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TWO-WEIGHT NORM INEQUALITIES FOR ROUGH FRACTIONAL INTEGRAL OPERATORS ON MORREY SPACES

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Abstract. We establish the two-weight norm inequalities for the rough fractional integral operators on Morrey spaces.

Keywords: two-weight norm inequalities, rough fractional integral operators, Morrey spaces.

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1. INTRODUCTION

This paper aims to establish the two-weight norm inequalities for the rough fractional integral operators on Morrey spaces. The rough fractional integral operators are introduced by Muckenhoupt and Wheeden in [15]. It is an extension of the fractional integral operators (Riesz potentials). For the applications of these operators, the reader may consult [15].

The mapping properties of the rough fractional integral operators on power-weighted Lebesgue spaces were obtained in [15]. It had been extended to weighted Lebesgue spaces in [4]. The results given in [4] are the one-weight norm inequalities for the rough fractional integral operators. The one-weight norm inequalities are special cases of two-weight norm inequalities. The Stein–Weiss inequality, which gives the mapping properties of the fractional integral operators on power weighted Lebesgue spaces, is one of the pioneer results on two-weight norm inequalities for fractional integral operators. The two-weight norm for the rough fractional integral operators had been established in [3].

The mapping property of fractional integral operators on Morrey type spaces is one of the important extensions for the study of the fractional integral operators. The classical results are the Spanne's result [19] and the Adams inequalities [1]. The mapping properties of the fractional integral operators on Morrey give applications on Schrödinger equation, see [18]. The mapping properties for the singular integral operators and the fractional integral operators on some generalizations of Morrey spaces such as weak Morrey space, Orlicz–Morrey spaces, Morrey spaces with variable exponents and Morrey–Banach spaces are given in [2, 5–7, 12, 13, 16, 17, 20, 21, 25].

The one-weight norm inequalities for maximal operator, singular integral operators and fractional integral operators on Morrey spaces are given by the mapping properties of these operators on the weighted Morrey spaces. These mapping properties are provided in [14]. The two-weight norm inequalities on Morrey spaces aand Herz spaces are given in [9, 22–24, 26]. In this paper, we extend the two-weight norm inequalities for the rough fractional integral operators on Lebesgue spaces to Morrey spaces.

This paper is organized as follows. Section 2 presents the definition of the rough fractional integral operators and its two-weight norm inequalities on Lebesgue spaces. The main result of this paper, the two-weight norm inequalities of the rough fractional integral operators on Morrey spaces are established in Section 3.

2. PRELIMINARIES

Let $0 < \alpha < n$ and Ω be a homogeneous function on $\mathbb{R}^n$ with degree zero, that is, for any $x \in \mathbb{R}^n$ and $\lambda > 0$

$$\Omega(\lambda x) = \Omega(x),$$

and $\Omega \in L^s(\mathbb{S}^{n-1})$, where $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$. The fractional integral operator with rough kernel is defined by

$$T_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)dy.$$

Notice that the above operator is also named as the fractional integral operator with homogeneous kernel, see [10, 15].

The rough fractional maximal operator $M_{\Omega,\alpha}$ is defined by

$$M_{\Omega,\alpha} f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|x-y|<r} |\Omega(x-y)||f(y)|dy.$$

Let $B(z,r) = \{x \in \mathbb{R}^n : |x-z| < r\}$ denote the open ball with center $z \in \mathbb{R}^n$ and radius $r > 0$. Let $\mathcal{B} = \{B(z,r) : z \in \mathbb{R}^n, r > 0\}$. For any Lebesgue measurable set E, let $|E|$ and χ_E be the Lebesgue measure and the characteristic function of E, respectively.

We now state the classes of weight functions used in this paper. We begin with the well known Muckenhoupt weight functions.

