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GENERAL MULTIPLICATIVE ZAGREB INDICES OF GRAPHS WITH GIVEN CLIQUE NUMBER

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Abstract. We obtain lower and upper bounds on general multiplicative Zagreb indices for graphs of given clique number and order. Bounds on the basic multiplicative Zagreb indices and on the multiplicative sum Zagreb index follow from our results. We also determine graphs with the smallest and the largest indices.

Keywords: clique number, multiplicative Zagreb index, chromatic number.

Mathematics Subject Classification: 05C35, 05C07.

1. INTRODUCTION

Molecular descriptors called topological indices are graph invariants that play a significant role in chemistry, materials science, pharmaceutical sciences and engineering, since they can be correlated with a large number of physico-chemical properties of molecules. In theoretical chemistry, topological indices are used for modelling properties of chemical compounds and biological activities in chemistry, biochemistry and nanotechnology.

Let $V(G)$ be the vertex set and let $E(G)$ be the edge set of a graph G. The order n of a graph G is the number of vertices of G. The degree of a vertex $v \in V(G)$, denoted by $d_G(v)$, is the number of edges incident with v. For $e \in E(G)$ let us denote by $G - e$ the subgraph of G obtained by deleting the edge e from $E(G)$. For two nonadjacent vertices $v_1, v_2 \in V(G)$, let us denote by $G + v_1v_2$ the graph obtained by adding the edge v_1v_2 to $E(G)$.

The chromatic number of a graph G is the smallest number of colours needed to colour the vertices of G so that no two adjacent vertices have the same colour. A clique of a graph G is a complete subgraph of G and the clique number $\omega(G)$ of G is the number of vertices in a maximum clique of G. Let $\chi_{n,k}$ be the set of connected graphs having order n and chromatic number k and let $W_{n,k}$ be the set of connected graphs having order n and clique number k.

We denote the complete graph and the path graph of order n by K_n and P_n, respectively. A k-partite graph is a graph whose vertices can be partitioned into k disjoint sets so that no two vertices within the same set are adjacent. Let us denote by $K_{n_1,n_2,\ldots,n_k}$ the complete k-partite graph with partite sets of orders $n_1, n_2, \ldots, n_k$. The Turán graph $T_{n,k}$ is a complete k-partite graph such that the orders of any two partite sets differ by at most 1. We denote by $K_k * P_{n-k}$ the graph obtained by joining one vertex of K_k to a pendant vertex of P_{n-k}. The graph $K_k \star S_{n-k}$ is obtained by joining one vertex of K_k to $n-k$ new vertices.

The first general multiplicative Zagreb index of a graph G is defined as

$$P_1^a(G) = \prod_{v \in V(G)} d_G(v)^a$$

the second general multiplicative Zagreb index is

$$P_2^a(G) = \prod_{v \in V(G)} d_G(v)^{a d_G(v)} = \prod_{uv \in E(G)} (d_G(u) d_G(v))^a$$

and the third general multiplicative Zagreb index of G,

$$P_3^a(G) = \prod_{uv \in E(G)} (d_G(u) + d_G(v))^a,$$

where $a \neq 0$ is any real number. These indices generalize basic multiplicative Zagreb indices. For $a = 1$, $P_1^1(G)$ is the Narumi-Katayama index and for $a = 2$, $P_1^2(G)$ is the first multiplicative Zagreb index

$$\prod\nolimits_1(G) = \prod_{v \in V(G)} d_G(v)^2.$$

For $a = 1$, $P_2^1(G)$ is the second multiplicative Zagreb index

$$\prod\nolimits_2(G) = \prod_{v \in V(G)} d_G(v)^{d_G(v)}.$$

For $a = 1$, $P_3^1(G)$ is the multiplicative sum Zagreb index

$$\prod\nolimits_1^*(G) = \prod_{uv \in E(G)} (d_G(u) + d_G(v)).$$

Multiplicative Zagreb indices for graphs of prescribed order and size were studied in [5], graphs of given order and chromatic number in [10]. bipartite graphs in [8]. molecular graphs in [4], graph operations in [6], some derived graphs in [1] and trees in [7]. Note that the Harary index of graphs with given matching number and clique number was investigated in [9].

The indices $\prod_1(G)$ $(= PM_1(G))$ and $\prod_2(G)$ $(= PM_2(G))$ for graphs of given clique number were considered in [3]. That paper contains mistakes. We mention only mistakes in main statements. The value of the second multiplicative Zagreb index for the Turán graph presented in their Lemma 5 and Corollary 7 is incorrect ($r - k$ must be replaced by $k - r$). The lower bound on the first multiplicative Zagreb index is incorrect. It should be $PM_1(G) \geq PM_1(K_k \star S_{n-k})$. The value of $PM_2(K_k(n-k)^1)$ presented in the last lines of the third section is incorrect if $n = k + 1$. Correct values are presented in our conclusion.

We study bounds on general multiplicative Zagreb indices for graphs of given order and clique number (chromatic number) and results on multiplicative Zagreb indices (including the multiplicative sum Zagreb index which has not been considered in combination with clique number before) are corollaries of our main results.

2. RESULTS

We study connected graphs of order n and clique number k (chromatic number k), thus we can assume that $k \geq 2$ since every connected graph having at least 2 vertices has clique number and chromatic number at least 2. We can also assume that $n > k$ since the only graph in $W_{n,k}$ and $\chi_{n,k}$ for $n = k$ is a complete graph.

