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ON k-SUMMABILITY OF FORMAL SOLUTIONS FOR CERTAIN PARTIAL DIFFERENTIAL OPERATORS WITH POLYNOMIAL COEFFICIENTS

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Abstract. We study the k-summability of divergent formal solutions for the Cauchy problem of certain linear partial differential operators with coefficients which are polynomial in t. We employ the method of successive approximation in order to construct the formal solutions and to obtain the properties of analytic continuation of the solutions of convolution equations and their exponential growth estimates.

Keywords: k-summability, Cauchy problem, power series solutions, successive approximation.

Mathematics Subject Classification: 35C10, 35C20, 35G10.

1. RESULT

Let a linear partial differential operator with t-dependent polynomial coefficients L be given by

$$L = \partial_t - P(t, \partial_x), \quad P(t, \partial_x) = \sum_{\alpha, i \in \mathbb{N}_0}^{\text{finite}} a_{\alpha i} t^i \partial_x^\alpha, \tag{1.1}$$

where $(t, x) \in \mathbb{C}^2$, $(\partial_t, \partial_x) = (\partial/\partial t, \partial/\partial x)$, $a_{\alpha i} \in \mathbb{C}$ and $\mathbb{N}_0$ denotes the set of non negative integers.

We consider the following Cauchy problem for L

$$\begin{cases} LU(t,x) \equiv (\partial_t - P(t, \partial_x))\, U(t,x) = 0, \\ U(0,x) = \varphi(x) \in \mathcal{O}_x, \end{cases} \tag{1.2}$$

where $\mathcal{O}_x$ denotes the set of holomorphic functions in a neighborhood of the origin $x = 0$. The Cauchy problem (1.2) has a unique formal solution of the form

$$\hat{U}(t,x) = \sum_{n=0}^{\infty} U_n(x)\frac{t^n}{n!}, \quad U_0(x) = \varphi(x). \tag{1.3}$$

We assume that for the operator $P = P(t, \partial_x)$

$$\max\{\alpha; a_{\alpha i} \neq 0\} \geq 2, \tag{A-1}$$

which is called non-Kowalevskian condition. In this case, the formal solution is divergent in general.

Our purpose in this paper is to study the k-summability of this divergent solution under some conditions for L. In order to explain the conditions we define the Newton polygon of L.

We define a domain $\mathrm{N}(\alpha, i)$ by

$$\mathrm{N}(\alpha,i) := \{(x,y) \in \mathbb{R}^2;\ x \leq \alpha,\ y \geq i\} \quad \text{for } a_{\alpha i} \neq 0,$$

and $\mathrm{N}(\alpha, i) := \phi$ for $a_{\alpha i} = 0$. Then the Newton polygon $\mathrm{N}(L)$ is defined by

$$\mathrm{N}(L) := \mathrm{Ch}\left\{\mathrm{N}(1,-1) \bigcup_{\alpha,i\in\mathbb{N}_0} \mathrm{N}(\alpha,i)\right\}, \tag{1.4}$$

where $\mathrm{Ch}\{\dots\}$ denotes the convex hull of points in $\mathrm{N}(1,-1) \cup_{\alpha,i} \mathrm{N}(\alpha,i)$. Here $\mathrm{N}(1,-1) := \{(x,y)\,;\, x \leq 1,\ y \geq -1\}$. We assume that

the Newton polygon $\mathrm{N}(L)$ has only one side of a positive slope with two end points $(1,-1)$ and (α_*, i_*), where $\alpha_* > 1$, $i_* \geq 0$ (see Figure 1). (A-2)

In this case, we notice $\alpha_* = \max\{\alpha;\, a_{\alpha i} \neq 0\} (\geq 2)$.

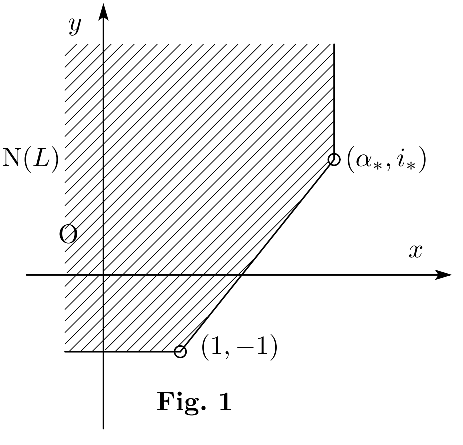

Fig. 1

We put

$$\frac{\alpha_*}{i_*+1} =: \frac{p}{q}, \quad (p,q)=1. \tag{1.5}$$

Then we assume

$$p = q = 1, \tag{A-3}$$

which means that $\alpha_* = i_* + 1$. Moreover, we assume that the indices (α, i) of the operator P satisfy the following inequalities.

$$i \leq i_* \quad \text{for } a_{\alpha i} \neq 0, \tag{A-4}$$

$$\frac{\alpha_*}{i_*+1} \geq \frac{\alpha}{i+1}. \tag{A-5}$$

We call this number $\alpha_*/(i_*+1)$ the modified order of L (cf. [6, 10, 14]).

Finally, we prepare the notation $S(d,\beta,\rho)$. For $d \in \mathbb{R}$, $\beta > 0$ and $\rho\,(0 < \rho \leq \infty)$, we define a sector $S = S(d,\beta,\rho)$ by

$$S(d,\beta,\rho) := \left\{ t \in \mathbb{C};\, |d - \arg t| < \frac{\beta}{2}, 0 < |t| < \rho \right\}, \tag{1.6}$$

where d, β and ρ are called the direction, the opening angle and the radius of S, respectively. We write $S(d,\beta,\infty) = S(d,\beta)$ for short.

Under the above preparations, our result is stated as follows.

Theorem 1.1. *We suppose the assumptions* (A-1)–(A-5). *Let $d \in \mathbb{R}$ be fixed and $d_j = d + (\arg a_{\alpha_* i_*} + 2\pi j)/\alpha_*$ for $j = 0, 1, \ldots, \alpha_* - 1$. Let*

$$\kappa = \frac{i_*+1}{\alpha_*-1}. \tag{1.7}$$

We assume that the Cauchy data $\varphi(x) \in \mathcal{O}_x$ can be analytically continued in the region $\bigcup_{j=0}^{\alpha_*-1} S(d_j,\varepsilon)$ for some positive ε, and has the following exponential growth estimate

$$|\varphi(x)| \leq C \exp\left(\delta |x|^{\frac{\alpha_*}{\alpha_*-1}}\right), \quad x \in \bigcup_{j=0}^{\alpha_*-1} S(d_j,\varepsilon), \tag{1.8}$$

for some positive constants C and δ. Then the divergent solution $\hat{U}(t,x)$ of the Cauchy problem (1.2) *is κ-summable in d direction.*

In appendix A, we shall characterize the exponential growth order of the entire function Cauchy data which guarantees the convergence in t-variable of the formal solutions. The obtained result is just corresponding to that in the condition (1.8).

On the k-summability of divergent solutions for non-Kowalevskian equations like heat equation, there are many studies for partial differential operators with constant coefficients (e.g. [7] for the heat equation, [2] for general equations and its references). But the study for equations with variable coefficients has not been developed yet. In the papers [3] and [4], the first author treated the equations with coefficients which are monomial in t. In this paper, we consider the equations with coefficients which are polynomial in t. Our theorem is a generalization of the result in [3].

We shall give the proof of Theorem 1.1 by using the method of successive approximation. This paper consists of the following construction. We give a review of k-summability in Section 2. In Section 3, we construct the formal solution of the original Cauchy problem. For the purpose, we give a decomposition of the operator P and construct exact formal successive approximation solutions which are based upon the decomposition of P. In Section 4, we give a result of Gevrey order of formal solutions and we define formal series associated with formal solutions of the Cauchy problems. In Section 5, we give the property of k-summability for the associated formal series. After that, we give a proof of Theorem 1.1 in Section 6 by using the property of formal successive approximation solutions. In Sections 7 and 8, we give proofs of the properties of summability given in Sections 5 and 6. Finally, we give the characterization of a class of Cauchy data for the convergence of formal solutions in Appendix A and the proof of Lemma 4.3 in Appendix B.

We remark that the summability and convergence theory of the formal solutions in this paper are written by the first author and the second author, respectively.

2. REVIEW OF k-SUMMABILITY

In this section, we give some notation and definitions in the way of Ramis or Balser (cf. W. Balser [1] for details).

Let $k > 0$, $S = S(d, \beta)$ and $B(\sigma) := \{x \in \mathbb{C}; |x| \le \sigma\}$. Let $v(t,x) \in \mathcal{O}(S \times B(\sigma))$ which means that $v(t,x)$ is holomorphic in $S \times B(\sigma)$. Then we define that $v(t,x) \in \mathrm{Exp}_t(k, S \times B(\sigma))$ if, for any closed subsector S' of S, there exist some positive constants C and δ such that

$$\max_{|x|\le\sigma} |v(t,x)| \le Ce^{\delta|t|^k}, \quad t \in S'. \tag{2.1}$$

For $k > 0$, we define that $\hat{v}(t,x) = \sum_{n=0}^{\infty} v_n(x)t^n \in \mathcal{O}_x[[t]]_{1/k}$ (we say $\hat{v}(t,x)$ is a formal power series of Gevrey order $1/k$) if $v_n(x)$ are holomorphic on a common closed disk $B(\sigma)$ for some $\sigma > 0$ and there exist some positive constants C and K such that for any n,

$$\max_{|x|\le\sigma} |v_n(x)| \le CK^n\Gamma\left(1 + \frac{n}{k}\right). \tag{2.2}$$

Here when $v_n(x) \equiv v_n$ (constants) for all n, we use the notation $\mathbb{C}[[t]]_{1/k}$ instead of $\mathcal{O}_x[[t]]_{1/k}$.

Let $k > 0$, $\hat{v}(t,x) = \sum_{n=0}^{\infty} v_n(x)t^n \in \mathcal{O}_x[[t]]_{1/k}$ and $v(t,x)$ be an analytic function on $S(d,\beta,\rho) \times B(\sigma)$. Then we define that

$$v(t,x) \cong_k \hat{v}(t,x) \quad \text{in } S = S(d,\beta,\rho), \tag{2.3}$$

if for any closed subsector S' of S, there exist some positive constants C and K such that for any $N \geq 1$, we have

$$\max_{|x|\leq\sigma}\left|v(t,x)-\sum_{n=0}^{N-1}v_n(x)t^n\right|\leq CK^N|t|^N\Gamma\left(1+\frac{N}{k}\right),\quad t\in S'. \tag{2.4}$$

For $k>0$, $d\in\mathbb{R}$ and $\hat{v}(t,x)\in\mathcal{O}_x[[t]]_{1/k}$, we say that $\hat{v}(t,x)$ is *k-summable* in d direction, and denote it by $\hat{v}(t,x)\in\mathcal{O}_x\{t\}_{k,d}$, if there exist a sector $S=S(d,\beta,\rho)$ with $\beta>\pi/k$ and an analytic function $v(t,x)$ on $S\times B(\sigma)$ such that $v(t,x)\cong_k\hat{v}(t,x)$ in S.

