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Critical infrastructure operation process related to climate-weather change process including extreme weather hazard

Keywords

critical infrastructure, operation, prediction, climate-weather change

Abstract

The operation process of the critical infrastructure is considered and its operation states are introduced. The semi-Markov process is used to construct a general probabilistic model of the critical infrastructure operation process. The semi-Markov process is used to construct a general probabilistic model of the climate-weather change process for the critical infrastructure operating area.

1. Introduction

The operation process of the critical infrastructure is considered and its operation states are introduced. The semi-Markov process is used to construct a general probabilistic model of the critical infrastructure operation process. To build this model, the vector of probabilities of the critical infrastructure operation process staying at the initial operation states, the matrix of probabilities of the critical infrastructure operation process transitions between the operation states, the matrix of conditional distribution functions and the matrix of conditional density functions of the critical infrastructure operation process conditional sojourn times at the operation states are defined.

The climate-weather change process for the critical infrastructure operating area is considered and its states are introduced. The semi-Markov process is used to construct a general probabilistic model of the climate-weather change process for the critical infrastructure operating area. To build this model the vector of probabilities of the climate-weather change process staying at the initials climate-weather states, the matrix of probabilities of the climate-weather change process transitions between the climate-weather states, the matrix of conditional distribution functions and the matrix of conditional density functions of the climate-weather change process conditional sojourn times at the climate-weather states are defined.

Further, these two precesses are joined into a general model of the critical infrastructure operation process related to climate weather change process.

The operation process of a critical infrastructure often has significant influence on its safety. Also, a critical infrastructure operating environment area climate-weather conditions are essential in its safety analysis. Usually, the critical infrastructure operation process and the climate-weather conditions at its operating area interact and have either an explicit or an implicit strong joint influence on the critical infrastructure safety. Thus, considering together those two processes influence on the critical infrastructure safety is of grate practical value.

To construct a joint mdel of those two processes, first, the semi-Markov approaches to a critical infrastructure operation process modeling and to climate-weather change process are separately developed. Next, those two separate models are linked into a jeneral joint model of a critical infrastructure operation process related to the climate-weather change process including extreme hazards. is build. climate to modelling a critical

infrastructure operation process including operating environment threats.
The method of defining the parameters of this general joint model of the critical infrastructure operation process is presented. The procedures of these parameters determination using the parametrers of the two separate models and their application to the critical infrastructure operation characteristics related to climate-weather change prediction are proposed as well.

2. Critical infrastructure operation process-modelling

2.1. Semi-Markov model of critical infrastructure operation process

We assume that the critical infrastructure during its operation process is taking $v, v \in N$, different operation states $z_1, z_2, \ldots, z_v$. Further, we define the critical infrastructure operation process $Z(t)$, $t \in < 0,+\infty)$, with discrete operation states from the set $\{z_1, z_2, \ldots, z_v\}$. Moreover, we assume that the critical infrastructure operation process $Z(t)$ is a semi-Markov process [Grabski, 2002], [Limnios, 2005], [Mercier, 2008], [Soszyńska, 2007], [Kołowrocki, Soszyńska, 2011] with the conditional sojourn times θ_{bl} at the operation states z_b when its next operation state is z_l, $b, l = 1,2,...,v$, $b \neq l$. Under these assumptions, the critical infrastructure operation process may be described by:

- the vector of the initial probabilities $p_b(0) = P(Z(0) = z_b)$, $b = 1,2,...,v$, of the critical infrastructure operation process $Z(t)$ staying at particular operation states at the moment $t = 0$

$$[p_b(0)]_{1\times v} = [p_1(0), p_2(0),..., p_v(0)]; \tag{1}$$

- the matrix of probabilities p_{bl}, $b, l = 1,2,...,v$, of the critical infrastructure operation process $Z(t)$ transitions between the operation states z_b and z_l

$$[p_{bl}]_{v\times v} = \begin{bmatrix} p_{11} \ p_{12} \cdots p_{1v} \\ p_{21} \ p_{22} \cdots p_{2v} \\ \cdots \\ p_{v1} \ p_{v2} \cdots p_{vv} \end{bmatrix}, \tag{2}$$

where by formal agreement

$p_{bb} = 0$ for $b = 1,2,...,v$;

- the matrix of conditional distribution functions $H_{bl}(t) = P(\theta_{bl} < t)$, $b, l = 1,2,...,v$, of the critical infrastructure operation process $Z(t)$ conditional sojourn times θ_{bl} at the operation states

$$[H_{bl}(t)]_{v\times v} = \begin{bmatrix} H_{11}(t)\ H_{12}(t) \ldots H_{1v}(t) \\ H_{21}(t)\ H_{22}(t) \ldots H_{2v}(t) \\ \ldots \\ H_{v1}(t)\ H_{v2}(t) \ldots H_{vv}(t) \end{bmatrix}, \tag{3}$$

where by formal agreement

$H_{bb}(t) = 0$ for $b = 1,2,...,v$,

We assume that the suitable and typical distributions suitable to describe the critical infrastructure operation process $Z(t)$ conditional sojourn times θ_{bl}, $b, l = 1,2,...,v$, $b \neq l$, in the particular operation states are that defined in [Kołowrocki, Soszyńska 2011] and [EU-CIRCLE Report D2.1-GMU2, 2016].

3. Critical infrastructure operation process – prediction

3.1. Prediction of critical infrastructure operation process characteristics

Assuming that we have identified the unknown parameters of the critical infrastructure operation process semi-Markov model:

– the initial probabilities $p_b(0)$, $b, = 1,2,...,v$, of the critical infrastructure operation process staying at the particular state z_b at the moment $t = 0$;
– the probabilities p_{bl}, $b, l = 1,2,...,v$, $b \neq l$, of the critical infrastructure operation process transitions from the operation state z_b into the operation state z_l;
– the distributions of the critical infrastructure operation process conditional sojourn times θ_{bl}, $b, l = 1,2,...,v$, $b \neq l$, at the particular operation states and their mean values $M_{bl} = E[\theta_{bl}]$, $b, l = 1,2,...,v$, $b \neq l$;

we can predict this process basic characteristics.
As the mean values of the conditional sojourn times θ_{bl} are given by [Kolowrocki, Soszyńska-Budny, 2011]

$$M_{bl} = E[\theta_{bl}] = \int_0^\infty t dH_{bl}(t) = \int_0^\infty t h_{bl}(t) dt, \tag{6}$$
$$b, l = 1,2,...,v, \ b \neq l,$$

then for the distinguished distributions (2.5)-(2.11) in [EU-CRCLE Report D2.1-GMU2, 2016], the mean values of the system operation process $Z(t)$ conditional sojourn times θ_{bl}, $b,l = 1,2,\ldots,v$, $b \neq l$, at the particular operation states are respectively given by (2.6)-(2.12) in [EU-CRCLE Report D2.1-GMU2, 2016].

From the formula for total probability, it follows that the unconditional distribution functions of the sojourn times θ_b, $b = 1,2,\ldots,v$, of the system operation process $Z(t)$ at the operation states z_b, $b = 1,2,\ldots,v$, are given by [Kołowrocki, Soszyńska-Budny, 2011]

$$H_b(t) = \sum_{l=1}^{v} p_{bl} H_{bl}(t), \; b = 1,2,\ldots,v. \tag{7}$$

Hence, the mean values $E[\theta_b]$ of the system operation process $Z(t)$ unconditional sojourn times θ_b, $b = 1,2,\ldots,v$, at the operation states are given by

$$M_b = E(\theta_b) = \sum_{l=1}^{v} p_{bl} M_{bl}, \; b = 1,2,\ldots,v, \tag{8}$$

where M_{bl} are defined by the formula (6) in a case of any distribution of sojourn times θ_{bl} and by the formulae (2.6)-(2.12) in the cases of particular defined respectively by (2.5)-(2.11) in [EU-CRCLE Report D2.1-GMU2, 2016] distributions of these sojourn times.

The limit values of the system operation process $Z(t)$ transient probabilities at the particular operation states

$$p_b(t) = P(Z(t) = z_b), \; t \in <0,+\infty), \; b = 1,2,\ldots,v, \tag{9}$$

are given by [Kołowrocki, Soszyńska-Budny, 2011]

$$p_b = \lim_{t\to\infty} p_b(t) = \frac{\pi_b M_b}{\sum_{l=1}^{v} \pi_l M_l}, \; b = 1,2,\ldots,v, \tag{10}$$

where M_b, $b = 1,2,\ldots,v$, are given by (8), while the steady probabilities π_b of the vector $[\pi_b]_{1xv}$ satisfy the system of equations

$$\begin{cases} [\pi_b] = [\pi_b][p_{bl}] \\ \sum_{l=1}^{v} \pi_l = 1. \end{cases} \tag{11}$$

In the case of a periodic system operation process, the limit transient probabilities p_b, $b = 1,2,\ldots,v$, at the operation states defined by (10), are the long term proportions of the system operation process $Z(t)$ sojourn times at the particular operation states z_b, $b = 1,2,\ldots,v$.