Lemma 2.1. *Let $n = n_1 + n_2 + \ldots + n_k$. Then for any $a \neq 0$ we have*

$$P_1^a(K_{n_1,n_2,\ldots,n_k}) = \prod_{i=1}^{k}(n-n_i)^{an_i},$$

$$P_2^a(K_{n_1,n_2,\ldots,n_k}) = \prod_{i=1}^{k}(n-n_i)^{an_i(n-n_i)},$$

$$P_3^a(K_{n_1,n_2,\ldots,n_k}) = \prod_{1\leq i<j\leq k}(2n-n_i-n_j)^{an_in_j}.$$

Proof. The vertices of $K_{n_1,n_2,\ldots,n_k}$ can be divided into k sets $N_1, N_2, \ldots, N_k$, where $|N_i| = n_i$ for $i = 1, 2, \ldots, k$, such that no two vertices in the same set are adjacent and any two vertices from different sets are adjacent. Every vertex in N_i has degree $n - n_i$, thus

$$P_1^a(K_{n_1,n_2,\ldots,n_k}) = (n-n_1)^{an_1}(n-n_2)^{an_2}\ldots(n-n_k)^{an_k} = \prod_{i=1}^{k}(n-n_i)^{an_i}$$

and

$$\begin{aligned}P_2^a(K_{n_1,n_2,\ldots,n_k}) &= (n-n_1)^{an_1(n-n_1)}(n-n_2)^{an_2(n-n_2)}\ldots(n-n_k)^{an_k(n-n_k)}\\ &= \prod_{i=1}^{k}(n-n_i)^{an_i(n-n_i)}.\end{aligned}$$

Since for $1 \leq i < j \leq k$ there are $n_i n_j$ edges connecting vertices in N_i (of degree $n - n_i$) and vertices in N_j of (degree $n - n_j$), we obtain

$$\begin{aligned} P_3^a(K_{n_1,n_2,\ldots,n_k}) &= (2n - n_1 - n_2)^{an_1n_2}(2n - n_1 - n_3)^{an_1n_3} \\ &\quad \ldots (2n - n_{k-1} - n_k)^{an_{k-1}n_k} \\ &= \prod_{1\leq i<j\leq k} (2n - n_i - n_j)^{an_in_j}. \end{aligned}$$

□

We show that by adding an edge to a graph G, we obtain a graph having larger general multiplicative Zagreb indices.

Lemma 2.2. *For a connected graph G with two nonadjacent vertices $v_1, v_2 \in V(G)$, we have $P_c^a(G) < P_c^a(G + v_1v_2)$ where $c = 1, 2, 3$ and $a > 0$.*

Proof. Let $G' = G + v_1v_2$. For $j = 1, 2$, we have

$$1 \leq d_G(v_j) < d_{G'}(v_j),$$

which means that

$$1 \leq d_G(v_j)^a < d_{G'}(v_j)^a.$$

Thus, $P_1^a(G) < P_1^a(G')$. Similarly,

$$1 \leq d_G(v_j)^{ad_G(v_j)} < d_{G'}(v_j)^{ad_{G'}(v_j)},$$

so $P_2^a(G) < P_2^a(G')$.

Since

$$2 \leq d_G(v_1) + d_G(v_2) < d_{G'}(v_1) + d_{G'}(v_2),$$

we obtain

$$1 < (d_G(v_1) + d_G(v_2))^a < (d_{G'}(v_1) + d_{G'}(v_2))^a \quad \text{and} \quad P_3^a(G) < P_3^a(G').$$

□

Similarly, the following lemma can be proved.

Lemma 2.3. *For a graph G with $e \in E(G)$, we have $P_c^a(G - e) < P_c^a(G)$ where $c = 1, 2, 3$ and $a > 0$.*

Let us study graphs of order n and chromatic number k.

Lemma 2.4. *Let $G \in \chi_{n,k}$ be a graph with the largest P_c^a index for $c = 1, 2, 3$ and $a > 0$. Then G is a complete k-partite graph.*

Proof. Since the chromatic number of G is k, its vertices can be divided into k colour sets. Note that there is no edge between vertices in the same set, so G is a k-partite graph with partite sets of orders $n_1, n_2, \ldots, n_k$. Since G has the largest P_c^a index, by Lemma 2.2, G is $K_{n_1,n_2,\ldots,n_k}$. □

Theorem 2.5 gives an upper bound on general multiplicative Zagreb indices for graphs of order n and chromatic number k.

Theorem 2.5. *Let $G \in \chi_{n,k}$. Then $P_c^a(G) \leq P_c^a(T_{n,k})$ for $c = 1, 2, 3$ and $a > 0$, with equality if and only if G is $T_{n,k}$.*

Proof. By Lemma 2.4, we know that a graph G' with the largest P_c^a index for $c = 1, 2, 3$, is a complete k-partite graph $K_{n_1,n_2,\dots,n_k}$. We prove by contradiction that G' is $T_{n,k}$ (and no other graph). Assume to the contrary that G' is $K_{n_1,n_2,\dots,n_k}$, where $n_i > n_j + 1$ for some $i, j \in \{1, 2, \dots, k\}$. Without loss of generality we can assume that $n_1 \geq n_2 + 2$ (which means that $n_1 - n_2 \geq 2$). Let H be the graph $K_{n_1-1,n_2+1,n_3,\dots,n_k}$. We distinguish three cases.