We remark that the function $v(t,x)$ above for a *k-summable* $\hat{v}(t,x)$ is unique if it exists. Therefore such a function $v(t,x)$ is called the *k-sum* of $\hat{v}(t,x)$ in d direction.

3. CONSTRUCTION OF FORMAL SOLUTION

3.1. DECOMPOSITION OF OPERATOR $P(t,\partial_x)$

In this subsection, we give a decomposition of the operator P.

For $j\geq 0$, we define

$$K_j=\{(\alpha,i);\ j=i+1-\alpha,\ a_{\alpha i}\neq 0\},$$

and we put

$$P_j(t,\partial_x):=\sum_{(\alpha,i)\in K_j}a_{\alpha i}t^i\partial_x^\alpha.$$

Especially, we obtain

$$K_0=\{(\alpha,i);0\leq i\leq i_*,\ \alpha=i+1\},\quad K_j=\{(\alpha,i);j-1\leq i\leq i_*,\ \alpha=i+1-j\}.$$

Therefore, we obtain

$$P(t,\partial_x)=\sum_{j=0}^{i_*+1}P_j(t,\partial_x).$$

3.2. THE SEQUENCE OF CAUCHY PROBLEMS

By employing the decomposition of the operator P given in previous subsection, we consider the following sequence of Cauchy problems.

$$\begin{cases}\partial_t u_0(t,x)=P_0(t,\partial_x)u_0(t,x),\\ u_0(0,x)=\varphi(x).\end{cases} \tag{E_0}$$

For $k \geq 1$, define

$$\begin{cases} \partial_t u_k(t,x) = P_0(t,\partial_x)u_k(t,x) + \displaystyle\sum_{j=1}^{\min\{i_*+1,k\}} P_j(t,\partial_x)u_{k-j}(t,x), \\ u_k(0,x) = 0. \end{cases} \tag{E_k}$$

For each k, the Cauchy problem (E_k) has a unique formal power series solution of the form

$$\hat{u}_k(t,x) = \sum_{n\geq 0} u_{k,n}(x)\frac{t^n}{n!}. \tag{Sol_k}$$

Then $\hat{U}(t,x) = \sum_{k\geq 0} \hat{u}_k(t,x)$ is the formal power series solution of the original Cauchy problem (1.2).

3.3. CONSTRUCTION OF FORMAL SOLUTIONS $\hat{u}_k(t,x)$

In this subsection, we construct the formal solutions $\hat{u}_k(t,x)$ of the Cauchy problems (E_k).

Lemma 3.1. *Let $k \geq 0$. For each k, the formal solution $\hat{u}_k(t,x)$ of the Cauchy problem (E_k) is given by*

$$\hat{u}_k(t,x) = \sum_{n\geq 0} u_{k,n}(x)\frac{t^n}{n!} = \sum_{n\geq 0} A_k(n)\varphi^{(n-k)}(x)\frac{t^n}{n!}, \tag{3.1}$$

where $\{A_k(n)\}$ satisfy the following recurrence formula:
When $k = 0$, one has

$$\begin{cases} A_0(n+1) = \displaystyle\sum_{K_0} a_{\alpha i}[n]_i A_0(n-i) \quad (n \geq 0), \\ A_0(0) = 1, \end{cases} \tag{R_0}$$

where we interpret as $A_k(n) = 0$ for all k if $n < 0$. Here the notation $[n]_i$ is defined by

$$[n]_i := \begin{cases} n(n-1)(n-2)\dots(n-i+1), & i \geq 1, \\ 1, & i = 0. \end{cases}$$

When $k \geq 1$, one has

$$\begin{cases} A_k(n+1) = \displaystyle\sum_{K_0} a_{\alpha i}[n]_i A_k(n-i) + \sum_{j=1}^{\min\{i_*+1,k\}} \sum_{K_j} a_{\alpha i}[n]_i A_{k-j}(n-i) \ (n \geq 0), \\ A_k(0) = 0. \end{cases} \tag{R_k}$$

By substituting (Sol_k) into the equation (E_k), we can see that $u_{k,n}(x) = A_k(n)\varphi^{(n-k)}(x)$, where $\{A_k(n)\}$ satisfy the recurrence formulas (R_k). We omit the details.

4. GEVREY ORDER OF FORMAL SOLUTIONS $\hat{u}_k$

We give a result of Gevrey order of formal solutions $\hat{u}_k(t,x)$ of (E_k).

Lemma 4.1. *Let $k \geq 0$ and let $\hat{u}_k(t,x)$ be the formal solutions of the Cauchy problem (E_k). Then we have $\hat{u}_k(t,x) \in \mathcal{O}_x[[t]]_{1/\kappa}$, $\kappa = (i_* + 1)/(\alpha_* - 1)$.*

This proposition was already proved by many mathematicians (for determination of Gevrey order from the Newton polygon of the differential equations, see [13, 16]). We omit the proof.

We introduce formal series associated with formal solutions $\hat{u}_k$.

We define

$$\hat{f}_k(t) := \sum_{n \geq 0} A_k(n) t^n, \tag{4.1}$$

which are the generating function of $\{A_k(n)\}$ for all k. Then we obtain the following result of Gevrey order of $\hat{f}_k$.

Lemma 4.2. *We have $\hat{f}_k(t) \in \mathbb{C}[[t]]_{1/\kappa}$, where*

$$\kappa = (i_* + 1)/(\alpha_* - 1) = (i_* + 1)/i_*. \tag{4.2}$$

Especially, we obtain the following lemma.

Lemma 4.3. *There exist positive constants A and B such that for all n, we have*

$$|A_k(n)| \leq AB^{n+k} n!^{\frac{1}{\kappa}}. \tag{4.3}$$

Lemma 4.2 follows from Lemma 4.3 or the fact that $\hat{f}_k(t)$ satisfy the following ordinary differential equations.

When $k = 0$, one has

$$\left(\sum_{K_0} a_{\alpha i} t^{i+1} [\delta_t + i]_i - 1 \right) \hat{f}_0(t) = -1, \tag{4.4}$$

where $\delta_t = td/(dt)$ denotes the Euler operator.

When $k \geq 1$, one has

$$\left(\sum_{K_0} a_{\alpha i} t^{i+1} [\delta_t + i]_i - 1 \right) \hat{f}_k(t) = - \sum_{j=1}^{\min\{i_*+1,k\}} \sum_{K_j} a_{\alpha i} t^{i+1} [\delta_t + i]_i \hat{f}_{k-j}(t). \tag{4.5}$$

We remark that the above ordinary differential equations are derived from the recurrence formulas of $\{A_k(n)\}$. Moreover, from the differential equations (4.4) and (4.5) we immediately see that $\hat{f}_k(t) = O(t^k)$ $(k \geq 0)$ by induction.

For the estimates of $A_k(n)$ in Lemma 4.3, we use the majorant method for the recurrence formulas of $\{A_k(n)\}$ (cf. [16]). We will give the proof in Appendix B.

5. PRELIMINARIES FOR PROOF OF THEOREM 1.1

In this section, we prepare some results which are employed to prove Theorem 1.1. First, we give the important lemma for the summability theory (cf. [1, 7]).

Lemma 5.1. *Let $\kappa > 0$, $d \in \mathbb{R}$ and $\hat{v}(t,x) = \sum_{n=0}^{\infty} v_n(x)t^n \in \mathcal{O}_x[[t]]_{1/\kappa}$. Then the following statements are equivalent:*

(i) $\hat{v}(t,x) \in \mathcal{O}_x\{t\}_{\kappa,d}$.

(ii) *We put*

$$v_B(s,x) = (\hat{\mathcal{B}}_\kappa \hat{v})(s,x) := \sum_{n=0}^{\infty} \frac{v_n(x)}{\Gamma(1+n/\kappa)} s^n, \tag{5.1}$$

which is a formal κ-Borel transformation of $\hat{v}(t,x)$, that is convergent in a neighborhood of $(s,x) = (0,0)$. Then $v_B(s,x) \in \mathrm{Exp}_s(\kappa, S(d,\varepsilon) \times B(\sigma))$ for some $\varepsilon > 0$ and $\sigma > 0$.

Next, we prepare lemmas for the summability of $\hat{f}_k(t)$ which is given by (4.1), whose proofs are given in Section 8.

Lemma 5.2. *Let $\hat{f}_k(t)$ be given by (4.1) and $\kappa = (i_*+1)/i_*$. Then $\hat{f}_k(t) \in \mathbb{C}\{t\}_{\kappa,\theta}$ for θ satisfying*

$$\theta \not\equiv -(\arg a_{\alpha_*,i_*} + 2\pi n)/(i_*+1) \pmod{2\pi} \quad (n = 0,1,\dots,i_*). \tag{5.2}$$

Equivalently we shall prove the following.

Lemma 5.3. *Let $\kappa = (i_*+1)/i_*$. Then $f_{kB}(s) = (\hat{\mathcal{B}}_\kappa \hat{f}_k)(s)$ has (i_*+1) singular points in s plane which are given by*

$$s = c_\kappa (a_{\alpha_*,i_*})^{-1/(i_*+1)} \omega_{i_*+1}^{-n}, \quad n = 0,1,\dots,i_*, \tag{5.3}$$

where $c_\kappa = (1/\kappa)^{1/\kappa}$ and $\omega_q = e^{2\pi i/q}$. Moreover, we obtain $f_{kB}(s) \in \mathrm{Exp}_s(\kappa, S(\theta,\varepsilon_0))$, where θ satisfies (5.2) and $\varepsilon_0 > 0$. Especially, we obtain the following estimates for $k \geq 0$

$$|f_{kB}(s)| \leq C_k |s|^k \exp(\delta |s|^\kappa), \quad s \in S(\theta,\varepsilon_0), \tag{5.4}$$

where δ is independent of k and $C_k \leq \hat{C}\hat{K}^k$ with some positive $\hat{C}$ and $\hat{K}$.

6. PROOF OF THEOREM 1.1

By employing Lemmas 5.1 and 5.3, we obtain the following results, which will be proved in the next section.