Other interesting characteristics of the system operation process $Z(t)$ possible to obtain are its total sojourn times $\hat{\theta}_b$ at the particular operation states z_b, $b = 1,2,\ldots,v$, during the fixed system opetation time. It is well known [Kołowrocki, Soszyńska-Budny, 2011] that the system operation process total sojourn times $\hat{\theta}_b$ at the particular operation states z_b, for sufficiently large operation time θ, have approximately normal distributions with the expected value given by

$$\hat{M}_b = E[\hat{\theta}_b] = p_b \theta, \; b = 1,2,\ldots,v, \tag{12}$$

where p_b are given by (10).

4. Climate-weather change process - modelling

4.1. Semi-Markov model of climate-weather change process

To model the climate-weather change process for the critical infrastructure operating area we assume that the climate-weather in this area is taking w, $w \in N$, different climate-weather states $c_1, c_2, \ldots, c_w$. Further, we define the climate-weather change process $C(t)$, $t \in <0,+\infty)$, with discrete operation states from the set $\{c_1, c_2, \ldots, c_w\}$. Assuming that the climate-weather change process $C(t)$ is a semi-Markov process it can be described by:

– the number of climate-weather states w, $w \in N$,

– the vector

$$[q_b(0)]_{1Xw} = [q_1(0), q_2(0), \ldots, q_w(0)]$$

of the initial probabilities

$$q_b(0) = P(C(0) = c_b), \; b = 1,2,\ldots, w,$$

of the climate-weather change process $C(t)$ staying at particular climate-weather states c_b at the moment $t = 0$;

– the matrix

$$[q_{bl}]_{wxw} = \begin{bmatrix} q_{11}\, q_{12} \cdots q_{1w} \\ q_{21}\, q_{22} \cdots q_{2w} \\ \cdots \\ q_{w1}\, q_{w2} \cdots q_{ww} \end{bmatrix} \tag{14}$$

of the probabilities of transitions q_{bl}, $b, l = 1,2,\ldots, w$, $b \neq l$, of the climate-weather change process $C(t)$ from the climate-weather states c_b to c_l, where by formal agreement

$q_{bb} = 0$ for $b = 1,2,\ldots, w$;

– the matrix

$$[C_{bl}(t)]_{wxw} = \begin{bmatrix} C_{11}(t)\, C_{12}(t) \ldots C_{1w}(t) \\ C_{21}(t)\, C_{22}(t) \ldots C_{2w}(t) \\ \ldots \\ C_{w1}(t)\, C_{w2}(t) \ldots C_{ww}(t) \end{bmatrix} \quad (15)$$

of the conditional distribution functions

$C_{bl}(t) = P(C_{bl} < t)$, $b, l = 1,2,\ldots, w$,
of the conditional sojourn times C_{bl} at the climate-weather states c_b when its next climate-weather state is
c_l, $b, l = 1,2,\ldots, w$, $b \neq l$, where by formal agreement

$C_{bb}(t) = 0$ for $b = 1,2,\ldots, w$,

or equivalently the matrix

$$[c_{bl}(t)]_{wxw} = \begin{bmatrix} c_{11}(t)\, c_{12}(t) \ldots c_{1w}(t) \\ c_{21}(t)\, c_{22}(t) \ldots c_{2w}(t) \\ \ldots \\ c_{w1}(t)\, c_{w2}(t) \ldots c_{ww}(t) \end{bmatrix} \quad (16)$$

of the conditional density functions of the climate-weather change process $C(t)$ conditional sojourn times C_{bl} at the climate-weather states corresponding to the conditional distribution functions $C_{bl}(t)$, where

$$c_{bl}(t) = \frac{d}{dt}[c_{bl}(t)] \text{ for } b, l = 1,2,\ldots, w, b \neq l, \quad (17)$$

and by formal agreement

$c_{bb}(t) = 0$ for $b = 1,2,\ldots, w$.

We assume that the suitable and typical distributions suitable to describe the climate-weather change process $C(t)$ conditional sojourn times C_{bl}, b, $l = 1,2,\ldots, w$, $b \neq l$, at the particular climate-weather states are given by (4.5)-(4.12) in [EU-CRCLE Report D2.1-GMU2, 2016]

5. Climate-weather change process – prediction

5.1. Prediction of climate-weather process characteristics

Assuming that we have identified the unknown parameters of the climate-weather change process semi-Markov model:
– the initial probabilities $q_b(0)$, $b = 1,2,\ldots,w$, of the climate-weather change process staying at the particular state c_b at the moment $t = 0$;
– the probabilities q_{bl}, b, $l = 1,2,\ldots,w$, $b \neq l$, of the climate-weather change process transitions from the climate-weather state c_b into the climate-weather state c_l;
– the distributions of the climate-weather change process conditional sojourn times C_{bl}, b, $l = 1,2,\ldots,w$, $b \neq l$, at the particular climate-weather states and their mean values $M_{bl} = E[C_{bl}]$, b, $l = 1,2,\ldots,w$, ;
we can predict this process basic characteristics.

As the mean values of the conditional sojourn times C_{bl} are given by [Kołowrocki, Soszyńska-Budny, 2011]

$$N_{bl} = E[C_{bl}] = \int_0^\infty t dC_{bl}(t) = \int_0^\infty t c_{bl}(t) dt, \quad (18)$$
$$b,l = 1,2,\ldots,w, \; b \neq l,$$

then for the distinguished distributions (4.5)-(4.12) in [Kolowrocki, Soszyńska-Budny 2011] , the mean values of the climate-weather change process $C(t)$ conditional sojourn times C_{bl}, $b,l = 1,2,\ldots,v$, $b \neq l$, at the particular operation states are respectively given by (4.14)-(4.21) [Kolowrocki, Soszyńska-Budny 2011].
From the formula for total probability, it follows that the unconditional distribution functions of the sojourn times C_b, $b = 1,2,\ldots,w$, of the climate-weather change process $C(t)$ at the climate-weather states c_b, $b = 1,2,\ldots,w$, are given by [Kołowrocki, Soszyńska-Budny, 2011]

$$C_b(t) = \sum_{l=1}^{v} q_{bl} C_{bl}(t), \; b = 1,2,\ldots,w.$$

Hence, the mean values $E[C_b]$ of the climate-weather change process $C(t)$ unconditional sojourn times C_b, $b = 1,2,\ldots,w$, at the climate-weather states are given by

$$N_b = E[C_b] = \sum_{l=1}^{v} q_{bl} N_{bl}, \ , \ b = 1,2,...,w, \quad (20)$$

where N_{bl} are defined by the formula (18) in a case of any distribution of sojourn times C_{bl} and by the formulae (4.14)-(4.21) in [EU-CRCLE Report D2.1-GMU2, 2016] in the cases of particular defined respectively by (4.5)-(4.12) [EU-CRCLE Report D2.1-GMU2, 2016] distributions of these sojourn times.

The limit values of the climate-weather change process $C(t)$ transient probabilities at the particular operation states

$$q_b(t) = P(C(t) = c_b), \ t \in <0,+\infty), \ b = 1,2,...,w, \quad (21)$$

are given by [Kołowrocki, Soszyńska-Budny, 2016]

$$q_b = \lim_{t \to \infty} q_b(t) = \frac{\pi_b N_b}{\sum_{l=1}^{w} \pi_l N_l}, \ b = 1,2,...,w, \quad (22)$$

where N_b, $b = 1,2,...,w$, are given by (20), while the steady probabilities π_b of the vector $[\pi_b]_{1xw}$ satisfy the system of equations

$$\begin{cases} [\pi_b] = [\pi_b][q_{bl}] \\ \sum_{l=1}^{v} \pi_l = 1. \end{cases} \quad (23)$$

In the case of a periodic climate-weather change process, the limit transient probabilities q_b, $b = 1,2,...,w$, at the climate-weather states defined by (22), are the long term proportions of the climate-weather change process $C(t)$ sojourn times at the particular climate-weather states c_b, $b = 1,2,...,w$.

Other interesting characteristics of the system climate-weather change process $C(t)$ possible to obtain are its total sojourn times $\hat{C}_b$ at the particular climate-weather states c_b, $b = 1,2,...,w$, during the fixed time. It is well known [Kołowrocki, Soszyńska-Budny, 2011] that the climate-weather change process total sojourn times $\hat{C}_b$ at the particular climate-weather states c_b, for sufficiently large time θ, have approximately normal distributions with the expected value given by

$$\hat{N}_b = E[\hat{C}_b] = q_b \theta, \ b = 1,2,...,w, \quad (24)$$

where q_b are given by (22).

