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DYNAMIC CONDITION-BASED MAINTENANCE POLICY FOR DEGRADING SYSTEMS DESCRIBED BY A RANDOM-COEFFICIENT AUTOREGRESSIVE MODEL: A COMPARATIVE STUDY

DYNAMICZNA STRATEGIA UTRZYMANIA RUCHU NA PODSTAWIE STANU TECHNICZNEGO DLA ULEGAJĄCYCH DEGRADACJI SYSTEMÓW OPISANYCH MODELEM AUTOREGRESYJNYM Z PARAMETRAMI LOSOWYMI – STUDIUM PORÓWNAWCZE

In this paper, we optimize a dynamic condition-based maintenance policy for a slowly degrading system subject to soft failure and condition monitoring at equidistant, discrete time epochs. A random-coefficient autoregressive model with time effect is developed to describe the system degradation. The system age, previous state observations, and the item-to-item variability of the degradation are jointly combined in the proposed degradation model. Stochastic behavior for both the age-dependent and the state-dependent term are considered, and a Bayesian approach for periodically updating the estimates of the stochastic coefficients is developed to combine information from a degradation database with real-time condition-monitoring information. Based on this degradation model, the dynamic maintenance policy is formulated and solved in a semi-Markov decision process framework. Incorporated with the same semi-Markov decision process framework is a novel approach for mean residual life estimation, which enables simultaneous residual life estimation with the optimization procedure. The effectiveness of using the proposed random-coefficient autoregressive model with time effect rather than the existing fixed-coefficient ones to describe system degradation is demonstrated through a comparative study based on a real degradation dataset. The advantages of using a dynamic maintenance policy are also revealed.

Keywords: *degradation modeling, autoregressive model, Bayesian method, residual life estimation, semi-Markov decision process, condition-based maintenance.*

W prezentowanej pracy dokonano optymalizacji dynamicznej, uwzględniającej stan techniczny obiektu strategii utrzymania ruchu dla wolno ulegającego degradacji systemu monitorowanego w równoodległych dyskretnych chwilach czasu (epokach) pod względem uszkodzeń parametrycznych oraz stanu technicznego. Do opisu degradacji systemu opracowano model autoregresyjny z parametrami losowymi uwzględniający wpływ czasu. Proponowany model degradacji bierze pod uwagę zarówno wiek systemu jak i wcześniejsze obserwacje stanu oraz zmienność degradacji pomiędzy obiektami. Rozważano zachowanie stochastyczne zarówno składnika zależnego od wieku jak i składnika zależnego od stanu; opracowano bayesowską metodę okresowej aktualizacji oszacowań współczynników stochastycznych, która pozwala łączyć informacje z bazy danych o degradacji z informacjami z monitorowania stanu w czasie rzeczywistym. W oparciu o otrzymany model degradacji, sformułowano dynamiczną politykę utrzymania ruchu; problem optymalizacji tej polityki rozwiązywano w ramach procesu decyzyjnego semi-Markowa. Do procesu decyzyjnego włączono nowatorską metodę obliczania trwałości resztkowej, co umożliwiło ocenę trwałości resztkowej jednocześnie z przeprowadzeniem procedury optymalizacyjnej. Skuteczność wykorzystania proponowanego modelu autoregresyjnego do opisu degradacji systemu porównywano ze skutecznością dotychczasowych modeli z parametrami stałymi w badaniu opartym na rzeczywistym zbiorze danych o degradacji. Wskazano również zalety stosowania proponowanej dynamicznej strategii utrzymania ruchu.

Słowa kluczowe: *modelowanie degradacji, model autoregresyjny, metoda bayesowska, ocena trwałości resztkowej, semi-markowski proces decyzyjny, utrzymanie na podstawie stanu technicznego.*

1. Introduction

To sustain high reliability is the goal of every system and product. However, no matter how good the system design is, the performance of every system and product will ultimately deteriorate due to wear, fatigue, environmental conditions and other causes. When the deterioration becomes too severe, it may cause system malfunction or failure, which may result in significantly high maintenance costs and worst of all, safety hazards. Therefore, both operators and maintainers tend to adopt a preventive maintenance strategy to prevent system breakdowns, in that the preventive maintenance strategy usually performs before failures and thus it has a higher economic and safety significance than corrective maintenance which only takes place when the failure is observed.

Condition-based maintenance (CBM) is a kind of preventive maintenance strategy. It recommends maintenance actions based on the health status of the operating system. In a typical CBM policy, the health status of the system monitored throughout its operating life determines whether a preventive maintenance should be performed. Compared to traditional time-based preventive maintenance strategy, which sets a periodic interval to perform preventive maintenance regardless of the health status of a system, CBM is more reliable and cost-effective. Some successful examples of implementing CBM in real systems have demonstrated its efficiency in preventing catastrophic failures and improving maintenance performance (e.g., [2, 3, 15]).

To analyze and optimize a CBM policy for a specific system, the essential procedure is to develop a degradation model to describe the deterioration behavior of the operating system. The degradation model can be developed based on discrete-state or continuous-state stochastic processes, or their combinations. [13] and [11] considered a three-state continuous-time discrete-state Markov chain to model wear process of the diesel engine in locomotives and the gear shaft in gearboxes, respectively. To relax the Markovian assumption, [17] developed a degradation model based on a non-homogeneous semi-Markov process to de- scribe the deterioration of wear process in the turbofan engines. Using these Markovian degradation models, the CBM optimization problem can be trans- formed into determining maintenance actions for all system states with different maintenance objectives considered, e.g., [4, 14, 18, 19]. When the evolution of degradation state is continuous over time, continuous-state stochastic degradation models are more suitable. Gamma process (e.g., [26]), Wiener process(e.g.,[12, 21]), inverse Gaussian process (e.g.,[5]) are very popular models of this kind, which often allow simple and even elegant solutions to inference, hypothesis testing, goodness of fit, and prediction problems.

However, most previously developed continuous-state stochastic degradation models assume the degradation trend is only driven by system age and not by its previous states. Nevertheless, even if this assumption is appropriate for many types of degradation processes, in a general situation it is more realistic and appropriate to assume that the degradation trend can depend on either the system state or its age, or both. For example, the crack propagation rate might be higher if the current crack length is larger and a longer time has passed since the crack started propagation. In recent years, some researchers have noticed the need of effective modeling approaches for describing age- and state-dependent degradation processes. The first worth-noting contribution comes from [9]. In their proposed age- and state-dependent degradation model, the degradation increment over an elementary time interval has a discretized gamma distribution which depends on both the current degradation state and the operating age. Recently, [8,10] proposed a new class of Markovian age- and state-dependent degradation models, the transformed gamma process, by which the conditional distribution of the degradation growth over a generic time interval can be formulated in an analytical closed form.

The above models work very well in real applications (see examples in [9, 8, 10]), but they are only suitable to represent strictly monotonic degradation processes. It is not difficult to discover that in many industrial cases, effective description of non-monotonic degradation process, e.g., due to minimal repair, reduced load, or self-recovering mechanism [28], is also needed. Examples can be found in rotating machines [7], batteries [29], electronic devices [22, 23], etc. Therefore, [27] proposed an age- and state-dependent degradation model based on Wiener process to describe non-monotonic degradation processes. They obtained an analytical approximated residual life distribution to facilitate the residual life estimation of an operating system. However, the model did not consider the presence of observed heterogeneity among different individuals. Another age- and state-dependent degradation model capable of describing non-monotonic degradation processes was proposed by [23]. [23] took advantage of the state-dependency characteristic of autoregressive models and added an age-dependent term to the autoregressive model to include the influence of time. This model is easy to implement in real applications and mathematically tractable. However, the model formulation in [23] is also not able to describe the heterogeneity among different individuals.

Therefore, in this paper, we will improve the autoregressive model with time effect proposed in [23], in order to extend its capability of describing degradation processes. To do this, we will (i) consider a more general formation of the autoregressive model with time effect by assuming both the age-dependent term and the state-dependent term have stochastic behaviors; and (ii) derive a Bayesian updating method to update the model coefficients during system operation, which combines the information across the population and the information coming from the real-time condition monitoring(CM). The model coefficient updates have explicit formulas to allow fast computation in each update, which is an advantage of this model, and currently cannot be achieved by other existing age- and state-dependent degradation models. Using this model, the procedure of estimating the mean residual life in [23] is no longer applicable, due to the fact that the explicit form for the failure time distribution is quite complicated to obtain in mathematical point of view. Thus, this estimation task will be achieved via a Monte Carlo simulation procedure. We will demonstrate through a comparative study using the same dataset in [23] that the proposed model formation is superior and the Bayesian updating procedure indeed improves the accuracy of residual life estimation.