Definition 2.1. For $1 < p < \infty$, a locally integrable function $u : \mathbb{R}^n \to [0,\infty)$ is said to be an A_p weight if

$$\sup_{B\in\mathcal{B}} \left(\frac{1}{|B|} \int_B u(x)dx \right) \left(\frac{1}{|B|} \int_B u(x)^{-\frac{p'}{p}} dx \right)^{\frac{p}{p'}} < \infty,$$

where $p' = \frac{p}{p-1}$ and $|B|$ denotes the Lebesgue measure of $B \in \mathcal{B}$.

Next, we recall the classes of weight functions used in [4] for the studies of the weighted norm inequalities of the rough fractional integral operators.

Definition 2.2. Let $1 < p, q < \infty$. For any nonnegative function u, we write $u \in A(p, q)$ if

$$\sup_{B\in\mathcal{B}} \left(\frac{1}{|B|}\int\limits_B u(x)^q dx\right)^{1/q} \left(\frac{1}{|B|}\int\limits_B u(x)^{-p'} dx\right)^{1/p'} < \infty.$$

The following are two classes of weight functions introduced in [3] for the studies of the two-weight norm inequalities of the rough fractional integral operators.

Definition 2.3. Let $1 < p < \infty$. For any pair of nonnegative functions u, v, we write $(u, v) \in A^*(p)$ if

$$\sup_{B\in\mathcal{B}} \left(\frac{1}{|B|}\int\limits_B u(x) dx\right) \left(\frac{1}{|B|}\int\limits_B v(x)^{-\frac{p'}{p}} dx\right)^{\frac{p}{p'}} < \infty.$$

Definition 2.4. Let $1 < p, q < \infty$. For any pair of nonnegative functions u, v, we write $(u, v) \in A^*(p, q)$ if

$$\sup_{B\in\mathcal{B}} \left(\frac{1}{|B|}\int\limits_B u^q(x) dx\right)^{1/q} \left(\frac{1}{|B|}\int\limits_B v^{-p'}(x) dx\right)^{1/p'} < \infty.$$

Let $1 < p < \infty$. For any nonnegative function u, the weighted Lebesgue space consists of those Lebesgue measurable functions satisfying

$$\|f\|_{L^p(u)} = \left(\int\limits_{\mathbb{R}^n} |f(x)|^p u(x) dx\right)^{1/p} < \infty.$$

When $u \equiv 1$, we write $L^p(u)$ by L^p.

We now present the two-weight norm inequalities for the rough fractional integral operator from [3, Theorem 1.1].

Theorem 2.5. *Let $s > 1$, $0 < \alpha < n$, $1 < p < n$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Suppose that Ω is a homogeneous function on $\mathbb{R}^n$ with degree zero and $\Omega \in L^s(\mathbb{S}^{n-1})$. If*

(1) $1 \le s' < p$, $(u^{s'}, v^{s'}) \in A^*(p/s', q/s')$ *and* $u^{s'}, v^{s'} \in A(p/s', q/s')$ *or*
(2) $s > q$, $(v^{-s'}, u^{-s'}) \in A^*(q'/s', p'/s')$ *and* $v^{-s'}, u^{-s'} \in A(q/s', p/s')$,

then there is a constant $C > 0$ such that

$$\|T_{\Omega,\alpha} f\|_{L^q(u^q)} \le C\|f\|_{L^p(v^p)}.$$

For the proof of the above result, the reader is referred to [3, Section 3].

3. MAIN RESULT

In this section, we establish the two-weight norm inequalities of the rough fractional integral operators on Morrey spaces. We present our result in terms of the mapping properties of the rough fractional integral operators on two different weighted Morrey spaces. Therefore, we begin with the definition of weighted Morrey spaces.

Let $\omega : \mathcal{B} \to (0,\infty)$. We say that ω is a Lebesgue measurable function if $\omega(B(\cdot,\cdot))$ is Lebesgue measurable on $\mathbb{R}^n \times (0,\infty)$.