Proposition 6.1. *Let d be fixed and define $(u_k)_B(s,x) = (\hat{\mathcal{B}}_\kappa \hat{u}_k)(s,x)$. We assume that the Cauchy data $\varphi(x)$ satisfies the same assumptions as in Theorem 1.1. Then for all k, we have*

$$\max_{|x|\leq\sigma} |(u_k)_B(s,x)| \leq CK^k \frac{|s|^k}{k!} \exp\left(\delta |s|^\kappa\right), \quad s \in S(d,\varepsilon) \tag{6.1}$$

by some positive constants C, K, δ and σ.

Corollary 6.2. *Let $\kappa = (i_* + 1)/(\alpha_* - 1)$ and $d \in \mathbb{R}$ be fixed. We assume that the Cauchy data $\varphi(x)$ satisfies the same assumptions as in Theorem 1.1. Then the formal solutions $\hat{u}_k(t,x)$ of (E_k) is κ-summable in d direction.*

We can prove Theorem 1.1 under these preparations.

Proof of Theorem 1.1. Let $\hat{U}(t,x) = \sum_{k\geq 0} \hat{u}_k(t,x)$ be the formal solution of original Cauchy problem (1.2). We finish the proof of Theorem 1.1 by showing that $U_B(s,x) = (\hat{\mathcal{B}}_\kappa \hat{U})(s,x) = \sum_{k\geq 0}(u_k)_B(s,x) \in \mathrm{Exp}_s(\kappa, S(d,\varepsilon))$.

By using Proposition 6.1, we obtain the desired estimate of $U_B(s,x)$ for $s \in S(d,\varepsilon)$.

$$\begin{aligned}\max_{|x|\leq\sigma} |U_B(s,x)| &\leq \sum_{k\geq 0} \max_{|x|\leq\sigma} |(u_k)_B(s,x)| \leq C\exp(\delta|s|^\kappa) \sum_{k\geq 0} \frac{(K|s|)^k}{k!} \\ &= C\exp(\delta|s|^\kappa)\cdot\exp(K|s|) \leq \widetilde{C}\exp(\tilde{\delta}|s|^\kappa) \quad (\because 1<\kappa)\end{aligned}$$

by some positive constants $\widetilde{C} > C$ and $\tilde{\delta} > \delta$. □

7. PROOF OF PROPOSITION 6.1

In this section, we shall give the proof of Proposition 6.1, since Corollary 6.2 follows from Proposition 6.1 immediately (cf. [7–9, 12]).

The formal solution $\hat{u}_k(t,x)$ of (E_k) is given by

$$\hat{u}_k(t,x) = \sum_{n\geq k} A_k(n)\varphi^{(n-k)}(x)\frac{t^n}{n!} = \sum_{n\geq 0} A_k(n+k)\varphi^{(n)}(x)\frac{t^{n+k}}{(n+k)!}.$$

Then the formal κ-Borel transformation of $\hat{u}_k$ is given by

$$(u_k)_B(s,x) = \sum_{n\geq 0} A_k(n+k)\varphi^{(n)}(x)\frac{s^{n+k}}{(n+k)!\Gamma(1+(n+k)/\kappa)}.$$

By using the Cauchy integral formula, we have for sufficiently small s and x

$$(u_k)_B(s,x) = \frac{1}{2\pi i}\oint_{|\zeta|=r_1} \frac{\varphi(x+\zeta)}{\zeta} \sum_{n\geq 0} \frac{n!}{(n+k)!}\frac{A_k(n+k)}{\Gamma(1+(n+k)/\kappa)}\frac{s^{n+k}}{\zeta^n}d\zeta.$$

Here when $k \geq 1$, we notice

$$\frac{n!}{(n+k)!} = \frac{B(n+1,k)}{(k-1)!} = \frac{1}{(k-1)!}\int_0^1 y^n(1-y)^{k-1}dy.$$

Therefore, for $k \geq 1$ we have

$$(u_k)_B(s,x) = \frac{1}{2\pi i}\int_0^1 \frac{(1-y)^{k-1}y^{-k}}{(k-1)!}dy \oint_{|\zeta|=r_1} \varphi(x+\zeta)\zeta^{k-1} \times f_{kB}\left(\frac{ys}{\zeta}\right)d\zeta,$$

where $|s| < c_\kappa |a_{\alpha_*,i_*}|^{-1/(i_*+1)} r_1$, $c_\kappa = (1/\kappa)^{1/\kappa}$ and

$$f_{kB}(X) = \sum_{n \geq 0} A_k(n+k) \frac{X^{n+k}}{\Gamma(1+(n+k)/\kappa)}.$$

We remark that $f_{kB}(X)$ has $(i_* + 1)$ singular points in X plane by Lemma 5.3. Therefore, we see that $f_{kB}(ys/\zeta)$ has $(i_* + 1)$ singular points in ζ plane which are given by

$$\zeta_n(s) := c_\kappa^{-1} a_{\alpha_*,i_*}^{1/(i_*+1)} ys\omega_{i_*+1}^n \quad (n = 0, 1, \ldots, i_*).$$

For a fixed s with $\arg s = d$, we have for $n = 0, 1, \ldots, i_*$

$$\arg \zeta_n(s) = d + \frac{\arg a_{\alpha_*,i_*} + 2\pi n}{i_* + 1} = d_n.$$

We consider the situation that $|s|$ becomes bigger along $\arg s = d$. In this case, we split the path of integral into (i_*+1) arcs γ_I and (i_*+1) arcs Γ_I $(I = 1, 2, \ldots, i_*+1)$, where each γ_I consists of the arc between points of argument $d_I - \varepsilon/3$ and $d_I + \varepsilon/3$, and each Γ_I consists of the arc between points of argument $d_I + \varepsilon/3$ and $d_{I+1} - \varepsilon/3$ with $d_{i_*+2} = d_1$. Since $\varphi(x)$ is analytic in $\cup_{j=0}^{i_*} S(d_j, \varepsilon)$, we can deform γ_I into paths $\gamma_{I,r(s)}$ which are taken along the ray $\arg \zeta = d_I - \varepsilon/3$ to a point with the modulus $r(s) = cy|s|(\sin(\varepsilon/3) + 1)$ (c is some constant), then along the circle $|\zeta| = r(s)$ to the ray $\arg \zeta = d_I + \varepsilon/3$ and back along this ray to the original arc. Therefore, we have

$$(u_k)_B(s,x) = \sum_{I=1}^{i_*+1} \frac{1}{2\pi i} \int_0^1 \frac{(1-y)^{k-1} y^{-k}}{(k-1)!} dy \left\{ \int_{\Gamma_I} + \int_{\gamma_{I,r(s)}} \right\} \varphi(x+\zeta) \zeta^{k-1} f_{kB}(ys/\zeta) d\zeta.$$

From the assumptions for the Cauchy data and Lemma 5.3, we obtain the analytic continuation into $S(d, \tilde{\varepsilon}) \times B(\sigma)$ for some positive $\tilde{\varepsilon}$ and σ. The desired exponential growth estimate of $(u_k)_B(s,x)$ is obtained by the following calculation.

$$\begin{aligned}
&\max_{|x| \leq \sigma} |(u_k)_B(s,x)| \\
&\leq \frac{1}{2\pi} \int_0^1 \frac{(1-y)^{k-1} y^{-k}}{(k-1)!} dy \\
&\quad \times C_\varphi e^{\delta_\varphi |s|^\kappa} \times \max_{r_1 \leq |\zeta| \leq r(s)} |\zeta|^{k-1} \times C_k \left| \frac{ys}{\zeta} \right|^k e^{\delta_f |s|^\kappa} \times \sum_{I=1}^{i_*+1} c(\sin(\varepsilon/3) + 1) y|s| \\
&\leq \{const\} \times \hat{K}^k |s|^{k+1} e^{\tilde{\delta}|s|^\kappa} \int_0^1 \frac{(1-y)^{k-1} y}{(k-1)!} dy \quad (\tilde{\delta} > \delta_\varphi + \delta_f) \\
&\leq \{const\} \hat{K}^k \frac{|s|^k}{k!} e^{\delta |s|^\kappa} \quad (\delta > \tilde{\delta}).
\end{aligned}$$

We remark that when $k = 0$, we obtain the similar estimate with the above one for $(u_0)_B(s,x)$.

8. PROOF OF LEMMA 5.3

We shall give the proof of Lemma 5.3. For the purpose, we will obtain the convolution equations of f_{kB}. After that, we will prove Lemma 5.3 by employing the method of successive approximation for the convolution equations.

8.1. A CANONICAL FORM FOR DIFFERENTIAL EQUATION OF $\hat{f}_k$

First, we reduce the differential equation of $\hat{f}_k(t)$ for each k to a certain canonical form (cf. [4]).

In the case $k = 0$, we know that $\hat{f}_0(t)$ satisfies the following differential equation

$$(L_0 - 1)\hat{f}_0(t) := \left(\sum_{K_0} a_{\alpha i} t^{i+1}[\delta_t + i]_i - 1\right)\hat{f}_0(t) = -1, \tag{8.1}$$

where $K_0 = \{(\alpha, i); 0 \le i \le i_*, \alpha = i+1\}$ and $\delta_t = t(d/dt)$ denotes the Euler operator.

Lemma 8.1. *We have*

$$[\delta_t + i]_i = \sum_{\ell=0}^{i} d_{i\ell}\delta_t^\ell, \tag{8.2}$$

where $d_{00} = 1$ *and*

$$d_{i,\ell} = d_{i-1,\ell-1} + i d_{i-1,\ell}, \quad 0 \le \ell \le i$$

with $d_{i-1,-1} = d_{i-1,i} = 0$. *Then* $d_{ii} = 1, d_{i,0} = i!$.

By using this lemma, we write the operator L_0 into the following form

$$L_0 = \sum_{i=0}^{i_*} a_{i+1,i} t^{i+1} \sum_{\ell=0}^{i} d_{i\ell}\delta_t^\ell = \sum_{i=0}^{i_*} a_{i+1,i} d_{i0} t^{i+1} + \sum_{i=1}^{i_*} a_{i+1,i} t^{i+1} \sum_{\ell=1}^{i} d_{i\ell}\delta_t^\ell.$$

Lemma 8.2. *We have*

$$t^{\kappa n}\delta_t^n = \sum_{\ell=1}^{n} D_{n\ell}(t^\kappa)^{n-\ell}(t^\kappa\delta_t)^\ell, \tag{8.3}$$

where $D_{11} = 1$ *and*

$$D_{n\ell} = -\kappa\ell D_{n-1,\ell} + D_{n-1,\ell-1}, \quad 1 \le \ell \le n$$

with $D_{n-1,n} = 0$ $(n \ge 1)$ *and* $D_{n-1,0} = \begin{cases} 1 & (n = 1) \\ 0 & (n \ge 2) \end{cases}$. *Then we have* $D_{nn} = 1$.