The proposed random-coefficient autoregressive model with time effect will then be applied to the optimization problem of a dynamic CBM policy. This maintenance policy is a commonly used control-limit policy in many industrial applications (e.g.,[13, 23]). Since we update model coefficients during system operation, we will consider the control-limit as a dynamic one, which is up-dated when new CM data becomes available. The optimization of this dynamic control-limit policy is achieved using a semi-Markov decision process(SMDP) framework. In this framework, we discover that the mean residual life of an operating system can be estimated simultaneously with the searching of the optimal control-limit. Therefore, it provides a novel idea of estimating the mean residual life for age- and state-dependent degradation models, and it also extends the application of the SMDP framework which is commonly considered as an approach only for policy decision problems. We will compare the mean residual life estimation results obtained by the SMDP-based approach with that by the Monte Carlo simulation approach, to reveal the advantage and disadvantage of the SMDP-based approach.

The rest of the paper is organized as follows. Section 2 introduces the general formation of the random-coefficient autoregressive model with time effect and the procedure to calculate the prior estimates of model coefficients. Section 3 develops a Bayesian updating framework to update the model coefficients for operating system. Section 4 describes the algorithms for optimizing the CBM policy and calculating the mean residual life for operating system. Section 5 gives numerical analysis. Conclusions and future research are given in Section 6.

2. The random-coefficient autoregressive model with time effect

Recall that the autoregressive model with time effect proposed in [23] deals with the degradation process whose degradation state can only be known at discrete inspection times. Suppose the degradation process starts from a known initial state Y_0, and is monitored through regular periodic inspections with inspection interval h. Let Y_1 denote the degradation state observed at inspection time $\{t\_n\} = nh, (n = 1, 2, 3, \ldots)$, then the autoregressive model with time effect has the following form defined as:

$$Y_n - \delta_0 = \beta t_n + \sum_{v=1}^{b} \varphi_v \left(Y_{n-v} - \delta_0\right) + \varepsilon_n \quad b = 1,2,3,\ldots, n = 1,2,3,\ldots \tag{1}$$

where b is the model order; δ_0 is a constant model coefficient; β and φ_v are also model coefficients; $\{\varepsilon_n\}$ are i.i.d. error terms and follow normal distribution $N\left(0,\sigma^2\right)$. To account for the heterogeneity among degradation paths of individual units, we consider a random-coefficient autoregressive model with time effect by supposing some (or all) of β and φ_v are possibly random.

[23] presented the procedure of choosing an appropriate model order b given historical data. In this paper, we will illustrate our method by considering the situation of $b = 1$. The other situations can be derived by the same procedure. For illustration purpose, we rewrite the random-coefficient autoregressive model with time effect as the following form:

$$Y_n - \delta_0 = \beta t_n + \varphi\left(Y_{n-1} - \delta_0\right) + \varepsilon_n,\ n = 1,2,3,\ldots \tag{2}$$

From the model, we can observe that when $\varphi \neq 1$, the current state $\mathrm{Y_n}$ depends on previous state $\mathrm{Y_{n-1}}$ and age t_n. To account for the heterogeneity among different individuals, assume β follows normal distribution with mean μ_β and variance σ_β^2, φ follows normal distribution with mean μ_φ and variance σ_β^2, and they have mutual covariance ρ. Therefore, the model of Eq.(2) has the model coefficients $\gamma = \left(\delta_0, \mu_\beta, \sigma_\beta^2, \mu_\varphi, \sigma_\varphi^2, \rho, \sigma^2\right)$, which are unknown and need to be estimated.

To estimate the constant model coefficient δ_0, we set $\delta = \delta_0 - \varphi_1\delta_0$, then the model of Eq.(2) can be rewritten as:

$$Y_n = \delta + \beta t_n + \varphi Y_{n-1} + \varepsilon_n, n = 1,2,3,\ldots \tag{3}$$

Assume that we have $.M.$ histories of the system. For the sth data history, we denote the number of inspections by m_s, the observed system states by $\left\{y_0, y_1^s, y_2^s, \ldots, y_{m_s}^s\right\}$ where y_0 is the same for all histories, and the inspection times by $\left\{t_1^s, t_2^s, \ldots, t_{m_s}^s\right\}$, $s = 1,2,\ldots,M$. So that for the M observed data histories, we have the regression representation $W = VA + E$, where:

$$\begin{aligned}
W' &= \left[y_{m_M}^M \ldots y_1^M \ldots y_{m_1}^1 \ldots y_1^1 \right]\\
E' &= \left[\varepsilon_{m_M}^N \ldots \varepsilon_1^N \ldots \varepsilon_{m_1}^1 \ldots \varepsilon_1^1 \right]\\
A' &= \left[\delta \quad \beta \quad \varphi\right]\\
V' &= \begin{bmatrix} 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 \\ t_{m_M}^M & \ldots. & t_1^M & \ldots & t_{m_M}^1 & \ldots & t_1^1 \\ y_{m_M-1}^M & \ldots & y_0 & \ldots & y_{m_M-1}^1 & \ldots & y_0 \end{bmatrix}
\end{aligned} \tag{4}$$

The least squares estimates for A is given by:

$$\hat{A} = \left(V'V\right)^{-1} V'W \tag{5}$$

Therefore,

$$\hat{\delta_0} = \frac{\hat{\delta}}{1 - \hat{\varphi}} \tag{6}$$

Let $X_n = Y_n - \delta_0$, $n = 1,2,3,\ldots$, the model of Eq.(2) can be transformed into the following form:

$$\begin{aligned} X_0 &= Y_0 - \delta_0, \\ X_n &= \beta t_n + \varphi X_{n-1} + \varepsilon_n, n = 1,2,3,\ldots \end{aligned} \tag{7}$$

Next, we have to estimate the distribution of β and φ. Let $\hat{\beta}_s$ and $\hat{\varphi}_s$ be the least squares estimates of parameter β and φ for each history s, $s = 1,2,\ldots,M$, then we have $\mu_\beta, \sigma_\beta^2, \mu_\varphi, \sigma_\varphi^2, \rho$ estimated by:

$$\begin{aligned}
\hat{\mu}_\beta &= \frac{1}{M}\sum_{s=1}^{M}\hat{\beta}_s,\ \hat{\mu}_\varphi = \frac{1}{M}\sum_{s=1}^{M}\hat{\varphi}_s,\\
\hat{\sigma}_\beta^2 &= \frac{1}{M-1}\sum_{s=1}^{M}\left(\hat{\beta}_s - \hat{\mu}_\beta\right)^2, \hat{\sigma}_\varphi^2 = \frac{1}{M-1}\sum_{s=1}^{M}\left(\hat{\varphi}_s - \hat{\mu}_\varphi\right)^2,\\
\hat{\rho} &= \frac{1}{M-1}\sum_{s=1}^{M}\left(\hat{\beta}_s - \hat{\mu}_\beta\right)\left(\hat{\varphi}_s - \hat{\mu}_\beta\right).
\end{aligned} \tag{8}$$

Using the least squares estimates $\hat{\beta}_s$ and $\hat{\varphi}_s$ of parameter β and φ for each history s, $s = 1,2,\ldots,M$, we are able to estimate the expected mean and stan- dard deviation of X_n^s conditioning on the initial degradation state X_0 for each data history, which are given by:

$$\begin{aligned}
E(X_n^s \mid X_0) &= \hat{\beta}_s \sum_{r=1}^{n} \hat{\varphi}^{n-r} t_r + \hat{\varphi}_s^n X_0,\\
Std(X_n^s \mid X_0) &= \sqrt{\sigma^2 \sum_{r=0}^{n-1} \hat{\varphi}_s^{2r}}.
\end{aligned} \tag{9}$$

Thus, let $x_n = y_n - \hat{\delta}_0$, the estimate of σ^2 is calculated using the following equation:

$$\hat{\sigma}^2 = \left(Q-1\right)^{-1} C'C, \tag{10}$$

where $Q = \sum_{s=1}^{M}\left(m_s - 1\right)$ is the total number of available observations of the degradation state, and:

$$C' = \sigma\left[\frac{x_{m_M}^M - E(X_{m_M}^M \mid X_0)}{Std(X_{m_M}^M \mid X_0)} \ \ldots\ \frac{x_1^M - E(X_1^M \mid X_0)}{Std(X_1^M \mid X_0)} \ \ldots\ \frac{x_1^1 - E(X_1^1 \mid X_0)}{Std(X_1^1 \mid X_0)}\right] \tag{11}$$