Definition 3.1. Let $1 < p < \infty$. For any Lebesgue measurable functions $u : \mathbb{R}^n \to (0,\infty)$ and $\omega : \mathcal{B} \to (0,\infty)$, the weighted Morrey space $M^p_\omega(u)$ consists of those Lebesgue measurable functions f satisfying

$$\|f\|_{M^p_\omega(u)} = \sup_{B\in\mathcal{B}} \frac{1}{\omega(B)} \|\chi_B f\|_{L^p(u)} < \infty.$$

Notice that there are two sets of conditions that guarantees the validity of Theorem 2.5. We first establish the mapping properties of the rough fractional integral operators on $M^p_\omega(v^p)$ and $M^q_\omega(u^q)$ when u and v satisfy Item (1) of Theorem 2.5. We start with the condition satisfied by the function ω.

Definition 3.2. Let $1 < p < \infty$ and u be a non-negative locally integrable function. For any $\omega : \mathcal{B} \to (0,\infty)$, we write $\omega \in \mathcal{W}_{p,u}$ if there exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$ and $r > 0$

$$\omega(B(x,2r)) \le C\omega(B(x,r)), \tag{3.1}$$

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^p(u)}}{\|\chi_{B(x,2^{j+1})}\|_{L^p(u)}} \omega(B(x,2^{j+1}r)) \le C\omega(B(x,r)).$$

We are now ready to present the first result on the two-weight norm inequalities of the rough fractional integral operators on Morrey spaces.

Theorem 3.3. *Let $n \in \mathbb{N}$, $s > n'$, $0 < \alpha < n$, $s' < p < n$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Suppose that Ω is a homogeneous function on $\mathbb{R}^n$ with degree zero and $\Omega \in L^s(\mathbb{S}^{n-1})$. Suppose that the Lebesgue measurable functions $u, v : \mathbb{R}^n \to (0,\infty)$ satisfy Item (1) of Theorem 2.5. If the Lebesgue measurable function $\omega : \mathbb{B} \to (0,\infty)$ belongs to $\mathcal{W}_{q,u^q}$, then there is a constant $C > 0$ such that for any $f \in M^p_\omega(v^p)$*

$$\|T_{\Omega,\alpha} f\|_{M^q_\omega(u^q)} \le C\|f\|_{M^p_\omega(v^p)}.$$

Proof. Let $B = B(z,r) \in \mathcal{B}$ and $f \in M^p_\omega(v^p)$. For any $z \in \mathbb{R}^n$ and $r > 0$, define

$$f_0 = \chi_{B(z,2r)} f \quad \text{and} \quad f_j = \chi_{B(z,2^{j+1}r)\setminus B(z,2^j r)} f, \quad j \in \mathbb{N}\setminus\{0\}.$$

We have $f(x) = f_0(x) + \sum_{j=1}^{\infty} f_j(x)$.

Theorem 2.5 asserts that $T_{\Omega,\alpha} : L^p(v^p) \to L^q(u^q)$. That is,

$$\|T_{\Omega,\alpha} f_0\|_{L^q(u^q)} \le C\|f_0\|_{L^p(v^p)}$$

and, hence, (3.1) gives

$$\frac{1}{\omega(B(z,r))}\|\chi_{B(z,r)}(T_{\Omega,\alpha}f_0)\|_{L^q(u^q)} \le C\frac{1}{\omega(B(z,2r))}\|\chi_{B(z,2r)}f\|_{L^p(v^p)}. \tag{3.2}$$

There is a constant $C>0$ such that, for any $j\ge 1$

$$\begin{aligned}&\chi_{B(z,r)}(x)|(T_{\Omega,\alpha}f_j)(x)|\\ &\le C\chi_{B(z,r)}(x)\int\limits_{B(z,2^{j+1}r)\backslash B(z,2^jr)}|\Omega(x-y)||x-y|^{-n+\alpha}|f(y)|dy.\end{aligned} \tag{3.3}$$