Case 1. $c = 1$. By Lemma 2.1,

$$\begin{aligned} P_1^a(G') &= (n-n_1)^{an_1}(n-n_2)^{an_2}\prod_{i=3}^{k}(n-n_i)^{an_i} \\ &= [(n-n_1)(n-n_2)]^{an_2}(n-n_1)^{a(n_1-n_2)}\prod_{i=3}^{k}(n-n_i)^{an_i} \end{aligned}$$

and

$$\begin{aligned} P_1^a(H) &= (n-(n_1-1))^{a(n_1-1)}(n-(n_2+1))^{a(n_2+1)}\prod_{i=3}^{k}(n-n_i)^{an_i} \\ &= [(n-n_1+1)(n-n_2-1)]^{an_2} \\ &\quad \cdot (n-n_1+1)^{a(n_1-n_2-1)}(n-n_2-1)^{a}\prod_{i=3}^{k}(n-n_i)^{an_i}. \end{aligned}$$

Since

$$(n-n_1+1)(n-n_2-1) = (n-n_1)(n-n_2) + n_1 - n_2 - 1 > (n-n_1)(n-n_2),$$

we have

$$[(n-n_1+1)(n-n_2-1)]^{an_2} > [(n-n_1)(n-n_2)]^{an_2}.$$

We also know that $(n-n_1+1)^{a(n_1-n_2-1)} > (n-n_1)^{a(n_1-n_2-1)}$ and $(n-n_2-1)^a > (n-n_1)^a$, thus $P_1^a(H) > P_1^a(G')$, a contradiction.

Case 2. $c = 2$. By Lemma 2.1,

$$P_2^a(G') = (n-n_1)^{an_1(n-n_1)}(n-n_2)^{an_2(n-n_2)}\prod_{i=3}^{k}(n-n_i)^{an_i(n-n_i)}$$

and

$$P_2^a(H) = (n-n_1+1)^{a(n_1-1)(n-n_1+1)}(n-n_2-1)^{a(n_2+1)(n-n_2-1)}\prod_{i=3}^{k}(n-n_i)^{an_i(n-n_i)}.$$

Then

$$
\begin{aligned}
\frac{P_2^a(H)}{P_2^a(G')} &= \frac{(n-n_1+1)^{an_1(n-n_1)}(n-n_1+1)^{a(2n_1-n-1)}}{(n-n_1)^{an_1(n-n_1)}} \\
&\quad \frac{(n-n_2-1)^{an_2(n-n_2)}(n-n_2-1)^{a(n-2n_2-1)}}{(n-n_2)^{an_2(n-n_2)}} \\
&\geq \left[\frac{(n-n_1+1)(n-n_2-1)}{(n-n_1)(n-n_2)}\right]^{an_2(n-n_2)} \frac{(n-n_2-1)^{a(n-2n_2-1)}}{(n-n_1+1)^{a(n-2n_1+1)}},
\end{aligned}
$$

since $\frac{n-n_1+1}{n-n_1} > 1$ and $n_1(n-n_1) \geq n_2(n-n_2)$ (note that the equality holds if only if $n = n_1 + n_2$). Since $(n-n_1+1)(n-n_2-1) > (n-n_1)(n-n_2)$, we have

$$
\left[\frac{(n-n_1+1)(n-n_2-1)}{(n-n_1)(n-n_2)}\right]^{an_2(n-n_2)} > 1.
$$

Therefore

$$
\frac{P_2^a(H)}{P_2^a(G')} > \frac{(n-n_2-1)^{a(n-2n_2-1)}}{(n-n_1+1)^{a(n-2n_1+1)}} > 1,
$$

since $n - n_2 - 1 \geq n - n_1 + 1$ and $n - 2n_2 - 1 > n - 2n_1 + 1$. Hence, $P_2^a(H) > P_2^a(G')$, a contradiction.

Case 3. $c = 3$. We have

$$
\begin{aligned}
\frac{P_3^a(H)}{P_3^a(G')} &= \frac{(2n-n_1-n_2)^{a(n_1-1)(n_2+1)} \prod_{1\leq p\leq k, p\neq 1,2}(2n-n_p-n_1+1)^{an_p(n_1-1)}}{(2n-n_1-n_2)^{an_1n_2} \prod_{1\leq p\leq k, p\neq 1,2}(2n-n_p-n_1)^{an_pn_1}} \\
&\quad \frac{\prod_{1\leq p\leq k, p\neq 1,2}(2n-n_p-n_2-1)^{an_p(n_2+1)}}{\prod_{1\leq p\leq k, p\neq 1,2}(2n-n_p-n_2)^{an_pn_2}} \\
&= \frac{(2n-n_1-n_2)^{a(n_1n_2+n_1-n_2-1)}}{(2n-n_1-n_2)^{an_1n_2}} \\
&\quad \prod_{1\leq p\leq k, p\neq 1,2} \frac{(2n-n_p-n_1+1)^{an_p(n_1-1)}}{(2n-n_p-n_1)^{an_pn_1}} \\
&\quad \prod_{1\leq p\leq k, p\neq 1,2} \frac{(2n-n_p-n_2-1)^{an_p(n_2+1)}}{(2n-n_p-n_2)^{an_pn_2}}
\end{aligned}
$$