By using Lemma 8.2, we have

$$\begin{aligned}
L_0 &= \sum_{i=0}^{i_*} a_{i+1,i} i! t^{i+1} + \sum_{i=1}^{i_*} a_{i+1,i} t^{i+1} \sum_{\ell=1}^{i} d_{i\ell} t^{-\kappa\ell} \cdot t^{\kappa\ell}\delta_t^\ell \quad (\because d_{i0} = i!) \\
&= \sum_{i=0}^{i_*} a_{i+1,i} i! t^{i+1} + \sum_{i=1}^{i_*} a_{i+1,i} t^{i+1} \sum_{\ell=1}^{i} d_{i\ell} t^{-\kappa\ell} \cdot \sum_{m=1}^{\ell} D_{\ell m} (t^\kappa)^{\ell-m} (t^\kappa \delta_t)^m \\
&= \sum_{i=0}^{i_*} a_{i+1,i} i! t^{i+1} + \sum_{i=1}^{i_*} a_{i+1,i} t^{i+1} \sum_{m=1}^{i} \sum_{\ell=m}^{i} d_{i\ell} D_{\ell m} t^{-\kappa m} (t^\kappa \delta_t)^m.
\end{aligned}$$

Here by putting

$$\mathcal{D}_{im} := \sum_{\ell=m}^{i} d_{i\ell} D_{\ell m},$$

we have

$$\begin{aligned}
L_0 &= \sum_{i=0}^{i_*} a_{i+1,i} i! t^{i+1} + \sum_{i=1}^{i_*} a_{i+1,i} t^{i+1} \sum_{m=1}^{i} \mathcal{D}_{im} t^{-\kappa m} (t^\kappa \delta_t)^m \\
&= \sum_{i=0}^{i_*} a_{i+1,i} i! t^{i+1} + \sum_{m=1}^{i_*} \sum_{i=m}^{i_*} a_{i+1,i} \mathcal{D}_{im} t^{i+1-\kappa m} (t^\kappa \delta_t)^m.
\end{aligned}$$

We put

$$A_m^{[0]}(t) := \sum_{i=m}^{i_*} a_{i+1,i} \mathcal{D}_{im} t^{i+1-\kappa m} \quad (0 \le m \le i_*).$$

Then we notice that

$$\begin{aligned}
A_0^{[0]}(t) &= \sum_{i=0}^{i_*} a_{i+1,i} \mathcal{D}_{i0} t^{i+1} = \sum_{i=0}^{i_*} a_{i+1,i} i! t^{i+1}, \\
A_{i_*}^{[0]}(t) &= a_{i_*+1,i_*} \mathcal{D}_{i_* i_*} t^{i_*+1-\kappa i_*} = a_{i_*+1,i_*},
\end{aligned}$$

because of

$$\begin{aligned}
\mathcal{D}_{i0} &= \sum_{\ell=0}^{i} d_{i\ell} D_{\ell 0} = d_{i0} D_{00} = i!, \\
\mathcal{D}_{i_* i_*} &= d_{i_* i_*} D_{i_* i_*} = 1 \quad \text{and} \quad i_* + 1 - \kappa i_* = 0.
\end{aligned}$$

Therefore, since

$$L_0 = a_{i_*+1,i_*} (t^\kappa \delta_t)^{i_*} + \sum_{m=0}^{i_*-1} A_m^{[0]}(t) (t^\kappa \delta_t)^m, \tag{8.4}$$

we can write the differential equation of $\hat{f}_0(t)$ into the following form

$$\left[a_{i_*+1,i_*}(t^\kappa\delta_t)^{i_*}-1\right]\hat{f}_0(t)=-1-\sum_{m=0}^{i_*-1}A_m^{[0]}(t)(t^\kappa\delta_t)^m\hat{f}_0(t). \tag{8.5}$$

Finally, we substitute $\hat{f}_0(t)=1+\tilde{f}_0(t)$ into the above equation. After some calculations, we replace $\tilde{f}_0$ by $\hat{f}_0$. Then we obtain the following canonical form for the differential equation of $\hat{f}_0$

$$\left[a_{i_*+1,i_*}(t^\kappa\delta_t)^{i_*}-1\right]\hat{f}_0(t)=-A_0^{[0]}(t)-\sum_{m=0}^{i_*-1}A_m^{[0]}(t)(t^\kappa\delta_t)^m\hat{f}_0(t). \tag{8.6}$$

In the case $k\geq 1$, we know that $\hat{f}_k(t)$ satisfies the following equation for each k

$$\begin{aligned}\left(\sum_{K_0}a_{\alpha i}t^{i+1}[\delta_t+i]_i-1\right)\hat{f}_k(t)&=-\sum_{j=1}^{\min\{i_*+1,k\}}\sum_{K_j}a_{\alpha i}t^{i+1}[\delta_t+i]_i\hat{f}_{k-j}(t)\\&=:-\sum_{j=1}^{\min\{i_*+1,k\}}L_j\hat{f}_{k-j}(t),\end{aligned} \tag{8.7}$$

where $K_j=\{(\alpha,i);j-1\leq i\leq i_*,\ \alpha=i+1-j\}$. Similarly, with the case $k=0$, we get the following

$$\begin{aligned}L_j&=\sum_{i=j-1}^{i_*}a_{i+1-j,i}t^{i+1}[\delta_t+i]_i=\sum_{i=j-1}^{i_*}a_{i-j+1,i}i!t^{i+1}+\sum_{i=j-1}^{i_*}a_{i-j+1,i}t^{i+1}\sum_{\ell=1}^{i}d_{i\ell}\delta_t^\ell\\&=\sum_{i=j-1}^{i_*}a_{i-j+1,i}i!t^{i+1}\\&\qquad+\left\{\sum_{m=1}^{j-1}\sum_{i=j-1}^{i_*}+\sum_{m=j}^{i_*}\sum_{i=m}^{i_*}\right\}a_{i-j+1,i}\mathcal{D}_{i,m}t^{i+1-\kappa m}(t^\kappa\delta_t)^m.\end{aligned}$$

Here for $j=1,2,\ldots,i_*+1$, we put

$$A_m^{[j]}(t):=\begin{cases}\displaystyle\sum_{i=j-1}^{i_*}a_{i-j+1,i}\mathcal{D}_{i,m}t^{i+1-\kappa m}, & 0\leq m\leq j-1,\\ \displaystyle\sum_{i=m}^{i_*}a_{i-j+1,i}\mathcal{D}_{i,m}t^{i+1-\kappa m}, & j\leq m\leq i_*\end{cases}$$

and for $j\geq i_*+2$ we define $A_m^{[j]}(t)\equiv 0$ for each m. Then we notice that

$$A_0^{[j]}(t)=\sum_{i=j-1}^{i_*}a_{i-j+1,i}i!t^{i+1},\quad A_{i_*}^{[j]}(t)=a_{i_*-j+1,i_*}.$$

Therefore, we obtain the following canonical form for the differential equation of $\hat{f}_k$

$$\begin{aligned}\left[a_{i_*+1,i_*}(t^\kappa\delta_t)^{i_*}-1\right]\hat{f}_k(t) &= -\sum_{m=0}^{i_*-1}A_m^{[0]}(t)(t^\kappa\delta_t)^m\hat{f}_k(t)\\ &\quad -\sum_{j=1}^{\min\{i_*+1,k\}}\sum_{m=0}^{i_*}A_m^{[j]}(t)(t^\kappa\delta_t)^m\hat{f}_{k-j}(t).\end{aligned} \tag{8.8}$$

For $1 \le k \le i_*+1$, we substitute $\hat{f}_0(t) = 1+\tilde{f}_0(t)$ into the above equation. After some calculations we replace $\tilde{f}_0(t)$ by $\hat{f}_0(t)$. Then we have the following canonical form

$$\begin{aligned}\left[a_{i_*+1,i_*}(t^\kappa\delta_t)^{i_*}-1\right]\hat{f}_k(t) &= -\sum_{m=0}^{i_*-1}A_m^{[0]}(t)(t^\kappa\delta_t)^m\hat{f}_k(t)\\ &\quad -A_0^{[k]}(t)-\sum_{j=1}^{k}\sum_{m=0}^{i_*}A_m^{[j]}(t)(t^\kappa\delta_t)^m\hat{f}_{k-j}(t).\end{aligned} \tag{8.9}$$

8.2. CONVOLUTION EQUATIONS

We shall obtain the convolution equations by operating the Borel transform to the canonical differential equations which are obtained in the previous subsection.

In the case $k = 0$, after operating the formal κ-Borel transformation to the equation (8.6) and differentiating the both sides, we substitute $D_s f_{0B}(s) = w_0(s)$ or $f_{0B}(s) = D_s^{-1}w_0(s)$, where $D_s = d/(ds)$ and $D_s^{-1} = \int_0^s$. Then the convolution equation for $w_0(s)$ is given by the following expression

$$\begin{aligned}&\left[\kappa^{i_*}a_{i_*+1,i_*}s^{i_*+1}-1\right]w_0(s)\\ &= D_s\left[-A_{0B}^{[0]}(s)-\sum_{m=0}^{i_*-1}A_{mB}^{[0]}(s) *_\kappa D_s^{-1}\kappa^m s^{\kappa m}w_0(s)\right],\end{aligned} \tag{8.10}$$

where $A_{mB}^{[0]}(s) = (\mathcal{B}_\kappa A_m^{[0]})(s)$ for $0 \le m \le i_*-1$.

Here the κ-convolution $a(s) *_\kappa b(s)$ with $a(0) = b(0) = 0$ is defined by the following integral

$$(a *_\kappa b)(s) = \int_0^s a\Big((s^\kappa - u^\kappa)^{1/\kappa}\Big)\frac{d}{du}b(u)du. \tag{8.11}$$

We remark that if $a(0) = b(0) = 0$, the convolution is commutative. Note that this formula is same with that in [1, Sec. 5.3] although the expression is a little different from it.