Since Ω is a homogeneous function on $\mathbb{R}^n$ with degree zero and $\Omega\in L^s(\mathbb{S}^{n-1})$, according to [10, (5.9)], we have

$$\chi_{B(z,r)}(x)|(T_{\Omega,\alpha}f_j)(x)| \le C\chi_{B(z,r)}(x)\frac{\|\chi_{B(z,2^{j+1}r)}f\|_{L^{s'}}}{|B(z,2^{j+1}r)|^{\frac{1}{s'}-\frac{\alpha}{n}}}. \tag{3.4}$$

As $p>s'$, the Hölder inequality asserts that

$$\begin{aligned}\|\chi_{B(z,2^{j+1}r)}f\|_{L^{s'}} &\le C\||\chi_{B(z,2^{j+1}r)}|f|^{s'}\|^{1/s'}_{L^{p/s'}(v^p)}\|\chi_{B(z,2^{j+1}r)}\|^{1/s'}_{L^{(p/s')'}(v^{-(p/s')'s'})}\\ &= C\|\chi_{B(z,2^{j+1}r)}f\|_{L^p(v^p)}\|\chi_{B(z,2^{j+1}r)}\|^{1/s'}_{L^{(p/s')'}(v^{-(p/s')'s'})}.\end{aligned} \tag{3.5}$$

In addition, since $(u^{s'},v^{s'})\in A^*(p/s',q/s')$, we have a constant $C>0$ such that for any $B(z,r)\in\mathcal{B}$,

$$\left(\frac{1}{|B(z,r)|}\int\limits_{B(z,r)}u(x)^q dx\right)^{s'/q}\left(\frac{1}{|B(z,r)|}\int\limits_{B(z,r)}v(x)^{-s'(p/s')'}dx\right)^{1/(p/s')'} < C.$$

Therefore,

$$\begin{aligned}\|\chi_{B(z,2^{j+1}r)}\|^{1/s'}_{L^{(p/s')'}(v^{-(p/s')'s'})} &= \left(\int\limits_{B(z,2^{j+1}r)}v(x)^{-s'(p/s')'}dx\right)^{1/s'(p/s')'}\\ &\le C|B(z,2^{j+1}r)|^{\frac{1}{q}+\frac{1}{s'(p/s')'}}\frac{1}{\|\chi_{B(z,2^{j+1}r)}\|_{L^q(u^q)}}.\end{aligned}$$

Notice that

$$\frac{1}{q}+\frac{1}{s'(p/s')'}=\frac{1}{q}+\frac{1}{s'}-\frac{1}{p}=\frac{1}{s'}-\frac{\alpha}{n}. \tag{3.6}$$

Consequently, (3.4), (3.5) and (3.6) give

$$\chi_{B(z,r)}(x)|(T_{\Omega,\alpha}f_j)(x)| \le C\chi_{B(z,r)}(x)\frac{\|\chi_{B(z,2^{j+1}r)}f\|_{L^p(v^p)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^q(u^q)}}.$$

Applying the norm $\|\cdot\|_{L^q(u^q)}$ on both sides of the above inequality, we have

$$\|\chi_{B(z,r)}T_{\Omega,\alpha}f_j\|_{L^q(u^q)} \leq C\|\chi_{B(z,r)}\|_{L^q(u^q)}\frac{\|\chi_{B(z,2^{j+1}r)}f\|_{L^p(v^p)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^q(u^q)}}. \tag{3.7}$$

We find that

$$\begin{aligned}&\frac{1}{u(z,r)}\|\chi_{B(z,r)}T_{\Omega,\alpha}f\|_{L^q(u^q)}\\&\leq \frac{1}{u(z,r)}\|\chi_{B(z,r)}Tf_0\|_{L^q(u^q)}+\sum_{j=1}^{\infty}\frac{1}{u(z,r)}\|\chi_{B(z,r)}Tf_j\|_{L^q(u^q)}.\end{aligned}$$