$$
\begin{aligned}
&> \prod_{1\le p\le k, p\ne 1,2} \frac{(2n-n_p-n_1+1)^{an_p(n_1-1)}}{(2n-n_p-n_1)^{an_pn_1}} \\
&\quad \prod_{1\le p\le k, p\ne 1,2} \frac{(2n-n_p-n_2-1)^{an_p(n_2+1)}}{(2n-n_p-n_2)^{an_pn_2}} \\
&= \prod_{1\le p\le k, p\ne 1,2} \left(1+\frac{1}{2n-n_p-n_1}\right)^{an_pn_1} (2n-n_p-n_1+1)^{-an_p} \\
&\quad \prod_{1\le p\le k, p\ne 1,2} \left(1-\frac{1}{2n-n_p-n_2}\right)^{an_pn_2} (2n-n_p-n_2-1)^{an_p} \\
&> \prod_{1\le p\le k, p\ne 1,2} \left(1+\frac{1}{2n-n_p-n_1}\right)^{an_pn_2} \left(1-\frac{1}{2n-n_p-n_2}\right)^{an_pn_2} \\
&\quad \prod_{1\le p\le k, p\ne 1,2} \left(\frac{2n-n_p-n_2-1}{2n-n_p-n_1+1}\right)^{an_p} \\
&\ge \prod_{1\le p\le k, p\ne 1,2} \left[\left(1+\frac{1}{2n-n_p-n_1}\right)\left(1-\frac{1}{2n-n_p-n_2}\right)\right]^{an_2n_p} > 1,
\end{aligned}
$$

since $n_1 \ge n_2+2$ and $(1+\frac{1}{2n-n_p-n_1})(1-\frac{1}{2n-n_p-n_2}) > 1$. Hence, $P_3^a(H) > P_3^a(G')$, a contradiction. □

In the proof of Theorem 2.7 we use the following result of Erdős [2].

Lemma 2.6. *Let G be a graph having clique number at most k. Then there exists a k-partite graph G'' such that $V(G'') = V(G)$ and $d_G(v) \le d_{G''}(v)$ for every vertex $v \in V(G)$.*

Now we obtain an upper bound on general multiplicative Zagreb indices for graphs of order n and clique number k.

Theorem 2.7. *Let $G \in W_{n,k}$. Then $P_c^a(G) \le P_c^a(T_{n,k})$ for $c = 1,2,3$ and $a > 0$, with equality if and only if G is $T_{n,k}$.*

Proof. Let G be any graph in $W_{n,k}$. By Lemma 2.6, there is a k-partite graph G'' where $V(G'') = V(G)$ and $d_G(v) \le d_{G''}(v)$ for every vertex $v \in V(G)$. Thus $d_G^a(v) \le d_{G''}^a(v)$, $(d_G(v))^{ad_G(v)} \le (d_{G''}(v))^{ad_{G''}(v)}$ and $(d_G(u)+d_G(v))^a < (d_{G''}(u)+d_{G''}(v))^a$, where $u, v \in V(G)$. This implies that $P_c^a(G) \le P_c^a(G'')$ for $c = 1,2,3$. By Lemma 2.2, $P_c^a(G'') \le P_c^a(G')$, where G' is a complete k-partite graph. Since every complete k-partite graph having order n is in $W_{n,k}$ and also in $\chi_{n,k}$, by Theorem 2.5, we obtain $P_c^a(G) \le P_c^a(T_{n,k})$ with equality if and only if G is $T_{n,k}$. □

Let us present a lower bound on the first general multiplicative Zagreb index for graphs having order n and clique number k.

Theorem 2.8. *Let $G \in W_{n,k}$. Then for $a > 0$ we have $P_1^a(G) \ge P_1^a(K_k \star S_{n-k})$ with equality if and only if G is $K_k \star S_{n-k}$.*

Proof. Let G' be a graph in $W_{n,k}$ having the smallest P_1^a index. Since $G' \in W_{n,k}$, we know that G' contains a complete graph K_k as a subgraph. Let us denote the vertices of this complete graph by $v_1, v_2, \ldots, v_k$. By Lemma 2.3, G' is a graph obtained from K_k by attaching trees to some vertices of K_k.

Claim 1. If G' contains a tree T attached to v_i $(1 \le i \le k)$, then T is a star.

Assume that T is not a star attached to v_i by its centre. Then T contains a vertex u $(u \neq v_i)$ with degree at least 2 adjacent to some pendent vertices. Let u' be a vertex of T with degree at least 2 which is furthest from v_i. Let $u_1, u_2, \ldots, u_p$ (where $p \ge 1$) be pendent vertices adjacent to u'. Let $G'' = G' - \{u'u_1, u'u_2, \ldots, u'u_p\} + \{v_iu_1, v_iu_2, \ldots, v_iu_p\}$. Then $d_{G'}(u') = p+1$, $d_{G''}(u') = 1$, $d_{G'}(v_i) = k_1$ and $d_{G''}(v_i) = k_1 + p$ where $k_1 \ge k$. Consequently,

$$P_1^a(G') - P_1^a(G'') = k_1^a(p+1)^a - (k_1+p)^a.$$

Clearly $k_1(p+1) = k_1p + k_1 > k_1 + p$, so $[k_1(p+1)]^a > (k_1+p)^a$ and $P_1^a(G') - P_1^a(G'') > 0$. Since $P_1^a(G') > P_1^a(G'')$, we have a contradiction. Hence, T is a star.

