equations, then
- seeking roots twice in range $r_i \in (0, 1\rangle$,

it is enough to solve one equation and seek its roots in range $r_i \in (0, +\infty)$.

6. Example

Determine the point of intersection K of two curves shown below.

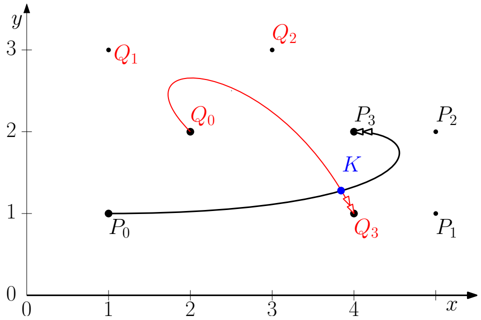

Fig. 6. Two intersecting cubic Bézier curves

Here are detailed calculations:

```
Points of the 1st curve:    P0(1,1)  P1(5,1)  P2(5,2)  P3(4,2)
Coefficients of the 1st curve:
 c3x =   3          c3y =  -2
 c2x = -12          c2y =   3
 c1x =  12          c1y =   0
 c0x =   1          c0y =   1
```

$$\begin{cases} x_1(t) = 3t^3 - 12t^2 + 12t + 1, \\ y_1(t) = -2t^3 + 3t^2 + 0t + 1. \end{cases}$$

```
Points of the 2nd curve:    Q0(2,2)  Q1(1,3)  Q2(3,3)  Q3(4,1)
Coefficients of the 2nd curve:
 d3x =  -4          d3y =  -1
 d2x =   9          d2y =  -3
 d1x =  -3          d1y =   3
 d0x =   2          d0y =   2
```

$$\begin{cases} x_2(t) = -4t^3 + 9t^2 - 3t + 2, \\ y_2(t) = -1t^3 - 3t^2 + 3t + 2. \end{cases}$$

Sylvester matrix:

$$\begin{bmatrix} 3+4k^3 & -12-9k^2 & 12+3k & -1 & 0 & 0 \\ 0 & 3+4k^3 & -12-9k^2 & 12+3k & -1 & 0 \\ 0 & 0 & 3+4k^3 & -12-9k^2 & 12+3k & -1 \\ -2+k^3 & 3+3k^2 & -3k & -1 & 0 & 0 \\ 0 & -2+k^3 & 3+3k^2 & -3k & -1 & 0 \\ 0 & 0 & -2+k^3 & 3+3k^2 & -3k & -1 \end{bmatrix}.$$

```
Coefficients of the 9th degree polynomial:
          coeff(0) =     -332          coeff(1) =    -2673
          coeff(2) =     3789          coeff(3) =     -567
          coeff(4) =     7074          coeff(5) =    -1107
          coeff(6) =    -6615          coeff(7) =     1215
          coeff(8) =    -6804          coeff(9) =     2619

Real roots of the 9th degree polynomial : 5
      ROOT 1 = 2.73698707    > 0, initially approved
      root 2 = -0.10727734   < 0, rejected
      ROOT 3 = 0.69536681    > 0, initially approved
      root 4 = -0.96568768   < 0, rejected
      ROOT 5 = 0.77505278    > 0, initially approved


   ROOT 1,    k = 2.73698707
==============================
FIRST EQUATION :
   3rd degree, Coefficients :
          a3= 85.01215564
          a2= -79.41988385
          a1= 20.21096120
          a0= -1.00
   3 Roots :
   (1)= 0.5218056   (2)= 0.0648624   (3)= 0.3475500
SECOND EQUATION :
   3rd degree, Coefficients :
          b3= 18.50303891
          b2= 25.47329462
```

```
         b1= -8.21096120
         b0= -1.00
   3 Roots :   (1)= 0.3475500   (2)= -1.6287869   (3)= -0.0954719
---------------------------
COMMON ROOT : 0.3475500
---------------------------
1st curve, t1 = 0.3475500 ,     0 <  t1 <= 1    , OK
2nd curve, t2 = 0.9512399 ,     0 <  t2 <= 1    , OK

Point of intersection:
   x = 3.8470508
   y = 1.2784112



   ROOT 3,    k = 0.69536681
==============================
FIRST EQUATION :
   3rd degree, Coefficients :
         a3= 4.34493675
         a2= -16.35181498
         a1= 14.08610042
         a0= -1.00
   3 Roots :
   (1)= 2.5067572   (2)= 0.0778887   (3)= 1.1787725
SECOND EQUATION :
   3rd degree, Coefficients :
         b3= -1.66376581
         b2= 4.45060499
         b1= -2.08610042
         b0= -1.00
   3 Roots :   (1)= 1.7823281   (2)= -0.2860817   (3)= 1.1787725
---------------------------
COMMON ROOT : 1.1787725
---------------------------

1st curve, t1 = 1.1787725 ,  outside limits, rejected.
```

```
   ROOT 5,    k = 0.77505278
==============================
FIRST EQUATION :
   3rd degree, Coefficients :
          a3= 4.86231797
          a2= -17.40636137
          a1= 14.32515835
          a0= -1.00
   3 Roots :
   (1)= 2.3766154   (2)= 0.0768249   (3)= 1.1264081
SECOND EQUATION :
   3rd degree, Coefficients :
          b3= -1.53442051
          b2= 4.80212046
          b1= -2.32515835
          b0= -1.00
   3 Roots :   (1)= 2.3766154   (2)= -0.2684614   (3)= 1.0214447
---------------------------
COMMON ROOT : 2.3766154
---------------------------
1st curve, t1 = 2.3766154 ,  outside limits, rejected.
```

Summing up for the analysed example, there exist only one point of intersection for $k \approx 0.34755$.