The least squares estimates $\hat{\beta}_s$ and $\hat{\varphi}_s$ for each history s, $s = 1,2,\ldots,M$ are given by:

$$\hat{A}_s = \left(V_s'V_s\right)^{-1} V_s'W_s \tag{12}$$

where:

$$\begin{aligned}
W_s' &= \left[x_{m_s}^s \ldots x_1^s\right],\quad E_s' = \left[\varepsilon_{m_s}^s \ldots \varepsilon_1^s\right],\quad A_s' = \left[\beta_s \quad \varphi_s\right]\\
V_s' &= \begin{bmatrix} t_{m_s}^s & \ldots & t_1^s \\ x_{m_s-1}^s & \ldots & x_0 \end{bmatrix}
\end{aligned} \tag{13}$$

Using Eq.(4)-Eq.(13), the estimates of model coefficients $\hat{\gamma}=\left(\hat{\mu}_{\beta},\hat{\sigma}_{\beta}^{2},\hat{\mu}_{\varphi},\hat{\sigma}_{\varphi}^{2},\hat{\rho},\hat{\sigma}^{2}\right)$ are obtained.

3. The Bayesian framework for adaptive model parameters via real-time CM data

The estimation procedure presented in Section 2 obtains the value of model coefficients for the whole population given historical data. However, to estimate the model coefficients for a specific system, it is more desirable to use the real-time CM observations collected during the system operation. In this section, we will develop a Bayesian framework for the update of model coefficients. According to the model in Section 2, the stochastic parameter $\{\beta,\varphi\}$ has the prior distribution of $\beta,\varphi\propto N\left(\mu_{\beta},\sigma_{\beta}^{2},\mu_{\varphi},\sigma_{\varphi}^{2},\rho\right)$. The estimates $\left\{\hat{\mu}_{\beta},\hat{\sigma}_{\beta}^{2},\hat{\mu}_{\varphi},\hat{\sigma}_{\varphi}^{2},\hat{\rho},\hat{\sigma}^{2}\right\}$ calculated by the procedure presented in Section 2 using historical data can be the prior estimates of $\left\{\mu_{\beta},\sigma_{\beta}^{2},\mu_{\varphi},\sigma_{\varphi}^{2},\rho,\sigma^{2}\right\}$.

Let $X_{1:r}=\{x_1,x_2,...,x_r\}$ where $x_r=y_r-\delta_0$, then given β and φ, the sampling distribution of $X_{1:r}$ is multi-variable normal as:

$$p(X_{1:r}\mid\beta,\varphi)=\frac{1}{\prod_{j=1}^{r}\sqrt{2\pi\sigma^2}}\times\exp\left[-\sum_{j=1}^{r}\frac{\left(x_j-\beta t_j-\varphi x_{j-1}\right)^2}{2\sigma^2}\right]. \quad (14)$$

Then the joint posterior estimate of β and φ conditional on $X_{1:r}$ is still normal resulted from the fact of the normal distribution assumption of β and φ. In other words, $\beta,\varphi\mid X_{1:r}\sim N\left(\mu_{\beta,r},\sigma_{\beta,r}^{2},\mu_{\varphi,r},\sigma_{\varphi,r}^{2},\rho_r\right)$. To be more precise, we have:

$$\begin{aligned}
p\left(\beta,\varphi|X_{1:r}\right)&\propto p\left(X_{1:r}|\beta,\varphi\right)\cdot p\left(\beta,\varphi\right)\\
&\propto exp\left[-\sum_{j=1}^{r}\frac{\left(x_j-\beta t_j-\varphi x_{j-1}\right)^2}{2\sigma^2}\right]\\
&\cdot exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\frac{\left(\beta-\mu_\beta\right)^2}{\sigma_\beta^2}-2\rho\frac{\left(\beta-\mu_\beta\right)\left(\varphi-\mu_\varphi\right)}{\sigma_\beta\sigma_\varphi}+\frac{\left(\varphi-\mu_\varphi\right)^2}{\sigma_\varphi^2}\right]\right\}\\
&\propto exp\{-\frac{1}{2\sigma^2\sigma_\beta^2\sigma_\varphi^2\left(1-\rho^2\right)}[\beta^2\left(\sigma_\beta^2\sigma_\varphi^2\left(1-\rho^2\right)\sum_{j=1}^{r}t_i^2+\sigma^2\sigma_\varphi^2\right)\\
&+\varphi^2\left(\sigma_\beta^2\sigma_\varphi^2\left(1-\rho^2\right)\sum_{j=1}^{r}x_{i-1}^2+\sigma^2\sigma_\beta^2\right)\\
&-2\beta\left(\sigma_\beta^2\sigma_\varphi^2\left(1-\rho^2\right)\sum_{j=1}^{r}t_ix_i+\sigma^2\mu_\beta\sigma_\varphi^2-\sigma^2\mu_\varphi\sigma_\beta\sigma_\varphi\rho\right)\\
&-2\varphi\left(\sigma_\beta^2\sigma_\varphi^2\left(1-\rho^2\right)\sum_{j=1}^{r}x_ix_{i-1}+\sigma^2\mu_\varphi\sigma_\beta^2-\sigma^2\mu_\beta\sigma_\beta\sigma_\varphi\rho\right)\\
&+2\beta\varphi\left(\sigma_\beta^2\sigma_\varphi^2\left(1-\rho^2\right)\sum_{j=1}^{r}t_ix_{i-1}-\sigma^2\sigma_\beta\sigma_\varphi\rho\right)]\}\\
&\propto\frac{1}{2\pi\sigma_{\beta,r}\sigma_{\varphi,r}\sqrt{1-\rho_r^2}}\exp\{-\frac{1}{2\left(1-\rho_r^2\right)}[\frac{\left(\beta-\mu_{\beta,r}\right)^2}{\sigma_{\beta,r}^2}\\
&-2\rho_r\frac{\left(\beta-\mu_{\beta,r}\right)\left(\varphi-\mu_{\varphi,r}\right)}{\sigma_{\beta,r}\sigma_{\varphi,r}}+\frac{\left(\varphi-\mu_{\varphi,r}\right)^2}{\sigma_{\varphi,r}^2}]\}
\end{aligned}\quad (15)$$

with:

$$\begin{aligned}
&\mu_{\beta,r}=\frac{A_2B_1-A_1C}{B_1B_2-C^2},\quad \mu_{\varphi,r}=\frac{A_1B_2-A_2C}{B_1B_2-C^2},\\
&\sigma_{\beta,r}^2=\frac{B_1D}{B_1B_2-C^2},\quad \sigma_{\varphi,r}^2=\frac{B_2D}{B_1B_2-C^2},\\
&\rho_r=\frac{-C}{\sqrt{B_1B_2}}.
\end{aligned}\quad (16)$$

where:

$$\begin{aligned}
A_1&=\sigma_\beta^2\sigma_\varphi^2\left(1-\rho^2\right)\sum_{j=1}^{r}x_{i-1}x_i+\sigma^2\mu_\varphi\sigma_\beta^2-\sigma^2\mu_\beta\sigma_\beta\sigma_\varphi\rho,\\
A_2&=\sigma_\beta^2\sigma_\varphi^2\left(1-\rho^2\right)\sum_{j=1}^{r}x_it_i+\sigma^2\mu_\beta\sigma_\varphi^2-\sigma^2\mu_\varphi\sigma_\beta\sigma_\varphi\rho,\\
B_1&=\sigma_\beta^2\sigma_\varphi^2\left(1-\rho^2\right)\sum_{j=1}^{r}x_{i-1}^2+\sigma^2\sigma_\beta^2,\\
B_2&=\sigma_\beta^2\sigma_\varphi^2\left(1-\rho^2\right)\sum_{j=1}^{r}t_i^2+\sigma^2\sigma_\varphi^2,\\
C&=\sigma_\beta^2\sigma_\varphi^2\left(1-\rho^2\right)\sum_{j=1}^{r}x_{i-1}t_i-\sigma^2\sigma_\beta\sigma_\varphi\rho,\\
D&=\sigma^2\sigma_\beta^2\sigma_\varphi^2\left(1-\rho^2\right).
\end{aligned}\quad (17)$$