6. Critical infrastructure operation process related to climate-weather change process-modelling

We assume, that the critical infrastructure during its operation process is taking $v, v \in N$, different operation states $z_1, z_2, \ldots, z_v$. Further, we define the critical infrastructure operation process $Z(t)$, $t \in <0,+\infty)$, with discrete operation states from the set $\{z_1, z_2, \ldots, z_v\}$. Moreover, we assume that the critical infrastructure operation process $Z(t)$ is a semi-Markov process that can be described by:

- the vector $[p_b(0)]_{1xv}$ of the initial probabilities $p_b(0)$, $b, = 1,2,...,v$, of the critical infrastructure operation process $Z(t)$ staying at particular operation state z_b, $b = 1,2,...,v$, at the moment $t = 0$;
- the matrix $[p_{bl}]_{vxv}$ of probabilities p_{bl}, $b, l = 1,2,...,v$, of the critical infrastructure operation process $Z(t)$ transitions between the operation states z_b and z_l, $b, l = 1,2,...,v$;
- the matrix $[H_{bl}(t)]_{vxv}$ of conditional distribution functions $[H_{bl}(t)$, $b, l = 1,2,...,v$, of the critical infrastructure operation process $Z(t)$ conditional sojourn times θ_{bl} at the operation states z_b under the condition that the next operation state will be z_l, $b, l = 1,2,...,v$.

We assume that the climate-weather change process $C(t)$, $t \in <0,+\infty)$, at the critical infrastructure operating area is taking w, $w \in N$, different climate-weather states $c_1, c_2, \ldots, c_w$. Further, we assume that the climate-weather change process $C(t)$ is a semi-Markov process and it can be described by:

– the vector $[q_b(0)]_{1Xw}$) of the initial probabilities $q_b(0)$, $b = 1,2,\ldots, w$, of the climate-weather change process $C(t)$ staying at particular climate-weather states c_b, $b = 1,2,\ldots, w$, at the moment $t = 0$;
– the matrix $[q_{bl}]_{wXw}$ of the probabilities q_{bl}, $b, l = 1,2,...,w$, of transitions of the climate-weather change process $C(t)$ from the climate-weather states c_b to the climate-weather state c_l, $b, l = 1,2,\ldots, w$;
– the matrix $[C_{bl}(t)]_{wxw}$ of the conditional distribution functions $C_{bl}(t)$, $b,l = 1,2,\ldots, w$, of the conditional sojourn times C_{bl} at the climate-weather states c_b when its next climate-weather state is c_l, b, l = 1,2,…, w.

6.1. Joint model of independent critical infrastructure operation process and climate-weather change process

Under the assumption that the critical infrastructure operation process $Z(t)$, $t \in <0,+\infty)$, and the climate-weather change process $C(t)$ are independent, we introduce the joint process of critical infrastructure operation process and climate-weather change process called the critical infrastructure operation process related to climate-weather change marked by

$$ZC(t),\ t \in <0,+\infty), \tag{25}$$

and we assume that it can take $\nu w, \nu, w \in N$, different operation states

$$zc_{11}, zc_{12}, \ldots, zc_{\nu w}, \quad i = 1,2,\ldots,\nu, \quad j = 1,2,\ldots,w,$$
(26)

We assume that the critical infrastructure operation process related to climate-weather change $ZC(t)$, at the moment $t \in <0,+\infty)$, is at the state zc_{ij}, $i = 1,2,\ldots,\nu$, $j = 1,2,\ldots,w$, if and only if at that moment, the operation process $Z(t)$ is at the operation states z_i, $i = 1,2,\ldots,\nu$, and the climate-weather change process $C(t)$ is at the climate-weather state c_j, $j = 1,2,\ldots,w$, what we mark as follows:

$$(ZC(t) = zc_{ij}) \Leftrightarrow (Z(t) = z_i \cap C(t) = c_j), \tag{27}$$
$$t \in <0,+\infty),\ i = 1,2,\ldots,\nu,\ j = 1,2,\ldots,w.$$

Further, we define the initial probabilities

$$pq_{ij}(0) = P(ZC(0) = zc_{ij}),\ i = 1,2,\ldots,\nu, \tag{28}$$
$$j = 1,2,\ldots,w,$$

of the critical infrastructure operation process related to climate-weather change $ZC(t)$, at the initial moment $t = 0$ at the operation and climate-weather state zc_{ij}, $i = 1,2,\ldots,\nu \in N$, $j = 1,2,\ldots,w$, and this way we have the vector

$$[pq_{ij}(0)]_{1x\nu w}$$

$$= \begin{bmatrix} pq_{11}(0), pq_{12}(0), \ldots, pq_{1w}(0); pq_{21}(0), \\ pq_{22}(0), \ldots, pq_{2w}(0); \ldots; pq_{\nu 1}(0), \\ pq_{\nu 2}(0), \ldots pq_{\nu w}(0) \end{bmatrix} \tag{29}$$

of the initial probabilities the critical infrastructure operation process related to climate-weather change $ZC(t)$ staying at the particular operation and climate-weather state at the initial moment $t = 0$.
From the assumption that the critical infrastructure operation process $Z(t)$ and climate-weather change process $C(t)$ are independent, it follows that

$$pq_{ij}(0) = P(ZC(0) = zc_{ij})$$
$$= P(Z(0) = z_i \cap C(0) = c_j)$$
$$= P(Z(0) = z_i) \cdot P(C(0) = c_j) = p_i(0) \cdot q_j(0), \tag{30}$$
$$i = 1,2,\ldots,\nu,\ j = 1,2,\ldots,w,$$

where $p_i(0), i = 1,2,\ldots,\nu$, and $q_j(0)$, $j = 1,2,\ldots,w$, are respectively defined in Section 2.1 and Section 4.1. [EU-CIRCLE Report D2.1-GMU2, 2016].
Hence, the vector of the initial probabilities the critical infrastructure operation process related to climate-weather change $ZC(t)$ defined by (5.29) takes the following form

$$[pq_{ij}(0)]_{1x\nu w} = [p_i(0)q_j(0)]_{1x\nu w}$$

$$= \begin{bmatrix} p_1(0)q_1(0), p_1(0)q_2(0), \ldots p_1(0) \\ q_{1w}(0); \ldots; p_\nu(0)q_1(0), p_\nu(0) \\ q_2(0), \ldots, p_\nu(0)q_w(0) \end{bmatrix}. \tag{31}$$

Further, we introduce the probabilities

$$pq_{ij\,kl},\ i = 1,2,\ldots,\nu,\ j = 1,2,\ldots,w, \tag{32}$$
$$k = 1,2,\ldots,\nu,\ l = 1,2,\ldots,w,$$

of the transitions of the critical infrastructure operation process related to climate-weather change $ZC(t)$ between the operation states

$$zc_{ij} \text{ and } zc_{kl},\ i = 1,2,\ldots,\nu, \tag{33}$$
$$j = 1,2,\ldots,w,\ k = 1,2,\ldots,\nu,\ l = 1,2,\ldots,w,$$

and get their following matrix form

$$[pq_{ij\,kl}]_{vwxvw} = \begin{bmatrix} pq_{11\,11} \; pq_{11\,12} \cdots pq_{11\,1w}; pq_{11\,21} pq_{11\,22} \cdots pq_{11\,2w}; \ldots; pq_{11\,v1} pq_{11\,v2} \cdots pq_{11\,vw} \\ pq_{12\,11} \; pq_{12\,12} \cdots pq_{12\,1w}; pq_{12\,21} pq_{12\,22} \cdots pq_{12\,2w}; \ldots; pq_{12\,v1} pq_{12\,v2} \cdots pq_{12\,vw} \\ \cdots \\ pq_{vw\,11} \; pq_{vw\,12} \cdots pq_{vw\,1w}; pq_{vw\,21} pq_{vw\,22} \cdots pq_{vw\,2w}; \ldots; pq_{vw\,v1} pq_{vw\,v2} \cdots pq_{vw\,vw} \end{bmatrix}. \tag{34}$$