7. Remarks

At first glance, the presented method has great potential. However, it has its own weaknesses, listed below. It also depends on other methods, especially on polynomial root – finding algorithms, which bring its own imperfections.

7.1. Common beginning

When both cubic Bézier curves have the same starting point, then:

$$\Delta_x = d_{0x} - c_{0x} = 0 \quad \text{and} \quad \Delta_y = d_{0y} - c_{0y} = 0,$$

then, from (10), all coefficients $\beta_i = 0$, hence the value of determinant (9) is equal to zero for any k and the method fails.

It can be somewhat remedied by switching the beginning and the end of the curve, as shown in Figure 7, because curves can also have a common ending point.

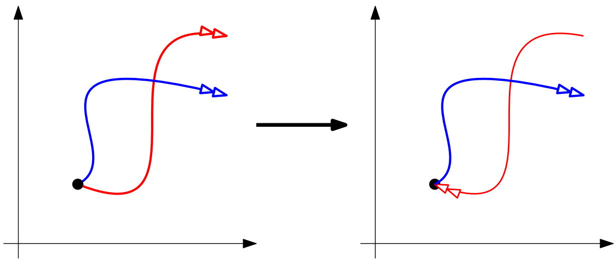

Fig. 7. Common begin of curves

A radical way to solve this issue would be to extend both curves in such way that their beginnings do not overlap. It is always possible to do so, see [1].

7.2. Rounding errors

If one of the roots of the equation is:

$$k = \sqrt[3]{\frac{c_{3x}}{d_{3x}}} \qquad \Big(\text{or} \quad k = \sqrt[3]{\frac{c_{3y}}{d_{3y}}}\Big),$$

then due to rounding error

$$c_{3x} - d_{3x}k^3 \neq 0 \qquad \big(\text{or} \quad c_{3y} - d_{3y}k^3 \neq 0\big).$$

Without special treatment of such cases there is possible to compute roots of a wrong polynomial with the highest coefficient value a_n close to zero. In such case, the result is going to be unacceptable.

7.3. Both curves belonging to one K-family

Definition 8 (K-family, [1]). *Let C and D determine two cubic Bézier curves:*

$$\mathbf{C} = \begin{bmatrix} c_{0x} & c_{1x} & c_{2x} & c_{3x} \\ c_{0y} & c_{1y} & c_{2y} & c_{3y} \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} d_{0x} & d_{1x} & d_{2x} & d_{3x} \\ d_{0y} & d_{1y} & d_{2y} & d_{3y} \end{bmatrix}.$$

Let $\mathbf{K}$ *be a transformation matrix:*

$$\mathbf{K} = \mathbf{K}(u, v) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ u & v & 0 & 0 \\ u^2 & 2uv & v^2 & 0 \\ u^3 & 3u^2v & 3uv^2 & v^3 \end{bmatrix}, \qquad u, v \in \mathbb{R}.$$

If exist (u, v) *such a:*

$$\mathbf{D} = \mathbf{KC},$$

then $\mathbf{C}$ *and* $\mathbf{D}$ *belong to one K-family.*

The presented method fails when both curves belong to one K-family. Such a case is shown on the left side of Figure 8. To prove that they are members of K-family, on the right it is shown another member of this family, which includes both curves. There is not known way to find a solution in this case, although such instances are extremely rare.

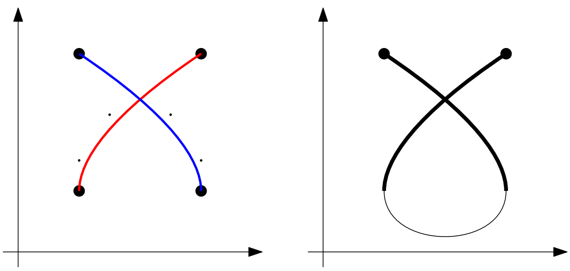

Fig. 8. K-group members

7.4. Errors in the polynomial root finding methods

Since we seek real roots of a polynomial, it is forbidden to use any method that may provide complex roots instead, like Bairstow method [7].

7.5. Errors due to wrong usage

The example shown in Figure 9 is a case of lower degree polynomial

$$c_{3x} = c_{3y} = 0 \qquad (\text{or} \;\; d_{3x} = d_{3y} = 0).$$

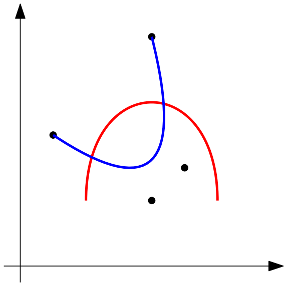

Fig. 9. Exemplary curves for lower order of matrix

In such a case we need to lower order of the Sylvester matrix, since shown examples work only for 3rd degree polynomials.

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