Claim 2. Only one vertex v_i $(1 \le i \le k)$ is adjacent to pendent vertices.

Assume that G' contains (at least) two different vertices v_i, v_j adjacent to pendent vertices. Let s be the number of pendent vertices adjacent to v_i and let t be the number of pendent vertices adjacent to v_j. We can denote the pendent vertices adjacent to v_j by $w_1, w_2, \ldots, w_t$. Let $G'' = G' - \{v_jw_1, v_jw_2, \ldots, v_jw_t\} + \{v_iw_1, v_iw_2, \ldots, v_iw_t\}$. Let $k_1 = k-1$. Then $d_{G'}(v_i) = k_1 + s$, $d_{G''}(v_i) = k_1 + s + t$, $d_{G'}(v_j) = k_1 + t$, $d_{G''}(v_i) = k_1$. Thus

$$P_1^a(G') - P_1^a(G'') = (k_1+s)^a(k_1+t)^a - (k_1+s+t)^a k_1^a.$$

Since $(k_1+s)(k_1+t) > (k_1+s+t)k_1$, we get $[(k_1+s)(k_1+t)]^a > [(k_1+s+t)k_1]^a$ and consequently $P_1^a(G') - P_1^a(G'') > 0$. Thus $P_1^a(G') > P_1^a(G'')$, which is a contradiction.

From Claims 1 and 2 it follows that $P_1^a(G) \ge P_1^a(K_k \star S_{n-k})$ with equality if and only if G is $K_k \star S_{n-k}$. □

Lower bounds on the second and third general multiplicative Zagreb indices for graphs of order n and clique number k are given in Theorems 2.9 and 2.10.

Theorem 2.9. *Let $G \in W_{n,k}$. Then for $a > 0$ we have $P_2^a(G) \ge P_2^a(K_k * P_{n-k})$ with equality if and only if G is $K_k * P_{n-k}$.*

Proof. Let G' be a graph in $W_{n,k}$ having the smallest P_2^a index. Since $G' \in W_{n,k}$, we know that G' contains a complete graph K_k as a subgraph. Let us denote the vertices of this complete graph by $v_1, v_2, \ldots, v_k$. By Lemma 2.3, G' is a graph obtained from K_k by attaching trees to some vertices of K_k.

Claim 1. If G' contains a tree T attached to v_i $(1 \le i \le k)$, then T is a path.

Assume that T is not a path attached to v_i by its end vertex. Then T contains (at least) 2 pendant vertices, say u, w. Let x be the closest vertex to u having degree at least 3 in G' and let x_1 be a vertex adjacent to x which belongs to the path connecting u and x (possibly $x_1 = u$). Let $G'' = G' - \{xx_1\} + \{wx_1\}$. We have $d_{G'}(x) = p \ge 3$,

$d_{G''}(x) = p-1$, $d_{G'}(w) = 1$ and $d_{G''}(w) = 2$. For any other vertex v, we have $d_{G'}(v) = d_{G''}(v)$. Thus,

$$\frac{P_2^a(G')}{P_2^a(G'')} = \frac{p^{ap}}{4^a(p-1)^{a(p-1)}} = \left[\frac{p}{4}\left(1+\frac{1}{p-1}\right)^{p-1}\right]^a > 1,$$

since $\frac{p}{4} \geq \frac{3}{4}$ and $(1+\frac{1}{p-1})^{p-1} \geq \frac{9}{4}$. We have $P_2^a(G') > P_2^a(G'')$, a contradiction.

Note that $d_{G'}(v_i) = k-1$ or k, where $1 \leq i \leq k$. Let

$$V' = \{v_i : 1 \leq i \leq k, d_{G'}(v_i) > k-1\}.$$

Claim 2. $|V'| = 1$.

Assume that $|V'| \geq 2$. Let $v_i, v_j \in V'$. Let $P_1 = v_i u_1 u_2 \ldots u_s$ and $P_2 = v_j w_1 w_2 \ldots w_t$ (where $s, t \geq 1$) be the paths attached to v_i and v_j, respectively. Let $G'' = G' - \{v_j w_1\} + \{u_s w_1\}$. If $d_{G'}(v_j) = 2$, then G' and G'' are paths, so we can assume that $d_{G'}(v_j) = k \geq 3$. Then $d_{G''}(v_j) = k-1$, $d_{G'}(u_s) = 1$ and $d_{G''}(u_s) = 2$. For any other vertex v, we have $d_{G'}(v) = d_{G''}(v)$. Thus

$$\frac{P_2^a(G')}{P_2^a(G'')} = \frac{k^{ak}}{4^a(k-1)^{a(k-1)}} > 1,$$

as in Claim 1. We have $P_2^a(G') > P_2^a(G'')$, a contradiction.