From the assumption that the critical infrastructure operation process $Z(t)$ and climate-weather change process $C(t)$ are independent, it follows that

$$pq_{ij\,kl} = p_{ik} p_{jl}, \; i = 1,2,\ldots,v, \; j = 1,2,\ldots,w, \; k = 1,2,\ldots,v, \; l = 1,2,\ldots,w, \tag{35}$$

where

$$p_{ik}, \; i = 1,2,\ldots,v, \; k = 1,2,\ldots,v, \text{ and } q_{jl}, \; j = 1,2,\ldots,w, \; l = 1,2,\ldots,w, \tag{36}$$

are respectively defined in Section 2.1 and Section 4.1 [EU-CIRCLE Report D2.1-GMU2, 2016].
Hence, the matrix of the probabilities of transitions between the critical infrastructure operation process related to climate-weather change $ZC(t)$ defined by (24) takes the following form

$$[pq_{ij\,kl}]_{vwxvw} = [p_{ik} q_{jl}]_{vwxvw}$$

$$= \begin{bmatrix} p_{11}q_{11} \; p_{11}q_{12} \cdots p_{11}q_{1w}; p_{12}q_{11} \; p_{12}q_{12} \cdots p_{12}q_{1w}; \ldots; p_{1v}q_{11} p_{1v}q_{12} \cdots p_{1v}q_{1w} \\ p_{11}q_{21} \; p_{11}q_{22} \cdots p_{11}q_{2w}; p_{12}q_{21} p_{12}q_{22} \cdots p_{12}q_{2w}; \ldots; p_{1v}q_{21} p_{1v}q_{22} \cdots p_{1v}q_{2w} \\ \cdots \\ p_{v1}q_{w1} \; p_{v1}q_{w2} \cdots p_{v1}q_{ww}; p_{v2}q_{w1} p_{v2}q_{w2} \cdots p_{v2}q_{ww}; \ldots; p_{vv}q_{w1} p_{vv}q_{w2} \cdots p_{vv}q_{ww} \end{bmatrix}. \tag{37}$$

The matrix of conditional distribution functions

$$HC_{ij\,kl}(t) = P(\theta C_{ij\,kl} < t), \; t \in <0,+\infty), \; i = 1,2,\ldots,v, \; j = 1,2,\ldots,w, \; k = 1,2,\ldots,v, \; l = 1,2,\ldots,w, \tag{38}$$

of the critical infrastructure operation process related to climate-weather change $ZC(t)$ conditional sojourn times $\theta C_{ij\,kl}$, $i = 1,2,\ldots,v$, $j = 1,2,\ldots,w$, $k = 1,2,\ldots,v$, $l = 1,2,\ldots,w$, at the operation state zc_{ik}, $i = 1,2,\ldots,v$, $k = 1,2,\ldots,v$, when the next operation state is zc_{jl}, $j = 1,2,\ldots,w$, $l = 1,2,\ldots,w$, takes the following form

$$[HC_{ij\,kl}(t)]_{vwxvw}$$

$$= \begin{bmatrix} HC_{11\,11}(t) \; HC_{11\,12}(t) \ldots HC_{11\,1w}(t); HC_{11\,21}(t) HC_{11\,22}(t) \ldots HC_{11\,2w}(t); \ldots; HC_{11\,v1}(t) HC_{11\,v2}(t) \ldots HC_{11\,vw}(t) \\ HC_{12\,11}(t) HC_{12\,12}(t) \ldots HC_{12\,1w}(t); HC_{12\,21}(t) HC_{12\,22}(t) \ldots HC_{12\,2w}(t); \ldots; HC_{12\,v1}(t) HC_{12\,v2}(t) \ldots HC_{12\,vw}(t) \\ \ldots \\ HC_{vw\,11}(t) \; HC_{vw\,12}(t) \ldots HC_{vw\,1w}(t); HC_{vw\,21}(t) HC_{vw\,22}(t) \ldots HC_{vw\,2w}(t); \ldots; HC_{vw\,v1}(t) HC_{vw\,v2}(t) \ldots HC_{vw\,vw}(t) \end{bmatrix} \tag{39}$$

and the matrix of their corresponding conditional density functions

$$hc_{ij\,kl}(t) = \frac{d}{dt}[HC_{ij\,kl}(t)] \text{ for } t \in <0,+\infty), \; i = 1,2,\ldots,v, \; j = 1,2,\ldots,w, \; k = 1,2,\ldots,v, \; l = 1,2,\ldots,w, \tag{40}$$

the form

$$[hc_{ij\,kl}(t)]_{vwxvw}$$
$$=\begin{bmatrix} hc_{1111}(t)\; hc_{1112}(t) \ldots hc_{111w}(t); hc_{1121}(t) hc_{1122}(t)..hc_{112w}(t);...;hc_{11v1}(t)hc_{11v2}(t)..hc_{11vw}(t) \\ hc_{1211}\,(t)hc_{1212}(t) \ldots hc_{121w}(t); hc_{1221}(t)hc_{1222}(t)..hc_{122w}(t);...;hc_{12v1}(t)hc_{12v2}(t)..hc_{12vw}(t) \\ \ldots \\ hc_{vw11}(t)\; hc_{vw12}(t) \ldots hc_{vw1w}(t); hc_{vw21}(t)hc_{vw22}(t)..hc_{vw2w}(t);...;hc_{vwv1}(t)hc_{vwv2}(t)..hc_{vwvw}(t) \end{bmatrix}. \quad (41)$$

From the assumption that the critical infrastructure operation process $Z(t)$ and climate-weather change process $C(t)$ are independent, it follows that

$$HC_{ij\,kl}(t) = P(\theta C_{ij\,kl} < t) = P(\theta_{ik} < t \cap C_{jl} < t) = H_{ik}(t)C_{jl}(t),\; t \in < 0,+\infty),$$
$$i = 1,2,...,v,\;\; j = 1,2,...,w,\;\; k = 1,2,...,v,\;\; l = 1,2,...,w, \quad (42)$$

and

$$hc_{ij\,kl}(t) = \frac{d}{dt}[HC_{ij\,kl}(t)] = \frac{d}{dt}[H_{ik}(t)C_{jl}(t)] = h_{ik}(t)C_{jl}(t) + H_{ik}(t)c_{jl}(t),\; t \in < 0,+\infty),$$
$$i = 1,2,...,v,\;\; j = 1,2,...,w,\;\; k = 1,2,...,v,\;\; l = 1,2,...,w, \quad (43)$$

where

$$H_{ik}(t),\; i = 1,2,...,v,\; k = 1,2,...,v, \text{ and } C_{jl}(t),\;\; j = 1,2,...,w,\;\; l = 1,2,...,w, \quad (44)$$

and

$$h_{ik}(t),\; i = 1,2,...,v,\; k = 1,2,...,v, \text{ and } c_{jl}(t),\;\; j = 1,2,...,w,\;\; l = 1,2,...,w, \quad (45)$$

are respectively defined in Chapter 2, Section 2.1 and Chapter 4, Section 4.1 [EU-CIRCLE Report D2.1-GMU2, 2016].

Hence, the matrix of the conditional distribution functions and the matrix of the conditional density functions of the critical infrastructure operation process related to climate-weather change $ZC(t)$ conditional sojourn times defined by (29) and (31) respectively take the following forms

$$[HC_{ij\,kl}(t)]_{vwxvw} = [H_{ik}(t)C_{jl}(t)]_{vwxvw}$$

$$=\begin{bmatrix} H_{11}(t)C_{11}(t)\; H_{11}(t)C_{12}(t) \ldots H_{11}(t)C_{1w}(t);...;H_{1v}(t)C_{11}(t)H_{1v}(t)C_{12}(t)...H_{1v}(t)C_{1w}(t) \\ H_{11}(t)C_{21}\,(t)H_{11}(t)C_{22}(t) \ldots H_{11}(t)C_{2w}(t);...;H_{1v}(t)C_{21}(t)H_{1v}(t)C_{22}(t)...H_{1v}(t)C_{1w}(t) \\ \ldots \\ H_{v1}(t)C_{w1}(t)\; H_{v1}(t)C_{w2}(t) \ldots H_{v1}(t)C_{ww}(t);...;H_{vv}(t)C_{w1}(t)H_{vv}(t)C_{w2}(t)...H_{vv}(t)C_{ww}(t) \end{bmatrix} \quad (46)$$

and

$$[hc_{ij\,kl}(t)]_{vwxvw} = [h_{ik}(t)C_{jl}(t) + H_{ik}(t)c_{jl}(t)]_{vwxvw}$$

$$= \begin{bmatrix} h_{11}(t)C_{11}(t) + H_{11}(t)c_{11}(t) \ldots h_{11}(t)C_{1w}(t) + H_{11}(t)c_{1w}(t);\ldots;h_{1v}(t)C_{11}(t) + H_{1v}(t)c_{11}(t)..h_{1v}(t)C_{1w}(t) + H_{1v}(t)c_{1w}(t) \\ h_{11}(t)C_{21}(t) + H_{11}(t)c_{21}(t) \ldots h_{11}(t)C_{2w}(t) + H_{11}(t)c_{2w}(t);\ldots;h_{1v}(t)C_{21}(t) + H_{1v}(t)c_{21}(t)..h_{1v}(t)C_{1w}(t) + H_{1v}(t)c_{1w}(t) \\ \ldots \\ h_{v1}(t)C_{w1}(t) + H_{v1}(t)c_{w1}(t) \ldots h_{v1}(t)C_{ww}(t) + H_{v1}(t)c_{ww}(t);\ldots;h_{vv}(t)C_{w1}(t) + H_{vv}(t)c_{w1}(t)..h_{vv}(t)C_{ww}(t) + H_{vv}(t)c_{ww}(t) \end{bmatrix} \quad (47)$$

We assume that the suitable and typical distributions suitable to describe the critical infrastructure operation process $Z(t)$ conditional sojourn times θ_{bl}, $b,l = 1,2,\ldots,v$, $b \neq l$, in the particular operation states are that defined in [Kołowrocki, Soszyńska-Budny, 2011], [EU-CIRCLE Report D2.1-GMU4-Part1, 2016] and [EU-CIRCLE Report D2.1-GMU4-Part2, 2016].