Using the conditional joint posterior distribution of β and φ, we are able to calculate the distribution of X_n given X_{n-1}. In fact, X_n follows a normal distribution with mean and variance given by:

$$\begin{aligned}
&E\left(X_n|X_{n-1}\right)=\mu_{\beta,n-1}t_n+\mu_{\varphi,n-1}X_{n-1},\\
&Var(X_n\mid X_{n-1})=\sigma_{\beta,n-1}^2t_n^2+\sigma_{\varphi,n-1}^2X_{n-1}^2+\sigma^2\\
&\qquad+2\rho t_nX_{n-1}\sigma_{\beta,n-1}\sigma_{\varphi,n-1}.
\end{aligned}\quad (18)$$

4. Residual life estimation and maintenance policy optimization

Our next goal is to determine the residual life of the operating system and the optimal control-limit to initialize preventive replacement. Since φ is a random variable, the derivation of an analytical form for the mean residual life and the failure time distribution, is encountering a large amount of difficulties. Therefore, we hereby consider at first a Monte Carlo simulation-based approach which generates a large sample of deterministic degradation paths to approximate the residual life distribution. On the other hand, for the maintenance optimization problem, we consider the commonly-used control-limit policy (e.g., [15, 12, 23, 6]), by which the system will be preventively replaced if its observed degradation state exceeds a control-limit (optimized) and it is left operational until next inspection if its degradation level is below the control-limit. After preventive replacement or corrective replacement (perform when the system fails), the system will go back to as-good-as-new state Y_0. Since the model coefficients update as real-time condition monitoring data becomes available, the optimal control-limit to initialize preventive replacement may have to change as well. Therefore, the optimal control-limit is dynamic.

We develop an optimization algorithm based on SMDP framework to obtain the dynamic control-limit, which is based on the algorithm proposed in [23] for fixed control-limit situation. Moreover, we discover that using the SMDP framework, the mean residual life and the dynamic optimal control-limit can be obtained simultaneously. To be specific, without any extra effort, the mean residual life can be obtained while calculating the optimal control-limit. It is interesting to find that the SMDP-based approach provides a novel way of calculating mean residual life, which is commonly considered as an approach only for policy decision problems. We will introduce this approach and discuss its advantages and disadvantages.

For both problems, we assume that failure occurs when the degradation signal reaches some given failure threshold ξ. When the degradation state Y_n reaches the threshold ξ, the system is no longer assumed to be able to function satisfactorily or safely and it should be correctively replaced, although no physical failure is observed. In this paper, we take the threshold value ξ as fixed and known, and assume the failure should be discovered by equidistant inspections.

4.1. Revisited: residual life estimation in the situation of fixed model coefficients

Before describing our approaches, we first revisit the residual life estimation approach under the situation of fixed β and φ for comparison purposes (refer to [23] for more details). For fixed β and φ, the conditional expected mean and variance of X_n can be obtained by conditioning on previous observations $X_0, X_1, ..., X_j$ for some integer $j < n$:

$$\begin{aligned} E_f(X_n \mid X_j) &= \beta \sum_{r=j+1}^{n} \varphi^{n-r} t_r + \varphi^{n-j} X_j \\ Var_f(X_n \mid X_j) &= \sigma^2 \sum_{r=0}^{n-j-1} \varphi^{2r}. \end{aligned} \tag{19}$$

Let $F_T(t_n \mid X_j)$ be the failure time distribution given the current observation X_j, and it is calculated by:

$$\begin{aligned} F_T(t_n \mid X_j) &= Pr\left(T \langle t_n | X_j\right) \\ &= Pr(X_n \geq \xi - \delta_0 \mid X_j) \\ &= 1 - \Phi\left(\frac{\xi - \delta_0 - E_f(X_n \mid X_j)}{\sqrt{Var_f(X_n \mid X_j)}}\right) \end{aligned} \tag{20}$$

Thus, the mean residual life at a given time $t_j = jh$ can be calculated by:

$$E(T \mid jh) = \sum_{n=j+1}^{+\infty} t_n [F_T(t_n \mid X_j) - F_T(t_{n-1} \mid X_j)] - jh. \tag{21}$$

4.2. Monte Carlo simulation-based approach for residual life estimation

When both β and φ are random variables, it is difficult to derive analytically the failure time distribution $F_T(t_n \mid X_j)$, the conditional expected mean $E(X_n \mid X_j)$ and variance $Var(X_n \mid X_j)$. When there is no closed-form expression for the above distributions, one can evaluate their estimates to any desired degree of precision using Monte Carlo simulation. This is done by generating a sufficiently large number of random sample paths from the assumed degradation model with the estimated coefficients. We use the following procedure:

1. Generate U simulated realizations of $\tilde{\beta}$ and $\tilde{\varphi}$ from $\beta, \varphi \mid X_{1:j} \sim N\left(\mu_{\beta,j}, \sigma^2_{\beta,j}, \mu_{\varphi,j}, \sigma^2_{\varphi,j}, \rho_j\right)$, where U is a large number (e.g., $U = 100{,}000$);
2. Given the current observation Y_j at inspection time $t_j = jh$, generate simulated random errors $\hat{\varepsilon}_n$ from $N\left(0, \sigma^2\right)$. For each of the U simulated paths, $X_n = Y_n - \delta_0$ for any $n > j$ is calculated by:

$$\begin{aligned} X_{j+1} &= \beta t_{j+1} + \varphi X_j + \hat{\varepsilon}_1, \\ X_{j+2} &= \beta t_{j+2} + \varphi X_{j+1} + \hat{\varepsilon}_2 \\ &= \beta t_{j+2} + \varphi\left(\beta t_{j+1} + \varphi X_j + \hat{\varepsilon}_1\right) + \hat{\varepsilon}_2, \\ X_n &= \beta t_n + \varphi X_{n-1} + \varepsilon_n,\ n = j+1, j+2, j+3, ... \end{aligned} \tag{22}$$

   Then the residual life T_l for the l th simulated path is determined by:

$$\hat{T}_l = h \times \underset{n}{\operatorname{argmin}} \left\{X_n - (\xi - \delta_0) \geq 0\right\} \tag{23}$$

3. Compute the corresponding $F_T(t_n \mid X_j)$ using U simulated paths for any desired values of $t_n = h, 2h, ...$

$$F_T(t_n \mid X_j) = \frac{\text{number of } \hat{T}_l \leq t_n}{U} \tag{24}$$

4. Calculate the mean residual life $E(T \mid jh)$ at current inspection time $t_j = jh$ by Eq.(21).

4.3. SMDP-based approach for maintenance optimization and mean residual life estimation

We consider using a SMDP framework to optimize the dynamic maintenance policy based on the proposed random-coefficient autoregressive model with time effect. In this maintenance policy, three costs are required, which are, a cost of C_P for a preventive replacement, a cost of C_F for a corrective replacement and a cost of C_{Obs} for an inspection; two replacement times are included, which are preventive replacement time T_F and corrective replacement time T_P. After these quantities are assigned, the economic consequence of using this maintenance policy can be reflected by the long-run expected average cost per unit time g.

The control-limit will be optimized each time when the model coefficients are updated using new available observations. Therefore, if new observations are available at inspection time $t_n = nh$, the objective of the maintenance policy is to find the optimal control-limit $\bar{w}_n^*$ to determine whether a preventive replacement should be initialized before next update of model coefficients, by minimizing the long-run expected average cost per unit time g. By renewal theory (see e.g.[20]), the cost minimization problem is equivalent to finding an optimal control-limit $\bar{w}_n^* \in \left(y_0, \xi\right]$ such that:

$$g\left(\bar{w}_n^*\right) = \frac{E_{\bar{w}_n^*}\left(CC\right)}{E_{\bar{w}_n^*}\left(CL\right)} = \inf \frac{E_{\bar{w}_n^*}\left(CC\right)}{E_{\bar{w}_n^*}\left(CL\right)}. \tag{25}$$

where CL and CC denote the cycle length and cycle cost for the systems whole lifecycle, respectively.

We surprisingly discover that the problem of estimating the mean residual life of the operating system can be incorporated into the problem of optimizing the dynamic control-limit. That is, if the control-limit $\bar{w}_n$ equals to the failure threshold ξ, $\bar{w}_n = \xi$, then the total length of the lifecycle CL under this policy is the same as that without any preventive policy. The system will ultimately go to the failure state. Thus, $CL - jh - T_F$ equals to the residual life estimated at $t_j = jh$. Then the expected total length of the lifecycle $E\left(CL\right) - jh - T_F$ equals exactly to the mean residual life $E(T \mid jh)$.