Therefore, (3.2) and (3.7) yield

$$\begin{aligned}&\frac{1}{u(z,r)}\|\chi_{B(z,r)}T_{\Omega,\alpha}f\|_{L^q(u^q)}\\&\leq C\|f_0\|_{M^p_\omega(v^p)}+C\sum_{j=1}^{\infty}\frac{u(z,2^{j+1}r)}{u(z,r)}\frac{\|\chi_{B(z,r)}\|_{L^q(u^q)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^q(u^q)}}\frac{\|\chi_{B(z,2^{j+1}r)}f\|_{L^p(v^p)}}{u(z,2^{j+1}r)}\\&\leq C\|f_0\|_{M^p_\omega(v^p)}+C\sum_{j=1}^{\infty}\frac{u(z,2^{j+1}r)}{u(z,r)}\frac{\|\chi_{B(z,r)}\|_{L^q(u^q)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^q(u^q)}}\|f\|_{M^p_\omega(v^p)}.\end{aligned}$$

Hence, (3.1) asserts that

$$\frac{1}{u(z,r)}\|\chi_{B(z,r)}T_{\Omega,\alpha}f\|_{M^q_\omega(u^q)} \leq C\|f\|_{M^p_\omega(v^p)}$$

for some $C>0$ independent of $z\in\mathbb{R}^n$ and $r>0$. By taking supremum over $B(z,r)\in\mathcal{B}$ on both sides of the above inequality, we obtain our desired result. □

Next, we present and establish the mapping properties of the rough fractional integral operators on $M^p_\omega(v^p)$ and $M^q_\omega(u^q)$ when u and v satisfy Item (2) of Theorem 2.5. We begin with the following characterization for the weight ω.

Definition 3.4. Let $1<p<\infty$, $\beta>0$, and v be a non-negative locally integrable function. For any $\omega:\mathcal{B}\to(0,\infty)$, we write $\omega\in\tilde{\mathcal{W}}_{p,v,\beta}$ if there exists a constant $C>0$ such that for any $x\in\mathbb{R}^n$ and $r>0$

$$\omega(B(x,2r)) \leq C\omega(B(x,r)), \tag{3.8}$$

$$\sum_{j=1}^{\infty}\frac{\|\chi_{B(z,2^jr)}\|_{L^{p'}(v^{-p'})}}{\|\chi_{B(z,r)}\|_{L^{p'}(v^{-p'})}}\frac{|B(z,r)|^\beta}{|B(z,2^jr)|^\beta}\omega(B(z,2^jr)) \leq C\omega(B(z,r)) \tag{3.9}$$

We now present the second result for the rough fractional integral operators on Morrey spaces.

Theorem 3.5. *Let $n \in \mathbb{N}$, $0 < \alpha < n$, $\max((n/\alpha)', q) < s$, $1 < p < n$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Suppose that Ω is a homogeneous function on $\mathbb{R}^n$ with degree zero and $\Omega \in L^s(\mathbb{S}^{n-1})$. Suppose that the Lebesgue measurable functions $u, v : \mathbb{R}^n \to (0,\infty)$ satisfy Item (2) of Theorem 2.5. If the Lebesgue measurable function $\omega : \mathbb{B} \to (0,\infty)$ belongs to $\tilde{\mathcal{W}}_{p,v,\frac{1}{s'}-\frac{\alpha}{n}}$, then there is a constant $C > 0$ such that for any $f \in M^p_\omega(v^p)$*

$$\|T_{\Omega,\alpha} f\|_{M^q_\omega(u^q)} \le C\|f\|_{M^p_\omega(v^p)}.$$

Roughly speaking, Item (2) of Theorem 2.5 the dual result of Item (1) of Theorem 2.5. Therefore, to obtain Theorem 3.5, we need to use the notion of blocks [8, Definition 2.7] and [10, Definition 4.2] because the block space is a pre-dual of the Morrey space.