We put

$$A_*(s) := 1-\kappa^{i_*}a_{i_*+1,i_*}s^{i_*+1}.$$

Similarly, in the case $k \geq 1$, after the formal κ-Borel transformation to the equations (8.9) and (8.8) and differentiating the both sides, we substitute $D_s f_{kB}(s) = w_k(s)$ or $f_{kB}(s) = D_s^{-1} w_k(s)$. Then for each k, the convolution equation for $w_k(s)$ is given by the following expression

$$\begin{aligned}
[-A_*(s)]\, w_k(s) = D_s \Bigg[&- A_{0B}^{[k]}(s) - \sum_{m=0}^{i_*-1} A_{mB}^{[0]}(s) *_\kappa D_s^{-1} \kappa^m s^{\kappa m} w_k(s) \\
&- \sum_{j=1}^{\min\{i_*+1,k\}} \sum_{m=0}^{i_*-1} A_{mB}^{[j]}(s) *_\kappa D_s^{-1} \kappa^m s^{\kappa m} w_{k-j}(s) \\
&- \sum_{j=1}^{\min\{i_*+1,k\}} a_{i_*-j+1, i_*} D_s^{-1} \kappa^{i_*} s^{i_*+1} w_{k-j}(s) \Bigg],
\end{aligned} \tag{8.12}$$

where $A_{mB}^{[j]}(s) = (\mathcal{B}_\kappa A_m^{[j]})(s)$ for $0 \leq m \leq i_* - 1,\ 1 \leq j \leq i_* + 1$.

We put

$$T_0(w_0)(s) := \frac{-1}{A_*(s)} \Big(\text{the right hand side of (8.10)} \Big),$$
$$T_k(w_k)(s) := \frac{-1}{A_*(s)} \Big(\text{the right hand side of (8.12)} \Big).$$

Then we remark that for all k

$$T_k : \mathbb{C}[[s]] \to \mathbb{C}[[s]],$$

where $\mathbb{C}[[s]]$ denotes the set of formal power series. Therefore, for all k, the function

$$w_k(s) = D_s f_{kB}(s) = D_s \sum_{n \geq 0} A_k(n) s^n / \Gamma(1 + n/\kappa)$$

is a unique holomorphic solution in a neighborhood of origin for the convolution equation (8.10) or (8.12).

8.3. PROOF OF LEMMA 5.3

In this subsection, we shall show that $w_k(s)$ has the exponential growth estimate of order at most κ in a sector with infinite radius. Therefore, $f_{kB}(s) = D_s^{-1} w_k(s)$ also has the same exponential growth estimate as that of $w_k(s)$. For the purpose, we employ the method of successive approximation for the convolution equation $w_k(s) = T_k(w_k)(s)$ (cf. [5]).

Case 1. $k = 0$. We define the functions $\{w_{0n}(s)\}$ by the following

$$\begin{cases} w_{00}(s) = \dfrac{1}{A_*(s)} D_s \left[A_{0B}^{[0]}(s) \right], \\ w_{0n}(s) = \dfrac{1}{A_*(s)} D_s \left[\displaystyle\sum_{m=0}^{i_*-1} A_{mB}^{[0]}(s) *_\kappa D_s^{-1} \kappa^m s^{\kappa m} w_{0,n-1}(s) \right] \quad (n \geq 1). \end{cases}$$

Case 2. $k \geq 1$. We define the functions $\{w_{kn}(s)\}$ by the following

$$\begin{cases} w_{k0}(s) = \dfrac{1}{A_*(s)} D_s \left[A_{0B}^{[k]}(s) + \displaystyle\sum_{j=1}^{\min\{i_*+1,k\}} a_{i_*-j+1,i_*} D_s^{-1} \kappa^{i_*} s^{i_*+1} w_{k-j,0}(s) \right], \\ w_{kn}(s) = \dfrac{1}{A_*(s)} D_s \left[\displaystyle\sum_{m=0}^{i_*-1} A_{mB}^{[0]}(s) *_\kappa D_s^{-1} \kappa^m s^{\kappa m} w_{k,n-1}(s) \right. \\ \qquad + \displaystyle\sum_{j=1}^{\min\{i_*+1,k\}} \sum_{m=0}^{i_*-1} A_{mB}^{[j]}(s) *_\kappa D_s^{-1} \kappa^m s^{\kappa m} w_{k-j,n-1}(s) \\ \qquad \left. + \displaystyle\sum_{j=1}^{\min\{i_*+1,k\}} a_{i_*-j+1,i_*} D_s^{-1} \kappa^{i_*} s^{i_*+1} w_{k-j,n}(s) \right] \quad (n \geq 1). \end{cases}$$

Then $w_k(s) = \sum_{n \geq 0} w_{kn}(s)$ is the convergent power series solution of the convolution equation $w_k(s) = \overline{T}_k(w_k)(s)$.

For sufficiently small r, we put

$$S(r) := S(\theta, \varepsilon_0) \setminus \{|s| < r\},$$

where $\theta \not\equiv -(\arg a_{i_*+1,i_*} + 2\pi j)/(i_* + 1) \pmod{2\pi}$ with $j = 0, 1, \ldots, i_*$.

We assume that for $s \in S(r)$,

$$\left| \frac{1}{A_*(s)} \right| \leq \frac{B_1}{1 + |s|^{i_*+1}}, \quad \left| D_s A_{mB}^{[j]}(s) \right| \leq B_2 |s|^{i_*-\kappa m}, \quad |a_{i_*-j+1,i_*}| \leq B_2, \\ \frac{|s|^{i_*}}{1 + |s|^{i_*+1}} \leq B_3 \tag{8.13}$$

with some positive constants B_1 B_2 and B_3 for $0 \leq m \leq i_* - 1, 0 \leq j \leq i_* + 1$.

Then the following proposition holds.

Proposition 8.3. *For each k, we have $w_{kn}(s) \in \mathcal{O}(S(r))$ for all n and*

$$|w_{kn}(s)| \leq C_k K^n \frac{|s|^{n+k}}{\Gamma\left(\frac{n+1}{\kappa}\right)}, \quad s \in S(r), \tag{8.14}$$

where $C_0 = B_1 B_2 B_3 \Gamma(1/\kappa)$ and $C_k = \beta^k C_0$ with

$$\beta = 2\left(\frac{1}{r} + B_1 B_2 B_3 \kappa^{i_*}\right)$$

and K is a positive constant such that

$$2 i_* (i_* + 2) B_1 B_2 B_3 \Gamma(1 + 1/\kappa) \kappa^{i_*-1} \Gamma(i_*) < K. \tag{8.15}$$

We remark that C_k and C_{k-1} satisfy the following recurrence formula

$$\left(\frac{1}{r}+B_1B_2B_3\kappa^{i_*}\right)C_{k-1}=\frac{1}{2}C_k(<C_k). \tag{8.16}$$

This proposition means that $w_k(s)$ has the exponential growth estimate of order at most κ in $S(\theta,\varepsilon_0)$. Therefore, Lemma 5.3 follows from this proposition immediately.

8.4. PROOF OF PROPOSITION 8.3

We shall prove Proposition 8.3 by induction on k and n.

First, we notice that for functions $A(s)$ and $W(s)$ with $A(0)=0$

$$\begin{aligned}
D_s\Big(A(s)*_\kappa D_s^{-1}W(s)\Big) &= D_s\int_0^s A\Big((s^\kappa-u^\kappa)^{1/\kappa}\Big)W(u)du\\
&= s^{\kappa-1}\int_0^s D_sA\Big((s^\kappa-u^\kappa)^{1/\kappa}\Big)(s^\kappa-u^\kappa)^{1/\kappa-1}W(u)du\\
&= s\int_0^1 D_sA\Big(s(1-t^\kappa)^{1/\kappa}\Big)(1-t^\kappa)^{1/\kappa-1}W(st)dt \quad (u=st).
\end{aligned} \tag{8.17}$$

Case 1. $k=0$.

When $n=0$, from (8.13) we have

$$|w_{00}(s)|\le\frac{B_1}{1+|s|^{i_*+1}}\cdot B_2|s|^{i_*}\le B_1B_2B_3=\frac{C_0}{\Gamma(1/\kappa)}.$$

When $n\ge 1$, by (8.17) and the induction assumption we have

$$\begin{aligned}
|w_{0n}(s)| &\le \frac{B_1}{1+|s|^{i_*+1}}\left\{\sum_{m=0}^{i*-1}|s|\right.\\
&\quad\left.\times\int_0^1 B_2\left|s(1-t^\kappa)^{1/\kappa}\right|^{i_*-\kappa m}(1-t^\kappa)^{1/\kappa-1}\kappa^m|st|^{\kappa m}C_0K^{n-1}\frac{|st|^{n-1}}{\Gamma(n/\kappa)}dt\right\}\\
&\le \frac{B_1B_2|s|^{i_*}}{1+|s|^{i_*+1}}C_0K^{n-1}\frac{|s|^n}{\Gamma(n/\kappa)}\sum_{m=0}^{i_*-1}\kappa^m\int_0^1(1-t^\kappa)^{i_*/\kappa+1/\kappa-m-1}t^{\kappa m+n-1}dt.
\end{aligned}$$

Next, we have the following formula

$$\int_0^1(1-t^\kappa)^{p-1}t^{q-1}dt=\frac{1}{\kappa}B\left(p,\frac{q}{\kappa}\right)=\frac{1}{\kappa}\frac{\Gamma(p)\Gamma(q/\kappa)}{\Gamma(p+q/\kappa)}.$$

By noticing $(i_*+1)/\kappa = i_*$, we have

$$|w_{0n}(s)| \le B_1B_2B_3C_0K^{n-1}\frac{|s|^n}{\Gamma(n/\kappa)}\frac{1}{\kappa}\left\{\sum_{m=0}^{i_*-1}\kappa^m\frac{\Gamma(i_*-m)\Gamma(m+n/\kappa)}{\Gamma(i_*+n/\kappa)}\right\}.$$

Lemma 8.4. *We have*

$$\frac{1}{\Gamma(n/\kappa)} \le \frac{\frac{n}{\kappa}\Gamma(1/\kappa)}{\Gamma(\frac{n+1}{\kappa})}, \tag{8.18}$$

$$\frac{\Gamma(p+c)}{\Gamma(q+c)} = \frac{1}{[q-1+c]_{q-p}}, \tag{8.19}$$

where $\kappa > 1$, $p, q \in \mathbb{N}$ *with* $p < q$ *and* $c > 0$.

The formula (8.18) is obtained by the lower estimate of $B(n/\kappa, 1/\kappa)$ and the formula (8.19) is obtained from the formula $\Gamma(1+z) = z\Gamma(z)$.