It follows that G' is $K_k * P_{n-k}$ and $P_2^a(G) \geq P_2^a(K_k * P_{n-k})$ with equality if and only if G is $K_k * P_{n-k}$. □

Theorem 2.10. *Let $G \in W_{n,k}$. Then for $a > 0$ we have $P_3^a(G) \geq P_3^a(K_k * P_{n-k})$ with equality if and only if G is $K_k * P_{n-k}$.*

Proof. Let G' be a graph in $W_{n,k}$ having the smallest P_3^a index. Since $G' \in W_{n,k}$, we know that G' contains a complete graph K_k as a subgraph. Let us denote the vertices of this complete graph by $v_1, v_2, \ldots, v_k$. By Lemma 2.3, G' is a graph obtained from K_k by attaching trees to some vertices of K_k.

Claim 1. If G' contains a tree T attached to v_i $(1 \leq i \leq k)$, then T is a path.

Assume that T is not a path attached to v_i by its end vertex. Let $v \in T$ be a vertex of degree $x \geq 3$ furthest from v_i (possibly $v = v_i$). Let u and w be two pendant vertices of T such that v is on the $u - v_i$ path and on the $w - v_i$ path. Let $vu_1u_2 \ldots u_{s-1}u$ be a path connecting v and u, and let $vw_1w_2 \ldots w_{t-1}w$ be a path connecting v and w.

For $s, t \geq 2$, let $G'' = G' - \{vw_1\} + \{uw_1\}$. We have $d_{G'}(v) = x \geq 3$, $d_{G''}(v) = x-1$, $d_{G'}(u) = 1$, $d_{G''}(u) = 2$ and for all the other vertices x we have $d_{G'}(x) = d_{G''}(x)$. We consider the edges $vw_1 \in E(G')$, $uw_1 \in E(G'')$ and the edges $vu_1, u_{s-1}u$ in G' and G'' and obtain

$$\frac{P_3^a(G')}{P_3^a(G'')} \geq \frac{3^a(x+2)^a(x+2)^a}{16^a(x+1)^a}.$$

We did not consider some edges of G' and G'' adjacent to v, because we know that the contribution of these edges in $P_3^a(G')$ is greater than their contribution in $P_3^a(G'')$ (since $d_{G'}(v) = x$ and $d_{G''}(v) = x - 1$).

$$\frac{3^a(x+2)^a(x+2)^a}{16^a(x+1)^a} = \left[\frac{3(x+1)(x+3)+3}{16(x+1)}\right]^a > \left[\frac{3(x+3)}{16}\right]^a > 1$$

for $x \geq 3$. Thus, $P_3^a(G') > P_3^a(G'')$, a contradiction.

The cases $s \geq 1, t = 1$ and $t \geq 1, s = 1$ can be solved using the same method, so without loss of generality we can assume that $s \geq 1, t = 1$. Let $G'' = G' - \{vw\} + \{uw\}$. We have $d_{G'}(v) = x \geq 3$, $d_{G''}(v) = x - 1$, $d_{G'}(u) = 1$, $d_{G''}(u) = 2$. If $s \geq 2$, we consider the edges $vw \in E(G')$, $uw \in E(G'')$ and $vu_1, u_{s-1}u$ in G' and G'' and we obtain

$$\frac{P_3^a(G')}{P_3^a(G'')} \geq \frac{3^a(x+1)^a(x+2)^a}{12^a(x+1)^a} = \left(\frac{x+2}{4}\right)^a > 1$$

for $x \geq 3$. If $s = 1$, we consider the edges $vw \in E(G')$, $uw \in E(G'')$ and vu in G' and G'' and get

$$\frac{P_3^a(G')}{P_3^a(G'')} \geq \frac{(x+1)^a(x+1)^a}{3^a(x+1)^a} = \left(\frac{x+1}{3}\right)^a > 1$$

for $x \geq 3$. So, again we obtain $P_3^a(G') > P_3^a(G'')$, a contradiction.

Note that $d_{G'}(v_i) = k - 1$ or k, where $1 \leq i \leq k$. Let

$$V' = \{v_i : 1 \leq i \leq k, d_{G'}(v_i) > k - 1\}.$$

Claim 2. $|V'| = 1$.

Assume that $|V'| \geq 2$. Let $v_i, v_j \in V'$. Let $P_1 = v_i u_1 u_2 \dots u_s$ and $P_2 = v_j w_1 w_2 \dots w_t$ (where $s, t \geq 1$) be the paths attached to v_i and v_j, respectively. Let $G'' = G' - \{v_j w_1\} + \{u_s w_1\}$. If $d_{G'}(v_j) = 2$, then G' and G'' are paths, so we can assume that $d_{G'}(v_j) = k \geq 3$. Then $d_{G''}(v_j) = k - 1$, $d_{G'}(u_s) = 1$ and $d_{G''}(u_s) = 2$. For any other vertex v, we have $d_{G'}(v) = d_{G''}(v)$.

If $s, t \geq 2$, we consider the edges $v_j w_1 \in E(G')$, $u_s w_1 \in E(G'')$ and the edges $u_s u_{s-1}, v_i v_j$ in G' and G'' and obtain

$$\begin{aligned}\frac{P_3^a(G')}{P_3^a(G'')} &\geq \frac{3^a(k+2)^a(2k)^a}{16^a(2k-1)^a} = \left[\frac{3(2k-1)(2k+5)}{32(2k-1)} + \frac{15}{32(2k-1)}\right]^a \\ &> \left[\frac{3(2k+5)}{32}\right]^a > 1\end{aligned}$$

for $k \geq 3$. (We did not consider some edges of G' and G'' adjacent to v_j, because we know that the contribution of these edges in $P_3^a(G')$ is greater than their contribution in $P_3^a(G'')$.) Thus $P_3^a(G') > P_3^a(G'')$, a contradiction.