Definition 3.6. Let $1 < p < \infty$, $u : \mathbb{R}^n \to (0,\infty)$ and $\omega : \mathcal{B} \to (0,\infty)$ be a Lebesgue measurable functions. A Lebesgue measurable function b is a (p, u, ω)-block if it is supported in a ball $B(z, r)$, $r > 0$, $z \in \mathbb{R}^n$ and

$$\|b\|_{L^p(u)} \le \frac{1}{\omega(B(z,r))}.$$

We write $b \in b_{p,\omega}(u)$ if b is a (p, u, ω)-block.

We present some supporting results for the proof of Theorem 3.5.

Proposition 3.7. *Let $1 < p < \infty$, $v : \mathbb{R}^n \to (0,\infty)$ and $\omega : \mathcal{B} \to (0,\infty)$ be a Lebesgue measurable functions. There exist constants $C, D > 0$ such that for any $f \in M^p_\omega(v^p)$*

$$C\|f\|_{M^p_\omega(v^p)} \le \sup_{b \in b_{p',\omega}(v^{-p'})} \left| \int_{\mathbb{R}^n} f(x)b(x)dx \right| \le D\|f\|_{M^p_\omega(v^p)}.$$

Proposition 3.8. *Let $1 < p < \infty$, $v : \mathbb{R}^n \to (0,\infty)$ and $\omega : \mathcal{B} \to (0,\infty)$ be a Lebesgue measurable functions. Let f be a Lebesgue measurable function. If*

$$\sup_{b \in b_{p',\omega}(v^{-p'})} \left| \int_{\mathbb{R}^n} f(x)b(x)dx \right| < \infty,$$

then $f \in M^p_\omega(v^p)$.

The corresponding results for the Morrey spaces with variable exponents and block spaces with variable exponents are given in [10, Propositions 4.5 and 4.6]. Since $(L^p(u^p))^* = L^{p'}(u^{-p'})$ and $(L^p(u^p))^{**} = L^p(u^p)$, where the duality between $f \in (L^p(u^p))^*$ and $g \in L^{p'}(u^{-p'})$ is given by $l_g(f) = \int_{\mathbb{R}^n} f(x)g(x)dx$, the proofs for Propositions 3.7 and 3.8 follow from [10, Propositions 4.5 and 4.6] with some simple modifications, for brevity, we leave it to the readers. By using the blocks, we can define and study the block space [8], which is a pre-dual of the Morrey space. As we do not need the block space in our study, we skip the details.

Proof of Theorem 3.5. Define $\tilde{\Omega}(x)=\overline{\Omega(-x)}$. Obviously, $\tilde{\Omega}$ is a homogeneous function of degree zero and $\tilde{\Omega}\in L^s(\mathbb{S}^{n-1})$. Furthermore, for any $f\in M^p_\omega(v^p)$ and $b\in b_{q',\omega}(u^{-q'})$, we have

$$\int_{\mathbb{R}^n} T_{\Omega,\alpha}f(x)b(x)dx=\int_{\mathbb{R}^n} f(x)T_{\tilde{\Omega},\alpha}b(x)dx \tag{3.10}$$

whenever the integrals are well defined.

In view of the above equality, we consider $T_{\tilde{\Omega},\alpha}b$, where $b\in b_{q',\omega}(u^{-q'})$.

Let $r>0$, $z\in\mathbb{R}^n$ and $b\in b_{q',\omega}(u^{-q'})$ be a block supported in $B(z,r)$. We write

$$b_0(x)=\chi_{B(z,2r)}(x)T_{\tilde{\Omega},\alpha}b(x)$$

and

$$b_j(x)=\chi_{B(z,2^{j+1}r)\setminus B(z,2^jr)}(x)T_{\tilde{\Omega},\alpha}b(x),\quad j\in\mathbb{N}.$$

Obviously, $T_{\tilde{\Omega},\alpha}b(x)=b_0(x)+\sum_{j=1}^\infty b_j(x)$.