By using Lemma 8.4, we have the following inequalities for $0 \le m \le i_*-1$

$$\frac{1}{\Gamma(n/\kappa)}\cdot\frac{\Gamma(m+n/\kappa)}{\Gamma(i_*+n/\kappa)} \le \frac{\frac{n}{\kappa}\Gamma(1/\kappa)}{\Gamma(\frac{n+1}{\kappa})}\cdot\frac{1}{[i_*-1+n/\kappa]_{i_*-m}} \le \frac{\Gamma(1/\kappa)}{\Gamma(\frac{n+1}{\kappa})}.$$

Therefore, we get

$$\begin{aligned}|w_{0n}(s)| &\le B_1B_2B_3C_0K^{n-1}\frac{|s|^n}{\Gamma((n+1)/\kappa)}\frac{\Gamma(1/\kappa)}{\kappa}\left\{\sum_{m=0}^{i_*-1}\kappa^m\Gamma(i_*-m)\right\}\\ &\le C_0K^n\frac{|s|^n}{\Gamma((n+1)/\kappa)}.\end{aligned}$$

For the last inequality, we used the following inequality

$$B_1B_2B_3\frac{\Gamma(1/\kappa)}{\kappa}\sum_{m=0}^{i_*-1}\kappa^m\Gamma(i_*-m) < K,$$

which follows from (8.15). This means that Proposition 8.3 is proved in the case $k=0$.

Case 2. $k \ge 1$. We assume that (8.14) holds up to $k-1$.

When $n=0$, from (8.13) we have

$$\begin{aligned}|w_{k0}(s)| &\le \frac{B_1}{1+|s|^{i_*+1}}\left\{B_2|s|^{i_*} + \sum_{j=1}^{\min\{i_*+1,k\}}B_2\kappa^{i_*}|s|^{i_*+1}C_{k-j}\frac{|s|^{k-j}}{\Gamma(1/\kappa)}\right\}\\ &\le B_1B_2B_3\frac{|s|^k}{\Gamma(1/\kappa)}\left\{\Gamma(1/\kappa)|s|^{-k} + \kappa^{i_*}\sum_{j=1}^{\min\{i_*+1,k\}}C_{k-j}|s|^{1-j}\right\}.\end{aligned}$$

When $1 \le k \le i_* + 1$, we have by noting $|s| \ge r$

$$
\begin{aligned}
& B_1B_2B_3\Gamma(1/\kappa)\frac{1}{r^k} + B_1B_2B_3\kappa^{i_*}\sum_{j=1}^{k} C_{k-j}\frac{1}{r^{j-1}} \\
&= C_0\frac{1}{r^k} + B_1B_2B_3\kappa^{i_*}\left(C_0\frac{1}{r^{k-1}} + C_1\frac{1}{r^{k-2}} + \ldots + C_{k-1}\right) \\
&= \frac{1}{r^{k-1}}\left(\frac{1}{r} + B_1B_2B_3\kappa^{i_*}\right)C_0 + B_1B_2B_3\kappa^{i_*}\left(C_1\frac{1}{r^{k-2}} + \ldots + C_{k-2}\frac{1}{r} + C_{k-1}\right) \\
&< \frac{1}{r^{k-1}}C_1 + B_1B_2B_3\kappa^{i_*}\left(C_1\frac{1}{r^{k-2}} + \ldots + C_{k-2}\frac{1}{r} + C_{k-1}\right) \quad (\because (8.16)) \\
&= \frac{1}{r^{k-2}}\left(\frac{1}{r} + B_1B_2B_3\kappa^{i_*}\right)C_1 + B_1B_2B_3\kappa^{i_*}\left(C_2\frac{1}{r^{k-3}} + \ldots + C_{k-2}\frac{1}{r} + C_{k-1}\right) \\
&< \frac{1}{r^{k-2}}C_2 + B_1B_2B_3\kappa^{i_*}\left(C_2\frac{1}{r^{k-3}} + \ldots + C_{k-2}\frac{1}{r} + C_{k-1}\right). \quad (\because (8.16))
\end{aligned}
$$

By repeating such calculation, we get

$$B_1B_2B_3\Gamma(1/\kappa)\frac{1}{r^k} + B_1B_2B_3\kappa^{i_*}\sum_{j=1}^{k} C_{k-j}\frac{1}{r^{j-1}} < C_k.$$

When $k \ge i_* + 2$, by using the relation

$$C_0\frac{1}{r^p} < C_p \quad (p \in \mathbb{N}),$$

we have

$$
\begin{aligned}
& B_1B_2B_3\Gamma(1/\kappa)\frac{1}{r^k} + B_1B_2B_3\kappa^{i_*}\sum_{j=1}^{i_*+1} C_{k-j}\frac{1}{r^{j-1}} \\
&= C_0\frac{1}{r^k} + B_1B_2B_3\kappa^{i_*}\sum_{j=1}^{i_*+1} C_{k-j}\frac{1}{r^{j-1}} \\
&< C_{k-i_*-1}\frac{1}{r^{i_*+1}} + B_1B_2B_3\kappa^{i_*}\left(C_{k-i_*-1}\frac{1}{r^{i_*}} + C_{k-i_*}\frac{1}{r^{i_*-1}} + \ldots + C_{k-1}\right) \\
&= \frac{1}{r^{i_*}}\left(\frac{1}{r} + B_1B_2B_3\kappa^{i_*}\right)C_{k-i_*-1} + B_1B_2B_3\kappa^{i_*}(C_{k-i_*}\frac{1}{r^{i_*-1}} + \ldots + C_{k-1}) \\
&< \ldots < C_k.
\end{aligned}
$$

Therefore, we have

$$|w_{k0}(s)| \le C_k\frac{|s|^k}{\Gamma(1/\kappa)}.$$

When $n \geq 1$, we put

$$\begin{aligned} w_{kn}(s) = \frac{1}{A_*(s)} D_s \Bigg[&\sum_{m=0}^{i_*-1} A_{mB}^{[0]}(s) *_\kappa D_s^{-1} \kappa^m s^{\kappa m} w_{k,n-1}(s) \\ &+ \sum_{j=1}^{\min\{i_*+1,k\}} \sum_{m=0}^{i_*-1} A_{mB}^{[j]}(s) *_\kappa D_s^{-1} \kappa^m s^{\kappa m} w_{k-j,n-1}(s) \\ &+ \sum_{j=1}^{\min\{i_*+1,k\}} a_{i_*-j+1,i_*} D_s^{-1} \kappa^{i_*} s^{i_*+1} w_{k-j,n}(s) \Bigg] \\ =: \frac{1}{A_*(s)} &\left\{ \sum_{m=0}^{i_*-1} I_{1,m}(s) + \sum_{j=1}^{\min\{i_*+1,k\}} \sum_{m=0}^{i_*-1} I_{2,j,m}(s) + \sum_{j=1}^{\min\{i_*+1,k\}} I_{3,j}(s) \right\}. \end{aligned} \tag{8.20}$$

We use the formula (8.17). For $0 \leq m \leq i_* - 1$, we have

$$\begin{aligned} \left| \frac{I_{1,m}(s)}{A_*(s)} \right| &\leq \frac{B_1}{1+|s|^{i_*+1}} \\ &\quad \times |s| \int_0^1 B_2 \left| s(1-t^\kappa)^{1/\kappa} \right|^{i_*-\kappa m} (1-t^\kappa)^{1/\kappa-1} \kappa^m |st|^{\kappa m} C_k K^{n-1} \frac{|st|^{n-1+k}}{\Gamma(n/\kappa)} dt \\ &\leq B_1 B_2 B_3 C_k K^{n-1} \frac{|s|^{n+k}}{\Gamma(n/\kappa)} \kappa^m \int_0^1 (1-t^\kappa)^{i_*/\kappa - m + 1/\kappa - 1} t^{\kappa m + n + k - 1} dt \\ &= B_1 B_2 B_3 C_k K^{n-1} \frac{|s|^{n+k}}{\Gamma(n/\kappa)} \kappa^{m-1} \frac{\Gamma(i_* - m)\Gamma(m + (n+k)/\kappa)}{\Gamma(i_* + (n+k)/\kappa)}. \end{aligned}$$

By using Lemma 8.4, we have

$$\left| \frac{I_{1,m}(s)}{A_*(s)} \right| \leq B_1 B_2 B_3 C_k K^{n-1} \frac{|s|^{n+k}}{\Gamma((n+1)/\kappa)} \frac{\Gamma(1/\kappa)}{\kappa} \kappa^m \Gamma(i_* - m).$$

For $1 \leq j \leq \min\{i_* + 1, k\}$ and $0 \leq m \leq i_* - 1$, we have

$$\begin{aligned} \left| \frac{I_{2,j,m}(s)}{A_*(s)} \right| &\leq \frac{B_1}{1+|s|^{i_*+1}} |s| \int_0^1 B_2 \left| s(1-t^\kappa)^{1/\kappa} \right|^{i_*-\kappa m} (1-t^\kappa)^{1/\kappa-1} \kappa^m |st|^{\kappa m} \\ &\qquad \times C_{k-j} K^{n-1} \frac{|st|^{n-1+k-j}}{\Gamma(n/\kappa)} dt \\ &\leq B_1 B_2 B_3 C_{k-j} K^{n-1} \frac{|s|^{n+k-j}}{\Gamma(n/\kappa)} \kappa^m \int_0^1 (1-t^\kappa)^{i_*-m-1} t^{\kappa m + n + k - j - 1} dt \\ &= B_1 B_2 B_3 C_{k-j} K^{n-1} \frac{|s|^{n+k-j}}{\Gamma(n/\kappa)} \kappa^{m-1} \frac{\Gamma(i_* - m)\Gamma(m + (n+k-j)/\kappa)}{\Gamma(i_* + (n+k-j)/\kappa)}. \end{aligned}$$

By using Lemma 8.4, we have the following inequalities for $0 \le m \le i_* - 1$:

$$\begin{aligned}&\frac{1}{\Gamma(n/\kappa)}\frac{\Gamma(m+(n+k-j)/\kappa)}{\Gamma(i_*+(n+k-j)/\kappa)}\\&\le \frac{\frac{n}{\kappa}\Gamma(1/\kappa)}{\Gamma(\frac{n+1}{\kappa})}\cdot\frac{1}{[i_*-1+(n+k-j)/\kappa]_{i_*-m}} \le \frac{\Gamma(1/\kappa)}{\Gamma(\frac{n+1}{\kappa})}.\end{aligned}$$

Therefore, we obtain

$$\left|\frac{I_{2,j,m}(s)}{A_*(s)}\right| \le B_1B_2B_3C_{k-j}K^{n-1}\frac{|s|^{n+k-j}}{\Gamma((n+1)/\kappa)}\frac{\Gamma(1/\kappa)}{\kappa}\kappa^m\Gamma(i_*-m).$$

Since $C_{k-j}\frac{1}{r^j} < C_k$, we have

$$\left|\frac{I_{2,j,m}(s)}{A_*(s)}\right| \le B_1B_2B_3C_kK^{n-1}\frac{|s|^{n+k}}{\Gamma((n+1)/\kappa)}\frac{\Gamma(1/\kappa)}{\kappa}\kappa^m\Gamma(i_*-m).$$