The cases $s \geq 1, t = 1$ and $t \geq 1, s = 1$ can be solved using the same technique, thus without loss of generality we can assume that $s \geq 1, t = 1$. We consider the edges $v_j w_1 \in E(G')$, $u_s w_1 \in E(G'')$ and $u_s u_{s-1}, v_i v_j$ in G' and G'' and get

$$\frac{P_3^a(G')}{P_3^a(G'')} \geq \frac{3^a(k+1)^a(2k)^a}{12^a(2k-1)^a} = \left[\frac{(k+1)2k}{4(2k-1)}\right]^a > 1$$

since $k+1 \geq 4$ and $2k > 2k-1$ for $k \geq 3$.

If $s = 1$ and $t = 1$, we consider the edges $v_j w_1 \in E(G')$, $u_1 w_1 \in E(G'')$ and $v_i u_1$ in G' and G'' and get

$$\frac{P_3^a(G')}{P_3^a(G'')} \geq \frac{(k+1)^a(k+1)^a}{3^a(k+2)^a} > 1$$

for $k \geq 3$. Thus $P_3^a(G') > P_3^a(G'')$, a contradiction.

It follows that G' is $K_k * P_{n-k}$ and $P_3^a(G) \geq P_3^a(K_k * P_{n-k})$ with equality if and only if G is $K_k * P_{n-k}$. □

3. CONCLUSION

if $a = 2$, then $P_1^2(G)$ is the first multiplicative Zagreb index $\prod_1(G)$ and from Theorems 2.7 and 2.8 we get the following corollary.

Corollary 3.1. *Let $G \in W_{n,k}$. Then $\prod_1(K_k \star S_{n-k}) \leq \prod_1(G) \leq \prod_1(T_{n,k})$ with equalities if and only if G is $K_k \star S_{n-k}$ and $T_{n,k}$, respectively.*

For $a = 1$, $P_2^1(G)$ is the second multiplicative Zagreb index $\prod_2(G)$. Theorems 2.7 and 2.9 yield Corollary 3.2.

Corollary 3.2. *Let $G \in W_{n,k}$. Then $\prod_2(K_k * P_{n-k}) \leq \prod_2(G) \leq \prod_2(T_{n,k})$ with equalities if and only if G is $K_k * P_{n-k}$ and $T_{n,k}$, respectively.*

For $a = 1$, $P_3^1(G)$ is the multiplicative sum Zagreb index $\prod_1^*(G)$. By Theorems 2.7 and 2.10 we get the following corollary.

Corollary 3.3. *Let $G \in W_{n,k}$. Then $\prod_1^*(K_k * P_{n-k}) \leq \prod_1^*(G) \leq \prod_1^*(T_{n,k})$ with equalities if and only if G is $K_k * P_{n-k}$ and $T_{n,k}$, respectively.*

In the proofs of Lemma 2.2, Theorems 2.5 (Case 1), 2.7 and 2.8, we use the following: if $1 \leq z_1 < z_2$ and $a > 0$, then $1 \leq z_1^a < z_2^a$. In the proof of Theorem 2.5 (Cases 2 and 3), z^a represents $\frac{P_c^a(H)}{P_c^a(G')}$. Since $z > 1$ and $a > 0$, we have $z^a > 1$. In the proofs of Theorems 2.9 and 2.10, $z^a > 1$ represents $\frac{P_c^a(G')}{P_c^a(G'')} > 1$, where $z > 1$ and $a > 0$.

Results about general multiplicative Zagreb indices for $a < 0$ can be obtained by use of similar proofs and these facts:

- if $z > 1$ and $a < 0$, then $0 < z^a < 1$,
- if $1 \leq z_1 < z_2$ and $a < 0$, then $0 < z_2^a < z_1^a \leq 1$.

Lemma 3.4. *For a connected graph G with two nonadjacent vertices $v_1, v_2 \in V(G)$, we have $P_c^a(G) > P_c^a(G + v_1v_2)$ where $c = 1, 2, 3$ and $a < 0$.*

Lemma 3.5. *For a graph G with $e \in E(G)$, we have $P_c^a(G - e) > P_c^a(G)$ where $c = 1, 2, 3$ and $a < 0$.*

Lemma 3.6. *Let $G \in \chi_{n,k}$ be a graph with the smallest P_c^a index for $c = 1, 2, 3$ and $a < 0$. Then G is a complete k-partite graph.*

A lower bound on general multiplicative Zagreb indices for graphs of order n and chromatic number k, where $a < 0$, is given in Theorem 3.7.