6.2. Joint model of dependent critical infrastructure operation process and climate-weather change process

Under the assumption that the critical infrastructure operation process $Z(t)$, $t \in <0,+\infty)$, and the climate-weather change process $C(t)$ are dependent, we introduce the joint process of critical infrastructure operation process and climate-weather change process called the critical infrastructure operation process related to climate-weather change marked by

$$ZC(t),\ t \in <0,+\infty), \quad (48)$$

and we assume that it can take $vw, v, w \in N$, different operation states

$$zc_{11}, zc_{12}, \ldots, zc_{vw}, \quad i = 1,2,\ldots,v, \quad j = 1,2,\ldots,w, \quad (49)$$

We assume that the critical infrastructure operation process related to climate-weather change $ZC(t)$, at the moment $t \in <0,+\infty)$, is at the state zc_{ij}, $i = 1,2,\ldots,v$, $j = 1,2,\ldots,w$, if and only if at that moment, the operation process $Z(t)$ is at the operation states z_i, $i = 1,2,\ldots,v$, and the climate-weather change process $C(t)$ is at the climate-weather state c_j, $j = 1,2,\ldots,w$, what we mark as follows:

$$(ZC(t) = zc_{ij}) \Leftrightarrow (Z(t) = z_i \cap C(t) = c_j), \quad (50)$$
$$t \in <0,+\infty),\ i = 1,2,\ldots,v,\ j = 1,2,\ldots,w.$$

Further, we define the initial probabilities

$$pq_{ij}(0) = P(ZC(0) = zc_{ij}),\ i = 1,2,\ldots,v, \quad (51)$$
$$j = 1,2,\ldots,w,$$

of the critical infrastructure operation process related to climate-weather change $ZC(t)$, at the initial moment $t = 0$ at the operation and climate-weather state zc_{ij}, $i = 1,2,\ldots,v \in N$, $j = 1,2,\ldots,w$, and this way we have the vector

$$[pq_{ij}(0)]_{1xvw} = \begin{bmatrix} pq_{11}(0), pq_{12}(0), \ldots, pq_{1w}(0); \\ pq_{21}(0), pq_{22}(0), \ldots, pq_{2w}(0); \\ \ldots; pq_{v1}(0), pq_{v2}(0), \ldots pq_{vw}(0) \end{bmatrix} \quad (52)$$

of the initial probabilities the critical infrastructure operation process related to climate-weather change $ZC(t)$ staying at the particular operation and climate-weather state at the initial moment $t = 0$.

In the case when the processess $Z(t)$ and $C(t)$ are dependent the initial probabilities existing in (52) can be expresed either by

$$pq_{ij}(0) = P(ZC(0) = zc_{ij})$$
$$= P(Z(0) = z_i \cap C(0) = c_j)$$
$$= P(Z(0) = z_i) \cdot P(C(0) = c_j \mid Z(0) = z_i)$$
$$= p_i(0) \cdot q_{j/i}(0),\ i = 1,2,\ldots,v,\ j = 1,2,\ldots,w, \quad (53)$$

where

$$p_i(0) = P(Z(0) = z_i),\ i = 1,2,\ldots,v, \quad (54)$$

are the initial probabilities of the operation process $Z(t)$ defined in Chapter 2 and

$$q_{j/i}(0) = P(C(0) = c_j \mid Z(0) = z_i), \quad (55)$$
$$i = 1,2,\ldots,v,\ j = 1,2,\ldots,w,$$

are conditional initial probabilities of the climate-weather change process $C(t)$ defined in Chapter 4 in case they are not conditional or by

$$pq_{ij}(0) = P(ZC(0) = zc_{ij}) = P(Z(0)$$
$$= z_i \cap C(0) = c_j)$$

$$= P(C(0) = C_j) \cdot P(Z(0) = z_i \mid C(0) = c_j)$$
$$= q_j(0) \cdot p_{i/j}(0),\ i = 1,2,\dots,v,\ \ j = 1,2,\dots,w, \qquad (56)$$

where

$$q_j(0) = P(C(0) = c_j),\ \ j = 1,2,\dots,w, \qquad (57)$$

are initial probabilities of the operation process $C(t)$ defined in Chapter 4 and

$$p_{i/j}(0) = P(Z(0) = z_i \mid C(0) = c_j), \qquad (58)$$
$$i = 1,2,\dots,v,\ \ j = 1,2,\dots,w,$$

are conditional initial probabilities of the climate-weather change process $Z(t)$ defined in Chapter 2 in case they are not conditional.

Further, we introduce the probabilities

$$pq_{ij\,kl},\ \ i = 1,2,\dots,v,\ \ j = 1,2,\dots,w, \qquad (59)$$
$$k = 1,2,\dots,v,\ l = 1,2,\dots,w,$$

of the transitions of the critical infrastructure operation process related to climate-weather change $ZC(t)$ between the operation states

$$zc_{ij} \text{ and } zc_{kl},\ \ i = 1,2,\dots,v, \qquad (60)$$
$$j = 1,2,\dots,w,\ \ k = 1,2,\dots,v,\ \ l = 1,2,\dots,w,$$

and get their following matrix form

$$[pq_{ij\,kl}]_{vwxvw} = \begin{bmatrix} pq_{1111}\ pq_{1112}\ \cdots\ pq_{111w};pq_{1121}pq_{1122}\cdots pq_{112w};\dots;pq_{11v1}pq_{11v2}\cdots pq_{11vw} \\ pq_{1211}\ pq_{1212}\ \cdots\ pq_{121w};pq_{1221}pq_{1222}\cdots pq_{122w};\dots;pq_{12v1}pq_{12v2}\cdots pq_{12vw} \\ \cdots \\ pq_{vw11}\ pq_{vw12}\ \cdots\ pq_{vw1w};pq_{vw21}pq_{vw22}\cdots pq_{vw2w};\dots;pq_{vwv1}pq_{vwv2}\cdots pq_{vwvw} \end{bmatrix}. \qquad (61)$$

In the case when the processess $Z(t)$ and $C(t)$ are dependent the probabilities of transitions between the operation states existing in (47) can be expresed either by

$$pq_{ij\,kl} = p_{ik} \cdot q_{jl/ik}\ \ i = 1,2,\dots,v, \qquad (62)$$
$$j = 1,2,\dots,w,\ \ k = 1,2,\dots,v,\ \ l = 1,2,\dots,w,$$

where

$$p_{ik},\ i = 1,2,\dots,v,\ k = 1,2,\dots,v, \qquad (63)$$

are transient probabilities of the operation process $Z(t)$ defined in Chapter 2 and

$$q_{jl/ik},\ \ i = 1,2,\dots,v,\ \ k = 1,2,\dots,v, \qquad (64)$$
$$j = 1,2,\dots,w,\ \ l = 1,2,\dots,w,$$

are conditional transient probabilities of the climate-weather change process $C(t)$ defined in Chapter 4 in case they are not conditional or by

$$pq_{ij\,kl} = q_{jl} \cdot p_{ik/jl}\ \ i = 1,2,\dots,v, \qquad (65)$$
$$j = 1,2,\dots,w,\ \ k = 1,2,\dots,v,\ \ l = 1,2,\dots,w,$$

where

$$q_{jl},\ \ j = 1,2,\dots,w,\ \ l = 1,2,\dots,w, \qquad (66)$$

are transient probabilities of the climate-weather change process $C(t)$ defined in Chapter 4 and

$$p_{ik/jl},\ \ i = 1,2,\dots,v,\ \ k = 1,2,\dots,v, \qquad (67)$$
$$j = 1,2,\dots,w,\ \ l = 1,2,\dots,w,$$

are conditional transient probabilities of the operation process $Z(t)$ defined in Chapter 2 in case they are not conditional.