For $1 \le j \le \min\{i_*+1, k\}$, we have

$$\begin{aligned}\left|\frac{I_{3,j}(s)}{A_*(s)}\right| &\le \frac{B_1}{1+|s|^{i_*+1}}B_2\kappa^{i_*}|s|^{i_*+1}C_{k-j}K^n\frac{|s|^{n+k-j}}{\Gamma((n+1)/\kappa)}\\&\le B_1B_2B_3\kappa^{i_*}C_{k-j}K^n\frac{|s|^{n+k+1-j}}{\Gamma((n+1)/\kappa)}.\end{aligned}$$

Since $B_1B_2B_3\kappa^{i_*}C_{k-i_*-1} < C_{k-i_*}$, we have

$$\begin{aligned}&B_1B_2B_3\kappa^{i_*}\sum_{j=1}^{\min\{i_*+1,k\}}C_{k-j}\frac{1}{r^{j-1}}\\&< C_{k-i_*}\frac{1}{r^{i_*}} + B_1B_2B_3\kappa^{i_*}\left(C_{k-i_*}\frac{1}{r^{i_*-1}} + C_{k-i_*+1}\frac{1}{r^{i_*-2}} + \ldots + C_{k-1}\right)\\&= \frac{1}{r^{i_*-1}}\left(\frac{1}{r} + B_1B_2B_3\kappa^{i_*}\right)C_{k-i_*} + B_1B_2B_3\kappa^{i_*}\left(C_{k-i_*+1}\frac{1}{r^{i_*-2}} + \ldots + C_{k-1}\right)\\&< \frac{1}{r^{i_*-1}}C_{k-i_*+1} + B_1B_2B_3\kappa^{i_*}\left(C_{k-i_*+1}\frac{1}{r^{i_*-2}} + \ldots + C_{k-1}\right)\\&< \ldots < \frac{1}{r}C_{k-1} + B_1B_2B_3\kappa^{i_*}C_{k-1} = \frac{1}{2}C_k.\end{aligned}$$

Therefore, we obtain

$$\sum_{j=1}^{\min\{i_*+1,k\}}\left|\frac{I_{3,j}(s)}{A_*(s)}\right| \le \frac{1}{2}C_kK^n\frac{|s|^{n+k}}{\Gamma((n+1)/\kappa)}.$$

Hence, we get

$$\begin{aligned}|w_{kn}(s)| \le C_kK^{n-1}\frac{|s|^{n+k}}{\Gamma((n+1)/\kappa)}\Bigg\{&B_1B_2B_3\Gamma(1+1/\kappa)\sum_{m=0}^{i_*-1}\kappa^m\Gamma(i_*-m)\\&+B_1B_2B_3\Gamma(1+1/\kappa)\sum_{j=1}^{i_*+1}\sum_{m=0}^{i_*-1}\kappa^m\Gamma(i_*-m) + \frac{K}{2}\Bigg\}.\end{aligned}$$

From the condition (8.15) for K we have

$$|w_{kn}(s)| \leq C_k K^n \frac{|s|^{n+k}}{\Gamma((n+1)/\kappa)}.$$

Under the above observations, the proof of Proposition 8.3 is completed.

APPENDIX

A. A CLASS OF CAUCHY DATA FOR THE CONVERGENCE OF FORMAL SOLUTIONS

A.1. RESULT

Let us consider the following Cauchy problem for a non-Kowalevskian equation with time dependent coefficients in a neighborhood of $t = 0$,

$$\begin{cases} \partial_t u(t,x) = \displaystyle\sum_{j=0}^{m} a_j(t)\partial_x^j u(t,x), \quad a_m(t) \not\equiv 0, \\ u(0,x) = \varphi(x) \in \mathbb{C}\{x\}, \end{cases} \tag{M1}$$

where $a_j(t) \in \mathbb{C}\{t\}$, the set of convergent series, and $(t,x) \in \mathbb{C}^2$.

The non-Kowalevskian condition means that $m \geq 2$ and $a_m(t) \not\equiv 0$. In this case, it is known that the Cauchy-Kowalevski theorem does not hold at the origin, which means that for a suitable data $\varphi(x) \in \mathbb{C}\{x\}$ the uniquely determined formal solution of the problem (M1) $u(t,x) = \sum_{l=0}^{\infty} u_l(x)t^l/l!$ $(u_0(x) = \varphi(x))$ is divergent (cf. [10,15]).

In this appendix we characterize a class of Cauchy data $\varphi(x) \in \mathbb{C}\{x\}$, for which the formal solution of the Cauchy problem (M1) converges in t-variable at $t = 0$.

Let $n_j \in \mathbb{N}$ be the order of zeros of $a_j(t)$ $(\in \mathbb{C}\{t\})$ at $t = 0$, that is,

$$a_j(t) = t^{n_j} b_j(t), \quad b_j(0) \neq 0.$$

For the modified order, we assume the following

$$\frac{m}{n_m + 1} = \max_{0 \leq j \leq m} \left\{ \frac{j}{n_j + 1} \right\}. \tag{M2}$$

This is a generalization of the conditions (A-3)–(A-5). Then we shall prove the following proposition.

Proposition A. 1. *Let $m > 1$. Then under the assumption* (M2)*, the Cauchy problem* (M1) *is uniquely solvable in locally holomorphic functions in t-variable at $t = 0$*

if and only if the Cauchy data $\varphi(x)$ is of Gevrey order at most $1/m$ which means $|\varphi^{(n)}(0)| \le AB^n(n!)^{1/m}$ for some positive constants A and B, or an entire function of exponential growth order at most $m/(m-1)$ that there exist C and $\delta > 0$ such that

$$|\varphi(x)| \le Ce^{\delta|x|^{m/(m-1)}}.$$

The technical condition (M2) is used only in the proof of the necessary part. Therefore, the sufficient part does hold without any condition.

A.2. PROOF OF SUFFICIENT PART

The sufficient part is a special case of the result in [11, Theorem 1] by M. Miyake which was studied from the most general framework. So we give here a direct and short proof associated with the problem.

Definition of Banach space $G_x^{(\sigma)}(R_t, R_x)$ $(0 < \sigma < 1, R_t > 0, R_x > 0)$:

$$v(t,x) = \sum_{l,n\ge 0} v_{l,n}\frac{t^l x^n}{l! \times n!} \in G_x^{(\sigma)}(R_t, R_x) \Longleftrightarrow \|v\|^{(\sigma)}_{R_t,R_x} := \sum_{l,n\ge 0} |v_{l,n}|\frac{(R_t)^l (R_x)^n}{(l+\sigma n)!} < \infty.$$

Note that for $v(t,x) \in G_x^{(\sigma)}(R_t, R_x)$ we have for any fixed $l \in \mathbb{N}$ that $|\partial_t^l \partial_x^n v(0,0)| = |v_{l,n}| \sim (\sigma n)!$ which means that $\partial_t^l v(0,x)$ is of Gevrey order σ in x-variable. Therefore, it holds that $|\partial_t^l v(0,x)| \le Ce^{\delta|x|^{1/(1-\sigma)}}$ for some positive constants C and δ.

We shall prove that the problem (M1) is uniquely solvable for any Cauchy data $\varphi(x) \in G_x^{(1/m)}(R_t, R_x)$ by taking a small enough R_t for any fixed R_x.

For the proof we may assume that $\varphi(x) \equiv 0$ without loss of generality. Then by putting $v(t,x) = \partial_t u(t,x)$ we have $u(t,x) = \partial_t^{-1} v(t,x)$ $(\partial_t^{-1} := \int_0^t)$, and the problem (M1) (for a non homogeneous equation) is reduced into the unique solvability of the following integro-differential equation in $G_x^{(1/m)}(R_t, R_x)$.

$$v(t,x) = \sum_{j=0}^{m} a_j(t)\partial_x^j \partial_t^{-1} v(t,x) + f(t,x), \quad f(t,x) \in G_x^{(1/m)}(R_t, R_x).$$

Then the assertion follows from the following lemma.

Lemma A. 2. *Let us define an integro-differential operator $\mathcal{L}$ by*

$$\mathcal{L} := \sum_{j=0}^{m} a_j(t)\partial_x^j \partial_t^{-1}.$$

Then the operator norm of $\mathcal{L}$ on $G_x^{(1/m)}(R_t, R_x)$ is estimated by

$$\|\mathcal{L}\| \le R_t \sum_{j=0}^{m} |a_j|(R_t) R_x^{-j},$$

where for $a(t)=\sum_{n=0}^{\infty}a_nt^n$ we define $|a|(t):=\sum_{n=0}^{\infty}|a_n|t^n$. This estimate shows that for any fixed $R_x>0$, $\mathcal{L}$ becomes a contraction operator on $G_x^{(1/m)}(R_t,R_x)$ by taking small R_t.

Proof. Let

$$g(t,x)=\sum_{l,n\geq 0}g_{l,n}t^lx^n/l!\times n!\in G_x^{(1/m)}(R_t,R_x),$$

and put

$$f(t,x)=t^i\partial_x^j\partial_t^{-1}g(t,x).$$

Then

$$f(t,x)=\sum_{l,n\geq 0}g_{l,n}\frac{t^{l+i+1}x^{n-j}}{(l+1)!\times(n-j)!}=\sum_{l\geq i+1,n\geq 0}g_{l-i-1,n+j}\frac{l!}{(l-i)!}\times\frac{t^lx^n}{l!\times n!}.$$

This shows that

$$\begin{aligned}\left\|t^i\partial_x^j\partial_t^{-1}g\right\|_{R_t,R_x}^{(1/m)}&=\sum_{l\geq i+1,n\geq 0}|g_{l-i-1,n+j}|\frac{l!}{(l-i)!}\frac{R_t^lR_x^n}{(l+n/m)!}\\&=\frac{R_t^{i+1}}{R_x^j}\sum_{l\geq i+1,n\geq 0}|g_{l-i-1,n+j}|\frac{R_t^{l-i-1}R_x^{n+j}}{((l-i-1+(n+j)/m)!}\\&\quad\times\frac{l!}{(l-i)!}\frac{(l-i-1+(n+j)/m)!}{(l+n/m)!}.\end{aligned}$$

Here we notice that

$$\frac{l!}{(l-i)!}\frac{(l-i-1+(n+j)/m)!}{(l+n/m)!}\leq\frac{(l-i+n/m-1+j/m)!}{(l-i+n/m)!}\leq 1,$$

by the assumption $j\leq m$. This proves the following estimate for the operator norm

$$\|t^i\partial_x^j\partial_t^{-1}\|\leq\frac{R_t^{i+1}}{R_x^j},$$

which implies Proposition A. 1. □

A.3. PROOF OF THE NECESSITY

We follow the argument in [10, Sections 3-5], where the Cauchy data $\varphi(x)\in\mathbb{C}\{x\}$ was constructed so that the formal solution of the Cauchy problem (M1) diverges under the similar but weaker condition than (M2) for more general equation.