Notice that $\frac{1}{q'}=1-\frac{1}{q}=1-\frac{1}{p}+\frac{\alpha}{n}$. That is, $\frac{1}{p'}=\frac{1}{q'}-\frac{\alpha}{n}$. Since $q'>s'$, $((v^{-1})^{s'},(u^{-1})^{s'})\in A^*(q'/s',p'/s')$ and $(v^{-1})^{s'},(u^{-1})^{s'}\in A(q/s',p/s')$, by applying the result of Item (1) of Theorem 2.5 on $T_{\tilde{\Omega},\alpha}$, we find that

$$\begin{aligned}\|b_0\|_{L^{p'}(v^{-p'})}&=\|\chi_{B(z,r)}T_{\tilde{\Omega},\alpha}b\|_{L^{p'}(v^{-p'})}\le\|T_{\tilde{\Omega},\alpha}b\|_{L^{p'}(v^{-p'})}\\&\le C\|b\|_{L^{q'}(u^{-q'})}\le C\frac{1}{\omega(B(z,r))}\end{aligned} \tag{3.11}$$

for some $C>0$.

Furthermore, in view of [10, (5.13)], we have

$$|b_j(x)|\le C\chi_{B(z,2^jr)}(x)\frac{\|b\|_{L^{s'}}}{|B(z,2^jr)|^{\frac{1}{s'}-\frac{\alpha}{n}}}. \tag{3.12}$$

As $q'>s'$, the Hölder inequality gives

$$\begin{aligned}\|b\|_{L^{s'}}&\le \||b|^{s'}\|^{\frac{1}{s'}}_{L^{q'/s'}(u^{-q'})}\|\chi_{B(z,r)}\|^{\frac{1}{s'}}_{L^{(q'/s')'}(u^{(q'/s')'s'})}\\&=\|b\|_{L^{q'}(u^{-q'})}\|\chi_{B(z,r)}\|^{\frac{1}{s'}}_{L^{(q'/s')'}(u^{(q'/s')'s'})}\\&\le C\frac{\|\chi_{B(z,r)}\|^{\frac{1}{s'}}_{L^{(q'/s')'}(u^{(q'/s')'s'})}}{\omega(B(z,r))}.\end{aligned}$$

By applying the norm $\|\cdot\|_{L^{p'}(v^{-p'})}$ on both sides of (3.12), we find that

$$\|b_j\|_{L^{p'}(v^{-p'})}\le\frac{C}{\omega(B(z,r))}\|\chi_{B(z,2^jr)}\|_{L^{p'}(v^{-p'})}\frac{\|\chi_{B(z,r)}\|^{\frac{1}{s'}}_{L^{(q'/s')'}(u^{(q'/s')'s'})}}{|B(z,2^jr)|^{\frac{1}{s'}-\frac{\alpha}{n}}}. \tag{3.13}$$

Since $(v^{-s'}, u^{-s'}) \in A^*(q'/s', p'/s')$, we have a constant $C > 0$ such that for any $z \in \mathbb{R}^n$ and $r > 0$

$$\left(\frac{1}{|B(z,r)|} \int\limits_{B(z,r)} v^{-p'} dx \right)^{s'/p'} \left(\frac{1}{|B(z,r)|} \int\limits_{B(z,r)} u^{s'(q'/s')'} dx \right)^{1/(q'/s')'} < C.$$

That is,

$$\|\chi_{B(z,r)}\|_{L^{(q'/s')'}(u^{(q'/s')'s'})}^{\frac{1}{s'}} \leq C \frac{|B(z,r)|^{\frac{1}{p'} + \frac{1}{s'(q'/s')'}}}{\|\chi_{B(z,r)}\|_{L^{p'}(v^{-p'})}}. \tag{3.14}$$

Moreover,

$$\frac{1}{p'} + \frac{1}{s'(q'/s')'} = \frac{1}{p'} + \frac{1}{s'} - \frac{1}{q'} = \frac{1}{s'} - \frac{\alpha}{n}.$$