The matrix of conditional distribution functions

$$HC_{ij\,kl}(t) = P(\theta C_{ij\,kl} < t),\ t \in <0,+\infty), \qquad (68)$$
$$i = 1,2,\dots,v,\ \ j = 1,2,\dots,w,\ \ k = 1,2,\dots,v,$$
$$l = 1,2,\dots,w,$$

of the critical infrastructure operation process related to climate-weather change $ZC(t)$ conditional sojourn times $\theta C_{ij\,kl}$, $i = 1,2,\dots,v$, $j = 1,2,\dots,w$, $k = 1,2,\dots,v$, $l = 1,2,\dots,w$, at the operation state zc_{ik},

$i = 1,2,...,\nu,$ $k = 1,2,...,\nu,$ when the next operation state is $zc_{jl},$ $j = 1,2,...,w,$ $l = 1,2,...,w,$ takes the following form

$$[HC_{ij\,kl}(t)]_{\nu w x \nu w}$$

$$= \begin{bmatrix} HC_{11\,11}(t)\; HC_{11\,12}(t) \ldots HC_{11\,1w}(t); HC_{11\,21}(t) HC_{11\,22}(t) .. HC_{11\,2w}(t);...; HC_{11\,\nu 1}(t) HC_{11\,\nu 2}(t) .. HC_{11\,\nu w}(t) \\ HC_{12\,11}(t) HC_{12\,12}(t) \ldots HC_{12\,1w}(t); HC_{12\,21}(t) HC_{12\,22}(t) .. HC_{12\,2w}(t);...; HC_{12\,\nu 1}(t) HC_{12\,\nu 2}(t) .. HC_{12\,\nu w}(t) \\ \ldots \\ HC_{\nu w\,11}(t)\; HC_{\nu w\,12}(t) \ldots HC_{\nu w\,1w}(t); HC_{\nu w\,21}(t) HC_{\nu w\,22}(t) .. HC_{\nu w\,2w}(t);...; HC_{\nu w\,\nu 1}(t) HC_{\nu w\,\nu 2}(t) .. HC_{\nu w\,\nu w}(t) \end{bmatrix} \quad (69)$$

and the matrix of their corresponding conditional density functions

$$hc_{ij\,kl}(t) = \frac{d}{dt}[HC_{ij\,kl}(t)] \text{ for } t \in <0,+\infty), \; i = 1,2,...,\nu, \; j = 1,2,...,w, \; k = 1,2,...,\nu, \; l = 1,2,...,w, \quad (70)$$

the form

$$[hc_{ij\,kl}(t)]_{\nu w x \nu w}$$

$$= \begin{bmatrix} hc_{11\,11}(t)\; hc_{11\,12}(t) \ldots hc_{11\,1w}(t); hc_{11\,21}(t) hc_{11\,22}(t) .. hc_{11\,2w}(t);...; hc_{11\,\nu 1}(t) hc_{11\,\nu 2}(t) .. hc_{11\,\nu w}(t) \\ hc_{12\,11}(t) hc_{12\,12}(t) \ldots hc_{12\,1w}(t); hc_{12\,21}(t) hc_{12\,22}(t) .. hc_{12\,2w}(t);...; hc_{12\,\nu 1}(t) hc_{12\,\nu 2}(t) .. hc_{12\,\nu w}(t) \\ \ldots \\ hc_{\nu w\,11}(t)\; hc_{\nu w\,12}(t) \ldots hc_{\nu w\,1w}(t); hc_{\nu w\,21}(t) hc_{\nu w\,22}(t) .. hc_{\nu w\,2w}(t);...; hc_{\nu w\,\nu 1}(t) hc_{\nu w\,\nu 2}(t) .. hc_{\nu w\,\nu w}(t) \end{bmatrix}. \quad (71)$$

In the case when the critical infrastructure operation process $Z(t)$ and climate-weather change process $C(t)$ are dependent, the distribution functions existing in (70) can be expressed either by

$$HC_{ij\,kl}(t) = P(\theta C_{ij\,kl} < t) = P(\theta_{ik} < t \cap C_{jl} < t)$$
$$= H_{ik}(t) C_{jl/ik}(t), \; t \in <0,+\infty), \quad (72)$$
$$i = 1,2,...,\nu, \; j = 1,2,...,w, \; k = 1,2,...,\nu,$$
$$l = 1,2,...,w,$$

where

$$H_{ik}(t), \; i = 1,2,...,\nu, \; k = 1,2,...,\nu, \quad (73)$$

are distribution functions defined of the sojourn lifetimes of the operation process $Z(t)$ defined in Chapter 2 and

$$C_{jl/ik}(t) = P(C_{jl} < t \mid \theta_{ik} < t), \; i = 1,2,...,\nu, \quad (74)$$
$$j = 1,2,...,w, \; k = 1,2,...,\nu, \; l = 1,2,...,w,$$

are conditional distributions of the sojourn lifetimes at the climate-weather states of the climate-weather change process $C(t)$ defined in Chapter 4 in case they are not conditional or by

$$HC_{ij\,kl}(t) = P(\theta C_{ij\,kl} < t) = P(\theta_{ik} < t \cap C_{jl} < t) \quad (75)$$
$$= C_{jl}(t) H_{ik/jl}(t), \; t \in <0,+\infty),$$
$$i = 1,2,...,\nu, \; j = 1,2,...,w, \; k = 1,2,...,\nu,$$
$$l = 1,2,...,w,$$

where

$$C_{jl}(t), \; j = 1,2,...,w, \; l = 1,2,...,w, \quad (76)$$

are distribution functions defined of the sojourn lifetimes at the climate-weather states of the climate-weather change process $C(t)$ defined in the Chapter 4 and

$$H_{ik/jl}(t) = P(\theta_{ik} < t \mid C_{jl} < t), \; i = 1,2,...,\nu, \quad (77)$$
$$j = 1,2,...,w, \; k = 1,2,...,\nu, \; l = 1,2,...,w,$$

are conditional distributions of the sojourn lifetimes at the operation states of the critical infrastructure operation process $Z(t)$ defined in Chapter 2 in case they are not conditional.

Hence, the density functions existing in (71) can be expressed either by

$$hc_{ij\,kl}(t) = \frac{d}{dt}[HC_{ij\,kl}(t)] = \frac{d}{dt}[H_{ik}(t)C_{jl/ik}(t)]$$
$$= h_{ik}(t)C_{jl/ik}(t) + H_{ik}(t)c_{jl/ik}(t), \ t \in <0,+\infty), \quad (78)$$

$i = 1,2,...,\nu, \ j = 1,2,...,w, \ k = 1,2,...,\nu,$
$l = 1,2,...,w,$

where

$H_{ik}(t), \ i = 1,2,...,\nu, \ k = 1,2,...,\nu,$ (79)
and $C_{jl/ik}(t), \ j = 1,2,...,w, \ l = 1,2,...,w,$

and

$h_{ik}(t), \ i = 1,2,...,\nu, \ k = 1,2,...,\nu,$ and $c_{jl/ik}(t),$ (80)
$j = 1,2,...,w, \ l = 1,2,...,w,$

are respectively defined in Section 2.1 and Section 4.1 or by

$$hc_{ij\,kl}(t) = \frac{d}{dt}[HC_{ij\,kl}(t)] = \frac{d}{dt}[C_{jl}(t)H_{ik/jl}(t)] \quad (81)$$
$$= c_{jl}(t)H_{ik/jl}(t) + C_{jl}(t)h_{ik/jl}(t), \ t \in <0,+\infty),$$

$i = 1,2,...,\nu, \ j = 1,2,...,w, \ k = 1,2,...,\nu,$
$l = 1,2,...,w,$

where

$C_{jl}(t), \ j = 1,2,...,w, \ l = 1,2,...,w,$ (82)
and $H_{ik/jl}(t), \ i = 1,2,...,\nu, \ k = 1,2,...,\nu,$

and

$c_{jl}(t), \ j = 1,2,...,w, \ l = 1,2,...,w,$ (83)
and $h_{ik/jl}(t), \ i = 1,2,...,\nu, \ k = 1,2,...,\nu,$

are respectively defined in Section 4.1 and Section 2.1 [EU-CIRCLE Report D2.1-GMU2, 2016].

We assume that the suitable and typical distributions suitable to describe the critical infrastructure operation process $Z(t)$ conditional sojourn times θ_{bl}, $b,l = 1,2,...,\nu, \ b \neq l,$ in the particular operation states are that defined in Section 4.2.4 [EU-CIRCLE Report D2.1-GMU2, 2016].