Since the system will ultimately fail, the expected total cost $E\left(CC\right)$ can be given by:

$$E\left(CC\right) = C_F + C_{Obs}\frac{E\left(CL\right)}{h} \tag{26}$$

On the other hand, if the long-run expected average cost per unit time g is known, $E\left(CC\right)$ can also be calculated by:

$$E\left(CC\right) = gE\left(CL\right) \tag{27}$$

Using Eq.(26) and Eq.(27), the residual life estimated at time $t_j = jh$ is given by:

$$\begin{gathered} E\left(CL\right) = \frac{C_F h}{gh - C_{Obs}}, \\ E(T \mid jh) = E\left(CL\right) - jh - T_F. \end{gathered} \tag{28}$$

Note that although the values of costs C_F, C_{Obs} and replacement times T_F are very important for maintenance optimization, for the residual life estimation, the costs and replacement times are intermediate quantities to obtain results. They can be set to any value. No matter what values they have, the system will ultimately go failure (the lifecycle length won't change) and have corrective maintenance. Therefore, when they change, the average cost g changes accordingly, which is given by $E\left(CC\right)/E\left(CL\right)$.

Now the only remaining question for the residual life estimation problem is, how to calculate g without knowing the exact value of $E\left(CL\right)$? For the maintenance optimization problem, we develop a SMDP framework to calculate g for each $\bar{w}_n \in \left(Y_0, \xi\right]$ and decide the optimal $\bar{w}_n^*$ at inspection time $t_n = nh$ by the minimum g^*. The whole searching procedure contains the situation of $\bar{w}_n = \xi$. Hence, we are able to obtain g when the control-limit is $\bar{w}_n = \xi$, simultaneously with the searching procedure of optimal control-limit $\bar{w}_n^*$.

Next, we will describe in detail how to calculate g using a SMDP framework and how to find the optimal dynamic control-limit $\bar{w}_n^*$ at each inspection time $t_n = nh$. To develop this SMDP framework, firstly, the possible range of Y_n is required to be discretized into a finite set of states. We define the state space as a combination of countable time points and value intervals. Denote the state space by $\Omega = \left(\mathbf{K}, \mathbf{H}\right)$, where $\mathbf{K}$ represents the discretized states of observed Y_n, and $\mathbf{H} = \left\{nh; n = 0,1,2,\ldots\right\}$ represents the inspection times. We set the possible smallest value of Y_n by $y_0^{'} = y_0 - 3\sigma$. Define $\left[\xi, +\infty\right)$ as the failure state $\mathbf{F}$, then we can divide the continuous state space of $\left[y_0^{'}, \xi\right]$ into L equidistant intervals with constant length $\Delta = \left(\xi - y_0^{'}\right)/L$. For the maintenance policy, we define the control-limit $\bar{w}_n$ by $\bar{k}\Delta$, for some fixed integer $0 < \bar{k} < L$, then the warning state $\mathbf{W}$ will be $\left[\bar{k}\Delta + y_0^{'}, L\Delta + y_0^{'}\right)$, and the healthy state $\mathbf{S}$ will be $\left[y_0^{'}, \bar{k}\Delta + y_0^{'}\right)$.

Secondly, the quantities in the SMDP should be determined, which are one-step transition probabilities of degradation states, one-step expected sojourn times and one-step expected costs. We also define an integer $\tilde{n}$ to be the total number of inspections for the system. If the system still operates without failure when the last inspection is performed, we will enforce a preventive replacement. The degrading system will surely be replaced before the last inspection if $\tilde{n}$ is large enough.

The one-step transition probabilities are calculated by Eq.(29), where $E(X_{n+1} \mid X_n)$ and $Std(X_{n+1} \mid X_n)$ are calculated by Eq.(18).

$$\begin{aligned}
p_{(k,n),(l,n+1)} &= \begin{cases} \Pr\left(Y_{n+1} \in \left[l\Delta + y_0^{'}, (l+1)\Delta + y_0^{'}\right], 0 \le k < L \mid Y_0 = y_0\right), \text{for } k = 0, n = 0 \\ \Pr\left(Y_{n+1} \in \left[l\Delta + y_0^{'}, (l+1)\Delta + y_0^{'}\right], 0 \le k < L \mid Y_n = \left(k + 0.5\right)\Delta + y_0^{'}, 0 \le k < \bar{k}\right), \text{for n} \ge 1 \end{cases} \\
&= \int_{l\Delta + y_0^{'}}^{(l+1)\Delta + y_0^{'}} \frac{1}{\sqrt{2\pi}\, Std(Y_{n+1} \mid Y_n)} \exp\left\{-\frac{1}{2}\left[\frac{y - E(Y_{n+1} \mid Y_n)}{Std(Y_{n+1} \mid Y_n)}\right]^2\right\} dy \\
&= \Phi\left(\frac{\left(l+1\right)\Delta + y_0^{'} - E(Y_{n+1} \mid Y_n)}{Std(Y_{n+1} \mid Y_n)}\right) - \Phi\left(\frac{l\Delta + y_0^{'} - E(Y_{n+1} \mid Y_n)}{Std(Y_{n+1} \mid Y_n)}\right) \\
&= \Phi\left(\frac{\left(l+1\right)\Delta + y_0^{'} - \delta_0 - E(X_{n+1} \mid X_n)}{Std(X_{n+1} \mid X_n)}\right) - \Phi\left(\frac{l\Delta + y_0^{'} - \delta_0 - E(X_{n+1} \mid X_n)}{Std(X_{n+1} \mid X_n)}\right) \\
p_{(k,n),(\mathbf{W},n+1)} &= \begin{cases} Pr\left(Y_{n+1} \in \mathbf{W} \mid Y_0 = y_0\right), \text{ for } k = 0, n = 0 \\ Pr\left(Y_{n+1} \in \mathbf{W} \mid Y_n = \left(k + 0.5\right)\Delta + y_0^{'}, 0 \le k < \bar{k}\right), \text{for n} \ge 1 \end{cases} \\
&= \sum_{l=\bar{k}}^{L} p_{(k,n),(l,n+1)} \\
p_{(k,n),(\mathbf{F},n+1)} &= \begin{cases} Pr\left(Y_{n+1} \in \mathbf{F} \mid Y_0 = y_0\right), \text{for } k = 0, n = 0 \\ Pr\left(Y_{n+1} \in \mathbf{F} \mid Y_n = \left(k + 0.5\right)\Delta + y_0^{'}, 0 \le k < \bar{k}\right), \text{for } n \ge 1 \end{cases} \\
&= 1 - \Phi\left(\frac{\xi - E(Y_{n+1} \mid Y_n)}{Std(Y_{n+1} \mid Y_n)}\right) = 1 - \Phi\left(\frac{\xi - \delta_0 - E(X_{n+1} \mid X_n)}{Std(X_{n+1} \mid X_n)}\right).
\end{aligned} \tag{29}$$

The expected sojourn times for the SMDP are given by:

$$\begin{aligned} &\tau\left(k,n\right) = h, k = 0,1,\ldots,\bar{k}; n = 0,1,\ldots,\tilde{n} - 1 \\ &\tau\left(\mathbf{F},n\right) = T_F, n = 0,1,\ldots,\tilde{n} \\ &\tau\left(\mathbf{W},n\right) = \tau\left(k,\tilde{n}\right) = T_P,\ k = 0,1,\ldots,\bar{k}; n = 0,1,\ldots,\tilde{n}. \end{aligned} \tag{30}$$

Similarly, the expected costs for the SMDP are given by:

$$\begin{aligned} &c\left(k,n\right) = C_{Obs}, k = 0,1,\ldots,\bar{k} \text{ and } n = 0,1,\ldots,\tilde{n} - 1 \\ &c\left(\mathbf{F},n\right) = C_F,\ n = 0,1,\ldots,\tilde{n} \\ &c\left(\mathbf{W},n\right) = c\left(k,\tilde{n}\right) = C_P,\ k = 0,1,\ldots,\bar{k},\ \text{and}\ n = 0,1,\ldots,\tilde{n}. \end{aligned} \tag{31}$$