Inequalities (3.13) and (3.14) give

$$\|b_j\|_{L^{p'}(v^{-p'})} \leq \frac{C}{\omega(B(z,r))} \frac{\|\chi_{B(z,2^j r)}\|_{L^{p'}(v^{-p'})}}{\|\chi_{B(z,r)}\|_{L^{p'}(v^{-p'})}} \frac{|B(z,r)|^{\frac{1}{s'} - \frac{\alpha}{n}}}{|B(z,2^j r)|^{\frac{1}{s'} - \frac{\alpha}{n}}}. \tag{3.15}$$

The Hölder inequality and (3.10) yield

$$\begin{aligned}
&\left| \int\limits_{\mathbb{R}^n} T_{\Omega,\alpha} f(x) b(x) dx \right| \leq \int\limits_{\mathbb{R}^n} |f(x) T_{\tilde{\Omega},\alpha} b(x)| dx \\
&\leq \int\limits_{B(z,r)} |f(x) b_0(x)| dx + \sum_{j=1}^{\infty} \int\limits_{B(z,2^j r) \setminus B(z,2^{j-1} r)} |f(x) b_j(x)| dx \\
&\leq \|\chi_{B(z,r)} f\|_{L^p(v^p)} \|b\|_{L^{p'}(v^{-p'})} + \sum_{j=1}^{\infty} \|\chi_{B(z,2^j r)} f\|_{L^p(v^p)} \|b_j\|_{L^{p'}(v^{-p'})}.
\end{aligned}$$

Consequently, (3.8), (3.11) and (3.15) give

$$\begin{aligned}
&\left| \int\limits_{\mathbb{R}^n} T_{\Omega,\alpha} f(x) b(x) dx \right| \\
&\leq \frac{\|\chi_{B(z,r)} f\|_{L^p(v^p)}}{\omega(B(z,r))} \\
&\quad + C \sum_{j=1}^{\infty} \frac{\|\chi_{B(z,2^j r)} f\|_{L^p(v^p)}}{\omega(B(z,2^j r))} \frac{\omega(B(z,2^j r))}{\omega(B(z,r))} \frac{\|\chi_{B(z,2^j r)}\|_{L^{p'}(v^{-p'})}}{\|\chi_{B(z,r)}\|_{L^{p'}(v^{-p'})}} \frac{|B(z,r)|^{\frac{1}{s'} - \frac{\alpha}{n}}}{|B(z,2^j r)|^{\frac{1}{s'} - \frac{\alpha}{n}}} \\
&\leq C \|f\|_{M^p_\omega(v^p)} \left(\sum_{j=1}^{\infty} \frac{\omega(B(z,2^j r))}{\omega(B(z,r))} \frac{\|\chi_{B(z,2^j r)}\|_{L^{p'}(v^{-p'})}}{\|\chi_{B(z,r)}\|_{L^{p'}(v^{-p'})}} \frac{|B(z,r)|^{\frac{1}{s'} - \frac{\alpha}{n}}}{|B(z,2^j r)|^{\frac{1}{s'} - \frac{\alpha}{n}}} \right).
\end{aligned}$$

By using (3.9), the integrals on (3.10) are well defined. By taking supremum over $b \in b_{q',\omega}(u^{-q'})$ over both sides of the above inequalities, Proposition 3.8 and (3.9) give

$$\|T_{\Omega,\alpha}f\|_{M^q_\omega(u^q)} = \sup_{b \in b_{q',\omega}(u^{-q'})} \left| \int_{\mathbb{R}^n} T_{\Omega,\alpha}f(x)b(x)dx \right| \le C\|f\|_{M^p_\omega(v^p)}.$$

□

Similarly, we also have the corresponding results for the rough fractional maximal operator $M_{\Omega,\alpha}$. As the results are similar to Theorems 3.3 and 3.5, for brevity, we omit the details. In addition, for the study of the singular integral operator with rough kernel on Morrey spaces, the reader is referred to [11].

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