7. Critical infrastructure operation process related to climate-weather change process - prediction

Assuming that we have identified the unknown parameters of the critical infrastructure operation process related to climate-weather change $ZC(t)$, $t \in <0,+\infty)$, that can take $\nu w, \nu, w \in N,$ different operation states $zc_{11}, zc_{12}, \ldots, zc_{\nu w}, \ i = 1,2,...,\nu,$ $j = 1,2,...,w,$ defined in Section 5.6 [EU-CIRCLE Report D2.1-GMU2, 2016] and described by :

- the vector $[pq_{ij}(0)]_{1x\nu w}$ of initial probabilities of the critical infrastructure operation process related to climate-weather change $ZC(t)$ staying at the initial moment $t = 0$ at the operation and climate-weather states $zc_{ij}, \ i = 1,2,...,\nu \in N, \ j = 1,2,...,w;$
- the matrix $[pq_{ij\,kl}]_{\nu wx\nu w}$ of the probabilities of transitions of the critical infrastructure operation process related to climate-weather change $ZC(t)$ between the operation states zc_{ij} and $zc_{kl}, \ i = 1,2,...,\nu, \ j = 1,2,...,w,$ $k = 1,2,...,\nu, \ l = 1,2,...,w;$
- the matrix $[HC_{ij\,kl}(t)]_{\nu wx\nu w}$ of the matrix of conditional distribution functions of the critical infrastructure operation process related to climate-weather change $ZC(t)$ conditional sojourn times $\theta C_{ij\,kl}, \ i = 1,2,...,\nu,$ $j = 1,2,...,w, \ k = 1,2,...,\nu, \ l = 1,2,...,w,$ at the operation state $zc_{ik}, \ i = 1,2,...,\nu,$ $k = 1,2,...,\nu,$ when the next operation state is $zc_{jl}, \ j = 1,2,...,w, \ l = 1,2,...,w,$

we can predict this process basic characteristics.

7.1. Critical infrastructure operation process related to climate-weather change process characteristics – independent critical infrastructure operation process and climate-weather change process

The mean values of the conditional sojourn times $\theta C_{ij\,kl}, \ i = 1,2,...,\nu, \ j = 1,2,...,w, \ k = 1,2,...,\nu,$ $l = 1,2,...,w,$ at the operation state $zc_{ik}, \ i = 1,2,...,\nu,$ $k = 1,2,...,\nu,$ when the next operation state is $zc_{jl},$ $j = 1,2,...,w, \ l = 1,2,...,w,$ are defined by [Kołowrocki, Soszyńska-Budny, 2011]

$$MN_{ij\,kl} = E[\theta C_{ij\,kl}] = \int_0^\infty t dHC_{ij\,kl}(t)dt$$
$$= \int_0^\infty thc_{ij\,kl}(t)dt, \quad (84)$$
$$i = 1,2,\ldots,v,\ \ j = 1,2,\ldots,w,\ \ k = 1,2,\ldots,v,$$
$$l = 1,2,\ldots,w.$$

In the case when the processess $Z(t)$ and $C(t)$ are independent, according to (37) the expessions (84) tasks the form

$$MN_{ij\,kl} = E[\theta C_{ij\,kl}]$$
$$= \int_0^\infty t[h_{ik}(t)C_{jl}(t) + H_{ik}(t)c_{jl}(t)]dt,\ i = 1,2,\ldots,v, \quad (85)$$
$$j = 1,2,\ldots,w,\ \ k = 1,2,\ldots,v,\ \ l = 1,2,\ldots,w.$$

Since from the formula for total probability, it follows that the unconditional distribution functions of the conditional sojourn times θC_{ij}, of the critical infrastructure operation process related to climate-weather change $ZC(t)$ at the operation states state

$zc_{ij},\ i = 1,2,\ldots,v,\ j = 1,2,\ldots,w,$ are given by

$$HC_{ij}(t) = \sum_{k=1}^{v}\sum_{l=1}^{w} p_{ij\,kl} HC_{ij\,kl}(t),\ \ t \in <0,+\infty), \quad (86)$$
$$i = 1,2,\ldots,v,\ \ j = 1,2,\ldots,w,$$

In the case when the processess $Z(t)$ and $C(t)$ are independent, according to (25) and (32) the expessions (76) tasks the form

$$HC_{ij}(t) = \sum_{k=1}^{v}\sum_{l=1}^{w} p_{ik} q_{jl} H_{ik}(t) C_{jl}(t),\ t \in <0,+\infty), \quad (87)$$
$$i = 1,2,\ldots,v,\ \ j = 1,2,\ldots,w,$$

From (86) it follows that the mean values $E[\theta C_{ij}]$ of the unconditional distribution functions of the conditional sojourn times θC_{ij}, of the critical infrastructure operation process related to climate-weather change $ZC(t)$ at the operation states zc_{ij}, $i = 1,2,\ldots,v,\ j = 1,2,\ldots,w,$ are given by

$$MN_{ij} = E[\theta C_{ij}] = \sum_{k=1}^{v}\sum_{l=1}^{w} p_{ij\,kl} MN_{ij\,kl}, \quad (88)$$
$$i = 1,2,\ldots,v,\ \ j = 1,2,\ldots,w,$$

where $MN_{ij\,kl}$ are given by the formula (84).

In the case when the processess $Z(t)$ and $C(t)$ are independent, considering (87) and (32) the expession (78) tasks the form

$$MN_{ij} = E[\theta C_{ij}] = \sum_{k=1}^{v}\sum_{l=1}^{w} p_{ik} q_{jl} MN_{ij\,kl}, \quad (89)$$
$$i = 1,2,\ldots,v,\ j = 1,2,\ldots,w,$$

where $MN_{ij\,kl}$ are given by the formula (85).

The transient probabilities of the critical infrastructure operation process related to climate-weather change $ZC(t)$ at the operation states zc_{ij}, $i = 1,2,\ldots,v,\ j = 1,2,\ldots,w,$ can be defined by

$$pq_{ij}(t) = P(ZC(t) = zc_{ij}),\ t \in <0,+\infty), \quad (90)$$
$$i = 1,2,\ldots,v,\ \ j = 1,2,\ldots,w.$$

In the case when the processess $Z(t)$ and $C(t)$ are independent the expession (90) for the transient probabilities can be expressed in the following way

$$pq_{ij}(t) = P(ZC(t) = zc_{ij}) = P(Z(t) = z_i \cap C(t) = c_j)$$
$$= P(Z(t) = z_i) \cdot P(C(t) = c_j) = p_i(t) \cdot q_j(t), \quad (91)$$
$$t \in <0,+\infty),\ i = 1,2,\ldots,v,\ \ j = 1,2,\ldots,w,$$

where

$$p_i(t) = P(Z(t) = z_i),\ t \in <0,+\infty),\ i = 1,2,\ldots,v, \quad (92)$$

are the transient probabilities of the operation process $Z(t)$ defined in Chapter 2 and

$$q_j(t) = P(C(t) = c_j),\ t \in <0,+\infty), \quad (93)$$
$$j = 1,2,\ldots,w,$$

are the transient probabilities of the climate-weather change process $C(t)$ defined in Chapter 4.

The limit values of the critical infrastructure operation process related to climate-weather change $ZC(t)$ at the operation states zc_{ij}, $i = 1,2,\ldots,v,$ $j = 1,2,\ldots,w,$ can be found from [Kołowrocki, Soszyńska-Budny, 2011]

$$pq_{ij} = \lim_{t\to\infty} \frac{\pi_{ij} MN_{ij}}{\sum_{i=1}^{v}\sum_{j=1}^{w} \pi_{ij} MN_{ij}},\ i = 1,2,\ldots,v,\ \ j = 1,2,\ldots,w, \quad (94)$$

where MN_{ij}, $i = 1,2,\ldots,v,\ \ j = 1,2,\ldots,w,$ are given by (89), while the steady probabilities π_{ij}, $i = 1,2,\ldots,v,$ $j = 1,2,\ldots,w,$ of the vector $[\pi_{ij}]_{1xvw}$ satisfy the system of equations

$$\begin{cases} [\pi_{ij}][pq_{ij\,kl}] = [\pi_{ij}] \\ \sum_{i=1}^{v}\sum_{j=1}^{w} \pi_{ij} = 1, \end{cases} \quad (95)$$

where $pq_{ij\,kl}$, $i=1,2,...,v$, $j=1,2,...,w$, $k=1,2,...,v$, $l=1,2,...,w$, are given by (25).

In the case of a periodic system operation process, the limit transient probabilities pq_{ij}, $i=1,2,...,v$, $j=1,2,...,w$, at the operation states given by (94), are the long term proportions of the critical infrastructure operation process $ZC_{ij}(t)$ sojourn times at the particular operation states zc_{ij}, $i=1,2,...,v$, $j=1,2,...,w$.