Therefore, at each inspection time $t_j = jh$, with new available observation Y_j and updated model parameters, the long-run expected average cost per unit time $g\left(\bar{w}_j\right)$ given the fixed control-limit $\bar{w}_j = \bar{k}\Delta$ can be obtained by solving the following system of linear equations:

$$v(0,0) = jC_{Obs} - g\left(\bar{w}_j\right) jh + v(a,j)$$

$$v(a,j) = c(a,j) - g\left(\bar{w}_j\right)\tau(a,j) + \sum_{l=0}^{\bar{k}} p_{(a,j),(l,j+1)} v(l,j+1)$$
$$+\, p_{(a,j),(\mathbf{W},j+1)} v(\mathbf{W},j+1) + p_{(a,j),(\mathbf{F},j+1)} v(\mathbf{F},j+1)$$

$$v(k,n) = c(k,n) - g\left(\bar{w}_j\right)\tau(k,n) + \sum_{l=0}^{\bar{k}} p_{(k,n),(l,n+1)} v(l,n+1)$$
$$+\, p_{(k,n),(\mathbf{W},n+1)} v(\mathbf{W},n+1) + p_{(k,n),(\mathbf{F},n+1)} v(\mathbf{F},n+1),$$
$$k = 0,1,\ldots,\bar{k} \text{ and } n = j+1, j+2, \ldots, \tilde{n}-1 \qquad (32)$$

$$v(\mathbf{W},n) = c(\mathbf{W},n) - g\left(\bar{w}_j\right)\tau(\mathbf{W},n) + v(0,0), n = j+1, 2, \ldots, \tilde{n}$$

$$v(\mathbf{W},n) = c(\mathbf{W},n) - g\left(\bar{w}_j\right)\tau(\mathbf{W},n) + v(0,0), n = j+1, 2, \ldots, \tilde{n}$$

$$v(k,\tilde{n}) = v(\mathbf{W},\tilde{n}), k = 0,1,\ldots,\bar{k}$$

$$v(p,q) = 0, \text{ for an arbitrarily selected single state } (p,q)$$

where $v(\bullet,\bullet)$ is the so-called relative value function plus a constant and $a = \lceil \left(Y_j - y_0'\right)/\Delta \rceil$. The first equation in Eq.(32) indicates that after $t_j = jh$ unit of time, the system runs from the initial state $(0,0)$ to the state of (a,j). The last equation in Eq.(32) guarantees that the solution to the system of linear equations in Eq.(32) is unique (see e.g.,[25]).

So that the optimal control-limit $\bar{w}_j^*$ at inspection time $t_j = jh$ and the corresponding minimum long-run expected average cost per unit time $g\left(\bar{w}_j^*\right)$ can be found by:

$$g\left(\bar{w}_j^*\right) = \inf_{\bar{w}_j \in \left[y_0', \xi\right]} \left\{ g\left(\bar{w}_j\right) \right\} \qquad (33)$$

Since only a single admissible action in each state is possible for a given control-limit, it is not necessary to formally apply the whole policy iteration procedure. We are only interested in computing the long-run average cost per unit time for a given control limit policy, and we chose SMDP for the computation, because we can make efficient use of the linear equations in step 1 of the policy iteration algorithm.

5. Case study and discussions

In this section, we will conduct a comparative study to demonstrate our proposed method. To reveal the effectiveness of the proposed random-coefficient autoregressive model with time effect, we will compare it with its fixed-coefficient counterpart using the same case studied in [22]. We will also use this case to illustrate our proposed dynamic maintenance policy and the approach of estimating the mean residual life for a functioning system.

5.1. The degradation dataset

The dataset is a real laser degradation data set presented by [16] (Example 13.6). It consists of 13 degradation histories of GaAs lasers (see Fig.1). Over the life of these lasers, degradation causes a decrease in light output. However, the lasers contain a feedback mechanism that maintains nearly constant light output by increasing operating current as the laser degrades. When operating current gets too high, the laser is considered to have failed. In applications, experts consider the laser failed if the operating current increases to ξ percent of its original value ($\xi \le 10$). To track the lasers degradation, in this data set, the operating currents were measured every $h = 20$ hours up to 4000 hours.

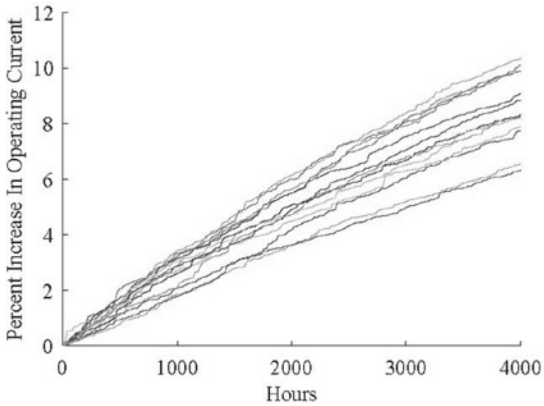

Fig. 1. Degradation paths in terms of the percent increase in operating current for the GaAs laser data set coming from [16]

5.2. Estimation of model coefficients

To demonstrate the present random-coefficient model formation is preferable to the fixed-coefficient one proposed in [23], we firstly use the 13 degradation histories as the training data to estimate the model coefficients $\gamma = \left(\delta_0, \mu_\beta, \sigma_\beta^2, \mu_\varphi, \sigma_\varphi^2, \rho, \sigma^2\right)$, using the two model formations respectively. According to the estimation procedure presented in Section 2 and in [23], we obtain the values of the model coefficients for both models, as listed in Table 1. Note that for the fixed-coefficient model, $\sigma_\beta^2 = 0$, $\sigma_\varphi^2 = 0$, and $\rho = 0$.

In Fig.2, a graphical proof is presented to directly show the superiority of the random-coefficient model over the fixed-coefficient model in capturing the lasers degradation behavior. This figure shows simulated degradation paths of 30 lasers based on the estimates in Table 1. It can be observed that the simulated paths using the random-coefficient model are very similar to the actual paths (refer to Fig.1). However, the simulated paths using the fixed-coefficient model are quite different, in that

1. The number of intersections in the actual paths is quite smaller than that in the simulated paths.
2. The decreasing phases are less pronounced than that in the simulated paths.
3. the variability of each path around its mean is quite smaller in the actual paths than that in the simulated paths.

Table 1. Parameters estimation results using the data of the training 13 lasers.

Model coefficients	Fixed-coefficient (refer to [23])	Random-coefficient
$\hat{\delta}_0$	-14.3722	-14.3722
$\hat{\mu}_\beta$	-1.4755×10^{-5}	-1.4750×10^{-5}
$\hat{\sigma}_\beta$	/	4.6099×10^{-6}
$\hat{\mu}_\varphi$	1.0038	1.0038
$\hat{\sigma}_\varphi$	/	6.6884×10^{-4}
$\hat{\rho}$	/	-0.9915
$\hat{\sigma}$	0.0636	0.0240

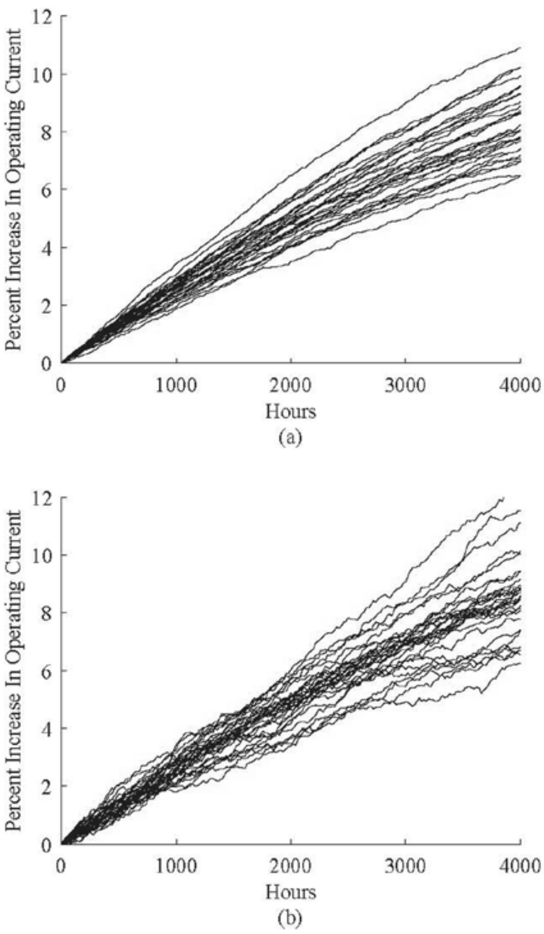

Fig. 2. Simulated degradation paths of 30 GaAs lasers based on (a) random-coefficient autoregressive model with time effect and (b) fixed-coefficient autoregressive model with time effect

All these differences indicate the heterogeneity does exist in this dataset and considering the model coefficient as random is more appropriate. The model with fixed coefficients places all uncertainties on the parameter σ, whose value we can observe from Table 1 is quite larger than the one in random-coefficient model. This is why the fluctuations in Fig.2(b) are more and larger. Moreover, in Fig.3, the estimated mean and the standard deviation of the percent increase in operating current calculated by the random-coefficient model (using mean and standard deviation of 3000 simulated paths) are compared with the corresponding empirical estimates calculated from the observed data. The results also indicate that the random-coefficient model fits well with this dataset.