By the assumption (M2), we write the equation in the form

$$\partial_tu(t,x)=\sum_{i=1}^{k}c_it^{n_i}\partial_x^{p_i}u(t,x)+\sum_{j=0}^{m}t^{l_j}b_j(t)\partial_x^ju(t,x),$$

which satisfies $c_i \in \mathbb{C}\backslash\{0\}$, $1 \le p_1 < p_2 < \ldots < p_k = m$ $(m \ge 2)$ and

$$\frac{p_1}{n_1+1} = \ldots = \frac{p_k}{n_k+1} = \frac{m}{n_m+1}, \quad \frac{m}{n_m+1} > \frac{j}{l_j+1} \quad (0 \le j \le m).$$

Let the formal solution of the Cauchy problem (M1) be given by

$$u(t,x) = \sum_{l=0}^{\infty} u_l(x) \frac{t^l}{l!}.$$

Then the coefficients $\{u_l(x)\}_{l=0}^{\infty}$ $(u_0(x) = \varphi(x))$ are expressed in the form

$$u_l(x) = S_l(\partial_x)\varphi(x), \quad \deg_\xi S_l(\xi) \le l \times \frac{m}{n_m+1} \left(= l \frac{p_k}{n_k+1} \right).$$

More precisely, let us define $L := \{l = \sum_{i=1}^{k} l_i \cdot (n_i+1)\,;\, l_i \in \mathbb{N}\}$. Then we easily examine that

$$l \notin L \quad \Longrightarrow \quad \deg_\xi(S_l(\xi)) < l \times \frac{m}{n_m+1},$$

$$l \in L \quad \Longrightarrow \quad \deg_\xi(S_l(\xi)) \le l \times \frac{m}{n_m+1} = \sum_{i=1}^{k} l_i \cdot p_i.$$

For $l \in L$, we put

$$S_l(\partial_x) = a_l\, \partial_x^{l \cdot m/(n_m+1)} \; + \text{ lower order term}, \quad a_l \in \mathbb{C}.$$

Then the following recurrence formula for $\{a_l\}_{l \in L}$ can be obtained (cf. [10, Sec. 5, (5.4)]):

$$a_l = \sum_{i=1}^{k} \frac{l!}{l \cdot (l-n_i-1)!}\, c_i \cdot a_{l-n_i-1}, \quad a_0 = 1.$$

Then, in [10, Sec. 4, (4.7)], the existence of a subsequence $\{l(j)\}_{j=0}^{\infty} \subset L$ $(l(0) = 0)$ with the following property was proved:

$$|a_{l(j)}| \ge A \frac{c^j}{\{d(n_k+1)\}^{q_j}} \times \frac{l(j)!}{[l(j)/(n_m+1)]!},$$

($[r]$ denotes Gauss' symbol for the integer part of $r \in \mathbb{R}$) by some positive A, c and d and q_j. Here q_j is indefinite but it satisfies $q_j \le [l(j)/(n_m+1)] \le l(j)$.

By admitting these facts, we construct a Cauchy data $\varphi(x)$ of Gevrey order q with $1/m < q < 1$ such that the formal solution of the Cauchy problem (M1) is divergent.

Let us define an entire function of Gevrey order $q < 1$ by

$$\varphi(x) = \sum_{j=0}^{\infty} e^{i\,\theta_j}\, \frac{x^{l(j)\cdot m/(n_m+1)}}{\left[\frac{l(j)\cdot m}{n_m+1}(1-q)\right]!}, \quad \theta_j \in \mathbb{R}.$$

Then we have

$$u_{l(j)}(0) = S_{l(j)}(\partial_x)\varphi(x)|_{x=0} = e^{i\,\theta_j}\, a_{l(j)}\, \frac{\left[\frac{l(j)\cdot m}{n_m+1}\right]!}{\left[\frac{l(j)\cdot m}{n_m+1}(1-q)\right]!} + f_j(\theta_i\,;\, 0 \le i \le j-1),$$

where f_j denotes a function depending on $\{\theta_i\}_{i=0}^{j-1}$. We define the argument θ_j by

$$\theta_j = \arg(f_j) - \arg(a_{l(j)}).$$

Then we have

$$\begin{aligned}|u_{l(j)}(0)| &\ge |a_{l(j)}| \frac{\left[\frac{l(j)\cdot m}{n_m+1}\right]!}{\left[\frac{l(j)\cdot m}{n_m+1}(1-q)\right]!} \\ &\ge A\frac{c^j}{\{d(n_k+1)\}^{q_j}} \times \frac{l(j)!}{[l(j)/(n_m+1)]!} \frac{\left[\frac{l(j)\cdot m}{n_m+1}\right]!}{\left[\frac{l(j)\cdot m}{n_m+1}(1-q)\right]!}.\end{aligned}$$

Hence, for the partial sum $\sum_{j=0}^{\infty} u_{l(j)}(0)t^{l(j)}/l(j)!$, by Stirling's formula, for a small positive constant ε we have

$$\begin{aligned}\frac{|u_{l(j)}(0)|}{l(j)!} &\ge \varepsilon^{1+l(j)} \times \left[l(j)\left\{\frac{m}{n_m+1} - \frac{1}{n_m+1} + \frac{m}{n_m+1}(q-1)\right\}\right]! \\ &= \varepsilon^{1+l(j)} \times \left[l(j) \times \frac{mq-1}{n_m+1}\right]!.\end{aligned}$$

This shows that if $q > 1/m$ the formal solution is divergent. This means that if the Cauchy problem (M1) has always a convergent solution it is necessary that the Cauchy data $\varphi(x)$ is an entire function of Gevrey order $\le 1/m$ which is equivalent that $\varphi(x)$ has exponential growth order at most $m/(m-1)$.

B. PROOF OF LEMMA 4.3

We give the proof of Lemma 4.3.

First, we put

$$B_k(n) = \frac{A_k(n)}{n!^{1/\kappa}}.$$

Then by dividing the both sides of (R_0) or (R_k) by $(n+1)!^{1/\kappa}$, $\{B_k(n)\}$ satisfy the following recurrence formula:

When $k = 0$, one has

$$\begin{cases} B_0(n+1) = \displaystyle\sum_{K_0} a_{\alpha i}[n]_i \frac{(n-i)!^{1/\kappa}}{(n+1)!^{1/\kappa}} B_0(n-i) \quad (n \ge 0), \\ B_0(0) = 1. \end{cases} \tag{S_0}$$

When $k \geq 1$, one has

$$\begin{cases} B_k(n+1) = \sum_{K_0} a_{\alpha i}[n]_i \dfrac{(n-i)!^{1/\kappa}}{(n+1)!^{1/\kappa}} B_k(n-i) \\ \qquad + \sum_{j=1}^{\min\{i_*+1,k\}} \sum_{K_j} a_{\alpha i}[n]_i \dfrac{(n-i)!^{1/\kappa}}{(n+1)!^{1/\kappa}} B_{k-j}(n-i) \quad (n \geq 0), \\ B_k(0) = 0. \end{cases} \tag{S_k}$$

Here we interpret as $B_k(n) = 0$ for all k if $n < 0$, and

$$K_0 = \{(\alpha, i); 0 \leq i \leq i_*, \alpha = i+1\}$$

and

$$K_j = \{(\alpha, i); j-1 \leq i \leq i_*, \alpha = i+1-j\}.$$

Next, we consider the majorant equations for $\{B_k(n)\}$. We note that

$$[n]_i \frac{(n-i)!^{1/\kappa}}{(n+1)!^{1/\kappa}} \leq 1 \quad \left(i \leq n, i_* \text{ and } \kappa = \frac{i_*+1}{i_*}\right)$$

and we put $R := \max\{|a_{\alpha i}|\}$.

We define $\{C_k(n)\}$ by the following recurrence formula:
When $k = 0$, we define $C_0(0) = 1$ and for $n \geq 0$

$$C_0(n+1) = R \sum_{i=0}^{i_*} C_k(n-i).$$

When $k \geq 1$, we define $C_k(0) = 0$ and for $n \geq 0$

$$C_k(n+1) = R \sum_{i=0}^{i_*} C_k(n-i) + R \sum_{j=1}^{\min\{i_*+1,k\}} \sum_{i=j-1}^{i_*} C_{k-j}(n-i).$$

Here we interpret as $C_k(n) = 0$ for all k if $n < 0$. Then we obtain $C_k(n) \geq |B_k(n)|$ for all k and n. Therefore it is enough to show that

$$C_k(n) \leq AB^{n+k} \quad \text{for all } \ k \text{ and } n, \tag{B.1}$$

where $A, B > 1$ and we take B such that

$$R(i_*+1)(i_*+3) \leq B.$$

Case 1. $k = 0$. It is trivial that the inequality (B.1) holds for $n = 0$. We assume that the inequalities (B.1) holds up to n. Then we have

$$C_0(n+1) = R \sum_{i=0}^{i_*} C_0(n-i) \leq R \sum_{i=0}^{i_*} AB^{n-i} \leq AB^n \times R(i_*+1) \leq AB^{n+1}.$$

Case 2. $k \geq 1$. We assume that the inequalities (B.1) holds up to $k-1$.

It is trivial that the inequality (B.1) holds for $n = 0$. We assume that the inequalities (B.1) holds up to n. Then we have

$$\begin{aligned} C_k(n+1) &= R\sum_{i=0}^{i_*} C_k(n-i) + R\sum_{j=1}^{\min\{i_*+1,k\}}\ \sum_{i=j-1}^{i_*} C_{k-j}(n-i) \\ &\leq R\sum_{i=0}^{i_*} AB^{n+k-i} + R\sum_{j=1}^{i_*+1}\sum_{i=j-1}^{i_*} AB^{n+k-i-j} \\ &= AB^{n+k}\times R\Big\{\sum_{i=0}^{i_*} B^{-i} + \sum_{j=1}^{i_*+1}\sum_{i=j-1}^{i_*} B^{-i-j}\Big\} \\ &\leq AB^{n+k}\times R(i_*+1)(i_*+3) \quad (\because B>1) \\ &\leq AB^{n+1+k}. \end{aligned}$$

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