Other interesting characteristics of the critical infrastructure operation process $ZC_{ij}(t)$ possible to obtain are its total sojourn times $\hat{\theta}\hat{C}_{ij}$, $i=1,2,...,v$, $j=1,2,...,w$, at the particular operation states zc_{ij}, $i=1,2,...,v$, $j=1,2,...,w$, during the fixed system opetation time. It is well known [Kołowrocki, Soszyńska-Budny, 2011] that the system operation process total sojourn times $\hat{\theta}\hat{C}_{ij}$, at the particular operation states zc_{ij}, for sufficiently large operation time θ, have approximately normal distributions with the expected value given by

$$\hat{M}\hat{N}_{ij} = E[\hat{\theta}\hat{C}_{ij}] = pq_{ij}\theta, \ i=1,2,...,v, \ j=1,2,...,w, \tag{96}$$

where pq_{ij}, $i=1,2,...,v$, $j=1,2,...,w$, are given by (94).

7.2. Critical infrastructure operation process related to climate-weather change process characteristics – dependent critical infrastructure operation process and climate-weather change process

The mean values of the conditional sojourn times $\theta C_{ij\,kl}$, $i=1,2,...,v$, $j=1,2,...,w$, $k=1,2,...,v$, $l=1,2,...,w$, at the operation state zc_{ik}, $i=1,2,...,v$, $k=1,2,...,v$, when the next operation state is zc_{jl}, $j=1,2,...,w$, $l=1,2,...,w$, are defined by [Kołowrocki, Soszyńska-Budny, 2011]

$$MN_{ij\,kl} = E[\theta C_{ij\,kl}] = \int_0^\infty t dHC_{ij\,kl}(t)dt = \int_0^\infty t hc_{ij\,kl}(t)dt, \ i=1,2,...,v, \ j=1,2,...,w, \ k=1,2,...,v, \ l=1,2,...,w. \tag{97}$$

Since from the formula for total probability, it follows that the unconditional distribution functions of the conditional sojourn times θC_{ij}, of the critical infrastructure operation process related to climate-weather change $ZC(t)$ at the operation states state zc_{ij}, $i=1,2,...,v$, $j=1,2,...,w$, are given by

$$HC_{ij}(t) = \sum_{k=1}^{v}\sum_{l=1}^{w} p_{ij\,kl} HC_{ij\,kl}(t), \ t \in <0,+\infty), \ i=1,2,...,v, \ j=1,2,...,w, \tag{98}$$

Hence, the mean values $E[\theta C_{ij}]$ of the unconditional distribution functions of the conditional sojourn times θC_{ij}, of the critical infrastructure operation process related to climate-weather change $ZC(t)$ at the operation states zc_{ij}, $i=1,2,...,v$, $j=1,2,...,w$, are given by

$$MN_{ij} = E[\theta C_{ij}] = \sum_{k=1}^{v}\sum_{l=1}^{w} p_{ij\,kl} MN_{ij\,kl}, \ i=1,2,...,v, \ j=1,2,...,w, \tag{99}$$

where $MN_{ij\,kl}$ are defined by the formula (87).

The transient probabilities of the critical infrastructure operation process related to climate-weather change $ZC(t)$ at the operation states zc_{ij}, $i=1,2,...,v$, $j=1,2,...,w$, can be defined by

$$pq_{ij}(t) = P(ZC(t) = zc_{ij}), \ t \in <0,+\infty), \ i=1,2,...,v, \ j=1,2,...,w. \tag{100}$$

In the case when the processess $Z(t)$ and $C(t)$ are dependent the transient probabilities can be expresed either by

$$\begin{aligned} pq_{ij}(t) &= P(ZC(t) = zc_{ij}) \\ &= P(Z(t) = z_i \cap C(t) = c_j) \\ &= P(Z(t) = z_i) \cdot P(C(t) = c_j \mid Z(t) = z_i) \\ &= p_i(t) \cdot q_{j/i}(t), \ t \in <0,+\infty), \ i=1,2,...,v, \ j=1,2,...,w, \end{aligned} \tag{101}$$

where

$$p_i(t) = P(Z(t) = z_i), \ t \in <0,+\infty), \ i=1,2,...,v, \tag{102}$$

are transient probabilities of the operation process $Z(t)$ defined in Chapter 2 and

$$q_{j/i}(t) = P(C(t) = c_j \mid Z(t) = z_i),\ t \in <0,+\infty), \quad (103)$$
$$i = 1,2,\ldots,v,\ \ j = 1,2,\ldots,w,$$

are conditional transient probabilities of the climate-weather change process $C(t)$ defined in Chapter 4 in case they are not conditional
or by

$$pq_{ij}(t) = P(ZC(t) = zc_{ij}) = P(Z(t) = z_i \cap C(t) = c_j)$$
$$= P(C(t) = C_j) \cdot P(Z(t) = z_i \mid C(t) = c_j)$$
$$= q_j(t) \cdot p_{i/j}(t),\ t \in <0,+\infty),\ i = 1,2,\ldots,v, \quad (104)$$
$$j = 1,2,\ldots,w,$$

where

$$q_j(t) = P(C(t) = c_j),\ t \in <0,+\infty), \quad (105)$$
$$j = 1,2,\ldots,w,$$

are transient probabilities of the operation process $C(t)$ defined in Chapter 4 and

$$p_{i/j}(t) = P(Z(t) = z_i \mid C(t) = c_j),\ t \in <0,+\infty),$$
$$i = 1,2,\ldots,v,\ \ j = 1,2,\ldots,w,$$

are conditional transient probabilities of the climate-weather change process $Z(t)$ defined in Chapter 2 in case they are not conditional.
The limit values of the critical infrastructure operation process related to climate-weather change $ZC(t)$ at the operation states zc_{ij}, $i = 1,2,\ldots,v$, $j = 1,2,\ldots,w$, can be found from [Kołowrocki, Soszyńska-Budny, 2011]

$$pq_{ij} = \lim_{t\to\infty} \frac{\pi_{ij} MN_{ij}}{\sum_{i=1}^{v}\sum_{j=1}^{w} \pi_{ij} MN_{ij}},\ i = 1,2,\ldots,v,\ \ j = 1,2,\ldots,w, \quad (106)$$

where MN_{ij}, $i = 1,2,\ldots,v$, $j = 1,2,\ldots,w$, are given by (99), while the steady probabilities π_{ij}, $i = 1,2,\ldots,v$, $j = 1,2,\ldots,w$, of the vector $[\pi_{ij}]_{1xvw}$ satisfy the system of equations

$$\begin{cases} [\pi_{ij}][pq_{ij\,kl}] = [\pi_{ij}] \\ \sum_{i-=1}^{v}\sum_{j=1}^{w} \pi_{ij} = 1. \end{cases} \quad (107)$$

In the case of a periodic system operation process, the limit transient probabilities pq_{ij}, $i = 1,2,\ldots,v$, $j = 1,2,\ldots,w$, at the operation states given by (107), are the long term proportions of the critical infrastructure operation process $ZC_{ij}(t)$ sojourn times at the particular operation states zc_{ij}, $i = 1,2,\ldots,v$, $j = 1,2,\ldots,w$.
Other interesting characteristics of the critical infrastructure operation process $ZC_{ij}(t)$ possible to obtain are its total sojourn times $\hat{\theta}\hat{C}_{ij}$, $i = 1,2,\ldots,v$, $j = 1,2,\ldots,w$, at the particular operation states zc_{ij}, $i = 1,2,\ldots,v$, $j = 1,2,\ldots,w$, during the fixed system opetation time. It is well known [Kołowrocki, Soszyńska-Budny, 2011] that the system operation process total sojourn times $\hat{\theta}\hat{C}_{ij}$, at the particular operation states zc_{ij}, for sufficiently large operation time θ, have approximately normal distributions with the expected value given by

$$\hat{M}\hat{N}_{ij} = E[\hat{\theta}\hat{C}_{ij}] = pq_{ij}\theta,\ \ i = 1,2,\ldots,v, \quad (108)$$
$$j = 1,2,\ldots,w,$$

where pq_{ij}, $i = 1,2,\ldots,v$, $j = 1,2,\ldots,w$, are given by (106).

8. Conclusions

The probabilistic model of the critical infrastructure operation process related to climate-wearher change process presented in this Chapter is the basis for further considerations in particular tasks of the project. First, this model will be used to construct the integrated general safety probabilistic model of the critical infrastructure related to its operation process and climate-weather process [EU-CIRCLE Report D.e3.3-GMU3, CIOP Model5, 2016].

Acknowledgements

The paper presents the results developed in the scope of the EU-CIRCLE project titled "A pan – European framework for strengthening Critical Infrastructure resilience to climate change" that has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 653824. http://www.eu-circle.eu/

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