Theorem 3.7. *Let $G \in \chi_{n,k}$. Then $P_c^a(G) \geq P_c^a(T_{n,k})$ for $c = 1, 2, 3$ and $a < 0$, with equality if and only if G is $T_{n,k}$.*

Let us state lower and upper bounds on general multiplicative Zagreb indices for graphs having order n and clique number k, where $a < 0$,

Theorem 3.8. *Let $G \in W_{n,k}$. Then $P_c^a(G) \geq P_c^a(T_{n,k})$ for $c = 1, 2, 3$ and $a < 0$, with equality if and only if G is $T_{n,k}$.*

Theorem 3.9. *Let $G \in W_{n,k}$. Then for $a < 0$ we have $P_1^a(G) \leq P_1^a(K_k \star S_{n-k})$ with equality if and only if G is $K_k \star S_{n-k}$.*

Theorem 3.10. *Let $G \in W_{n,k}$. Then for $c = 2, 3$ and $a < 0$ we have $P_c^a(G) \leq P_c^a(K_k * P_{n-k})$ with equality if and only if G is $K_k * P_{n-k}$.*

Finally, we compute values of general multiplicative Zagreb indices for extremal graphs. For $2 \leq k < n$ we have $n = k\lfloor \frac{n}{k} \rfloor + r$ where $0 \leq r < k$, thus we can assume that $T_{n,k}$ has $k - r$ partite sets of order $\lfloor \frac{n}{k} \rfloor$ and r partite sets of order $\lceil \frac{n}{k} \rceil$.

Since $T_{n,k}$ contains $(k - r)\lfloor \frac{n}{k} \rfloor$ vertices of degree $n - \lfloor \frac{n}{k} \rfloor$ and $r\lceil \frac{n}{k} \rceil$ vertices of degree $n - \lceil \frac{n}{k} \rceil$, we obtain

$$P_1^a(T_{n,k}) = \left(n - \left\lfloor \frac{n}{k} \right\rfloor\right)^{a(k-r)\lfloor \frac{n}{k} \rfloor} \left(n - \left\lceil \frac{n}{k} \right\rceil\right)^{ar\lceil \frac{n}{k} \rceil}$$

and

$$P_2^a(T_{n,k}) = \left(n - \left\lfloor \frac{n}{k} \right\rfloor\right)^{a(k-r)\lfloor \frac{n}{k} \rfloor (n - \lfloor \frac{n}{k} \rfloor)} \left(n - \left\lceil \frac{n}{k} \right\rceil\right)^{ar\lceil \frac{n}{k} \rceil (n - \lceil \frac{n}{k} \rceil)}.$$

Moreover, $T_{n,k}$ contains

(a) $(k - r)\lfloor \frac{n}{k} \rfloor \cdot r\lceil \frac{n}{k} \rceil$ edges with one end vertex of degree $n - \lfloor \frac{n}{k} \rfloor$ and the other end vertex of degree $n - \lceil \frac{n}{k} \rceil$,

(b) $\binom{k-r}{2}\lfloor \frac{n}{k} \rfloor^2$ edges with both end vertices having degree $n - \lfloor \frac{n}{k} \rfloor$,

(c) $\binom{r}{2}\lceil \frac{n}{k} \rceil^2$ edges with both end vertices having degree $n - \lceil \frac{n}{k} \rceil$.

Thus, we obtain

$$\begin{aligned} P_3^a(T_{n,k}) = &\left(2n - \left\lfloor \frac{n}{k} \right\rfloor - \left\lceil \frac{n}{k} \right\rceil\right)^{ar(k-r)\lfloor \frac{n}{k} \rfloor \lceil \frac{n}{k} \rceil} \left(2n - 2\left\lfloor \frac{n}{k} \right\rfloor\right)^{a\binom{k-r}{2}\lfloor \frac{n}{k} \rfloor^2} \\ &\cdot \left(2n - 2\left\lceil \frac{n}{k} \right\rceil\right)^{a\binom{r}{2}\lceil \frac{n}{k} \rceil^2}. \end{aligned}$$

The graph $K_k \star S_{n-k}$ contains $k - 1$ vertices having degree $k - 1$, one vertex having degree $n - 1$ and $n - k$ vertices having degree 1, therefore we obtain

$$P_1^a(K_k \star S_{n-k}) = (n - 1)^a (k - 1)^{a(k-1)}.$$

Since $K_k * P_{n-k}$ contains $k-1$ vertices having degree $k-1$, one vertex having degree k, one vertex having degree one and $n-k-1$ vertices having degree 2, we obtain

$$P_2^a(K_k * P_{n-k}) = k^{ak}(k-1)^{a(k-1)^2} 2^{2a(n-k-1)}.$$

The graph $K_k * P_{n-k}$ contains $\binom{k-1}{2}$ edges with both end vertices having degree $k-1$ and $k-1$ edges with one end vertex having degree k and the other end vertex having degree $k-1$.

If $n = k+1$, then $K_k * P_{n-k}$ contains one edge with one end vertex having degree k and the other end vertex having degree one.

If $n \geq k+2$, then $K_k * P_{n-k}$ contains one edge with one end vertex having degree k and the other end vertex having degree 2, one edge containing one end vertex having degree 2 and the other end vertex having degree one and $n-k-2$ edges containing both end vertices having degree 2.

Therefore, if $n = k+1$, then

$$P_3^a(K_k * P_{n-k}) = \left[(k+1)(2k-2)^{\binom{k-1}{2}}(2k-1)^{k-1}\right]^a,$$

and if $n \geq k+2$, then

$$P_3^a(K_k * P_{n-k}) = \left[3(k+2)4^{n-k-2}(2k-2)^{\binom{k-1}{2}}(2k-1)^{k-1}\right]^a.$$

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