5.3. Update of model coefficients using the Bayesian framework

In order for the model coefficients to better accommodate a specific functioning laser, we use the Bayesian framework proposed in Section 3 to update the model coefficients once new observations are available. To demonstrate the advantages of this update procedure, we randomly select 4 degradation histories which relatively degrade slowly in the dataset as the training units, to estimate the initial values of the model coefficients. Then we select 1 degradation history which relatively degrades quickly in the dataset as the testing unit, assuming this unit is the functioning laser. The training units and the testing unit are plotted in Fig.4.

With the joint prior distribution of model coefficients $\{\beta,\varphi\}$, we are able to find the joint posterior distribution of $\{\beta,\varphi\}$ for the functioning laser any time we obtain a new observation, i.e., at any inspection time $t_n = nh$. Fig.5 presents the evolution of the posterior means for β, and φ respectively, given the observation data from the testing unit. Given these posterior means, we can then compute the expected percent increase at the next sampling time using the following equation:

$$E_{\hat{\gamma}}(Y_n \mid Y_{n-1}) = \hat{\mu}_{\beta,n-1}t_n + \hat{\mu}_{\varphi,n-1}\left(Y_{n-1} - \hat{\delta}_0\right) + \hat{\delta}_0 \tag{34}$$

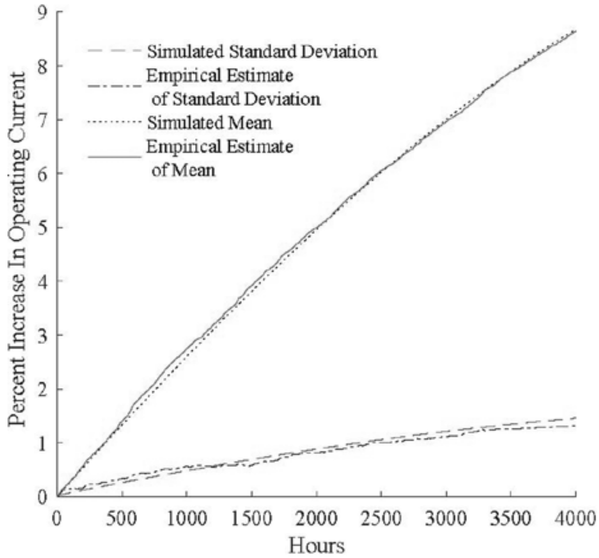

Fig. 3. Curves of the estimated mean and standard deviation of the percent increase in operating current compared with the corresponding empirical estimates

Fig.6 shows the observed and the expected percent increase in the operating current plotted against time for the testing unit. The results show that using the Bayesian updating framework, the expected degradation path follows well with the actual degradation path. To demonstrate the advantages of using the Bayesian updating framework, the root mean squared error (RMSE) for the testing unit under the random-coefficient model with and without Bayesian updating procedure are calculated respectively, which is defined by Eq.(35). For the model without Bayesian updating procedure, the RMSE is 0.0464, while for the model with Bayesian updating procedure, the RMSE is 0.0440. The results show that the Bayesian updating procedure increases the accuracy of predicting the degradation state of functioning laser.

$$RMSE_1 = \sqrt{\frac{1}{Q}\sum_{n=1}^{Q}[Y_n - E_{\hat{\gamma}}(Y_n \mid Y_{n-1})]^2} \tag{35}$$

where $Q = 200$ is the total number of observations of the testing unit.

5.4. Optimization of the dynamic maintenance policy

With the posterior estimates of model coefficients, we are able to find the dynamic control-limit to initialize preventive maintenance. To illustrate the whole optimization procedure, we use the 4 training units in Fig.4 to obtain the initial values of model coefficients. Then, the dynamic policy is optimized using the algorithm proposed in Sec-

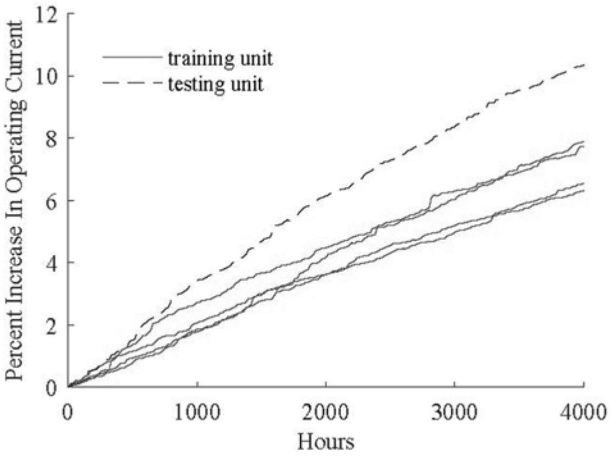

Fig. 4. Degradation paths in terms of the percent increase in operating current for the training units and the testing unit.

tion 4, with the help of real-time observations coming from the testing unit and the posterior estimates of model coefficients.

Suppose that the failure threshold is $\xi = 4$. We determine the optimal control-limit every 20 inspections with the following replacement times and cost data: $T_P = T_F = 20$ hours, $C_F = \$3000$, $C_P = \$1000$ and $C_{Obs} = \$100$. We partition the continuous degradation interval $\left[Y_0 = 0, \xi = 4\right]$ into 64 sub- intervals, and set the maximum number of inspections to $\hat{n} = 128$. For every 20 inspections,

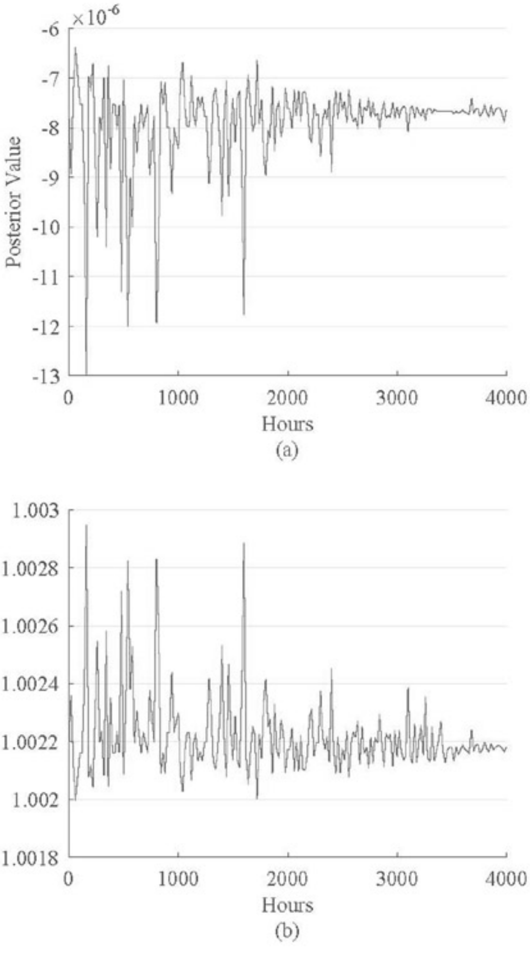

Fig. 5. Evolution of the posterior means for (a) β and (b) φ using the testing unit

that is, at inspection time $t_n = nh$, $n = 20, 40, 60, \ldots$, we obtain the minimum long-run expected average cost per hour $g\left(\bar{\omega}_n^*\right)$ with the corresponding optimal control-limit $\bar{\omega}_n^*$. The dynamic optimal control-limit for the maintenance policy is shown in Fig.7. It reveals that for this dataset, the optimal control-limit neither shows a monotone nondecreasing trend as [6] and [24], nor shows a decreasing trend as [1]. It firstly decreases then raises back to its initial value. This going-back control-limit seems to be a proof to doubt the significance of using this dynamic maintenance policy. However, from the behavior of the dynamic control-limit, we actually can see the increased urgency for preventive maintenance when degradation of the functioning laser begins to deviate fast from the historical degradation paths (refer to Fig.4 to see the faster degradation of the testing unit). The following increased behavior of the control-limit can be explained by increased accuracy (less variability) in predicting the future degradation trend. Furthermore, this result only reflects the situation under current maintenance situation (maintenance costs, maintenance times and etc.).

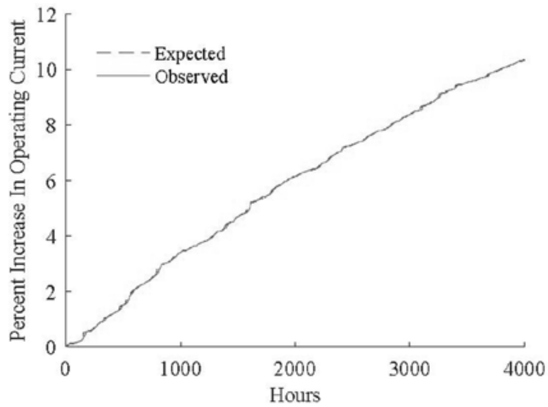

Fig. 6. Expected and observed percent increase in operating current vs. time for the testing unit

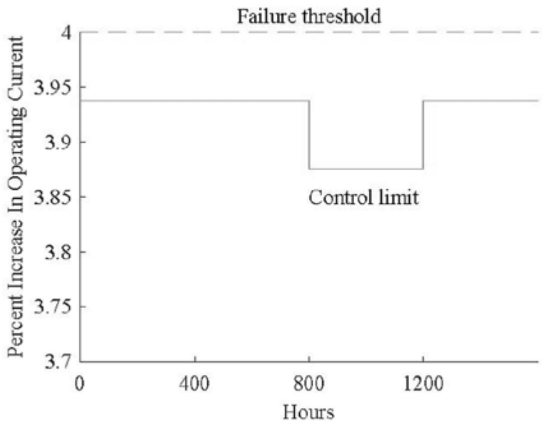

Fig. 7. Optimization results of the dynamic control-limit

5.5. Real-time estimation of residual life

We then use the posterior means and variances of model coefficients to compute the mean residual life given observations obtained up to that point in time. We firstly use the Monte Carlo simulation-based approach and make a comparison between the results obtained by the proposed random-coefficient model with Bayesian updating procedure and the fixed-coefficient model. The model coefficients are calculated using the training units presented in Fig.4 and updated by observations of the testing unit in Fig.4. We use the following RMSE to evaluate their performances:

$$RMSE_2 = \sqrt{\frac{1}{Q}\sum_{j=1}^{Q}[E_{true}(T \mid jh) - E(T \mid jh)]^2} \tag{36}$$

where $E_{true}(T \mid jh)$ and $E(T \mid jh)$ are true remaining time to observe failure at $t_j = jh$ and expected mean residual life estimated at $t_j = jh$, respectively, and Q is the total number of observations before system failure.

The results of RMSE are: for the fixed-coefficient model, $RMSE_2 = 432$, while for the random-coefficient model with Bayesian updating procedure, $RMSE_2 = 427$. The results support that the random-coefficient model with Bayesian updating procedure improves the performance of residual life estimation. To give a vivid impression of the results, we plot in Fig.8 the estimation results at each inspection time for both the fixed-coefficient model and the random-coefficient model with Bayesian updating procedure. The figure shows that although the estimates of the residual life become more accurate with more available CM observations for both models, the random-coefficient model with Bayesian up-dating procedure seems to give more estimates which are closer to the true residual life, especially in the middle stage of the degradation.

Next, we present the procedure of using our proposed SMDP-based approach to estimate the mean residual life. In this case study, with the process of optimizing the control-limit at inspection time $t_n = nh$, $n = 0, 20, 40, 60$, we are able to obtain the mean residual life estimates given the observations up until t_n, using the long-run expected average cost per hour $g(\bar{\omega}_n)$ when $\bar{\omega}_n = \xi$ and the equations Eq.(26) - Eq.(28). The results are shown in Table 2. The true residual life and the estimated mean residual life obtained by Monte Carlo simulation-based approach are also listed in Table 2. It shows that the estimation results by SMDP-based approach are very close to the results obtained by Monte Carlo simulation-based approach, indicating that the SMDP-based approach for residual life estimation is reliable. The slight difference may due to the different approximation methods used in these two approaches.

Although the SMDP-based approach is feasible for residual life estimation, we still have to point out that it is only an approximate approach for residual life estimation and its computation cost is too large. It could estimate mean residual life simultaneously and quickly with the optimization process of the dynamic maintenance policy analyzed in this paper, but it is not an efficient approach if it is used independently. However, for a system using control-limit policy to conduct condition-based maintenance, engineers are more enthusiastic to know the system remaining time to the preventive control-limit than the system remaining time to failure. To approximately estimate the system remaining time to the control-limit, our proposed SMDP-based approach provides a very direct and easy way, by using the results of the minimum long-run expected average cost per unit time .

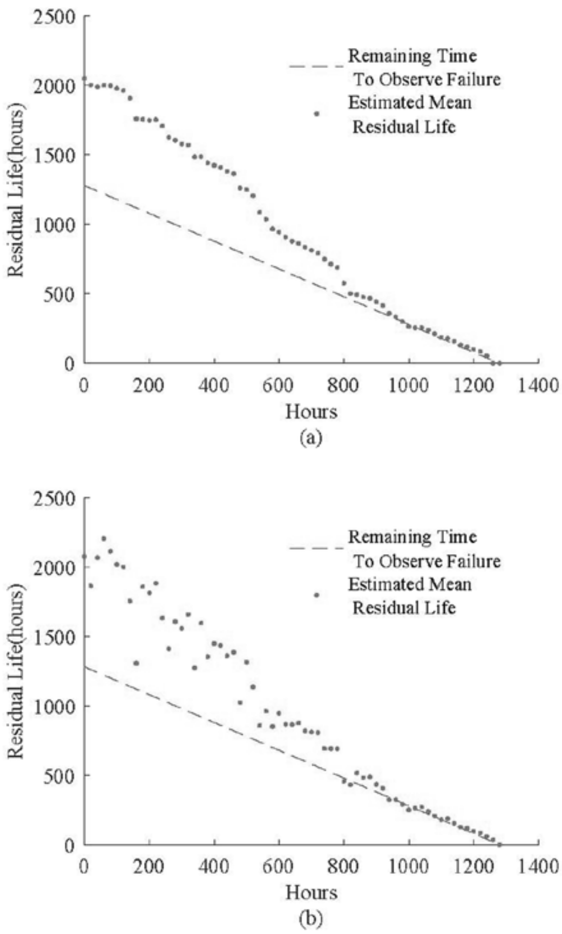

Fig. 8. The actual remaining time to observe failure and the estimated mean residual life for the testing unit using (a) the fixed-coefficient autoregressive model with time effect and (b) the random-coefficient autoregressive model with time effect and Bayesian updating procedure

Table 2. The results of mean residual life estimation using SMDP-based approach.

Inspection time (hours)	0	400	800	1200
True residual life	1280	900	500	100
Estimate in Fig.8(b)	2074	1450	456	98
Estimate using SMDP-based approach	2062	1499	464	135

6. Conclusion

In this paper, we have presented a dynamic CBM policy using random-coefficient autoregressive model with time effect. This random-coefficient autoregressive model with time effect has a more general formation than the fixed-coefficient counterpart ([23]) by considering stochastic behavior for both the age-dependent and the state-dependent term. Furthermore, a Bayesian approach for automatically updating the estimates of the stochastic coefficients is developed to combine information from a degradation database with real-time CM information. With the normal assumption for the prior distribution of those stochastic coefficients, the updates have explicit formulas. This implies that each update can be performed with a single computation, which leads to an extremely fast and simple updating procedure.

We believe that the comparison results presented in Section 5 clearly indicate the value of using the improved random-coefficient autoregressive model with time effect and the Bayesian approach to incorporate real-time CM information.

The dynamic CBM policy is optimized using a SMDP framework. This optimization approach is based on [23] but surpasses it by considering real-time CM information. We have demonstrated through a case study of GaAs lasers that the dynamic control-limit maintenance policy is more sensitive to the urgency of preventive maintenance when the functioning system exhibits distinct difference from the degradation database. Moreover, using this SMDP framework, we have also explored the possibility of using SMDP to estimate mean residual life for a functioning system, for the first time in literature. The comparison between Monte Carlo simulation-based approach and the SMDP-based approach verifies the feasibility of the latter approach. However, we note that the computation cost of the SMDP-based approach hinders its independent application in residual life estimation problems. Only when it is combined with optimization problems of dynamic maintenance policy could its efficiency be revealed. Further research topics include developing residual life distributions for this random-coefficient autoregressive model and exploring more dynamic CBM policies based on this model. Extension of the model to describe and control partially observable degrading processes with soft failures is also a suitable one.

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