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COLOURINGS OF $(k-r,k)$-TREES

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Abstract. Trees are generalized to a special kind of higher dimensional complexes known as (j,k)-trees ([L.W. Beineke, R.E. Pippert, *On the structure of* (m,n)*-trees*, Proc. 8th S-E Conf. Combinatorics, Graph Theory and Computing, 1977, 75–80]), and which are a natural extension of k-trees for $j=k-1$. The aim of this paper is to study $(k-r,k)$-trees ([H.P. Patil, *Studies on k-trees and some related topics*, PhD Thesis, University of Warsaw, Poland, 1984]), which are a generalization of k-trees (or usual trees when $k=1$). We obtain the chromatic polynomial of $(k-r,k)$-trees and show that any two $(k-r,k)$-trees of the same order are chromatically equivalent. However, if $r \neq 1$ in any $(k-r,k)$-tree G, then it is shown that there exists another chromatically equivalent graph H, which is not a $(k-r,k)$-tree. Further, the vertex-partition number and generalized total colourings of $(k-r,k)$-trees are obtained. We formulate a conjecture about the chromatic index of $(k-r,k)$-trees, and verify this conjecture in a number of cases. Finally, we obtain a result of [M. Borowiecki, W. Chojnacki, *Chromatic index of k-trees*, Discuss. Math. 9 (1988), 55–58] as a corollary in which k-trees of Class 2 are characterized.

Keywords: chromatic polynomial, partition number, colouring, tree.

Mathematics Subject Classification: 05C75.

1. INTRODUCTION

All graphs considered here are finite and simple. We follow the terminology of [2, 10]. Given a graph G, $V(G)$ and $E(G)$ will denote the vertex set and the edge set of G, respectively. The *order* of G is the number of vertices of G. For a labeled graph G of order p, $f(G,t)$ denotes the number of different proper colourings of the vertices of G using either all or some of the colours from a set of t colours with colour difference on each edge of G. It is well-known in the literature that the function $f(G,t)$, which is popularly known as the chromatic polynomial of G, is of the form:

$$f(G,t) = \sum_{m=0}^{p} (-1)^{p-m} a_m \, t^m, \text{ where } a_m \geq 0.$$

A graph is *triangulated* if every cycle of length greater than three possesses a chord.

2. STRUCTURE AND THE VERTEX-PARTITION NUMBER OF $(k-r,k)$-TREES

Multidimensional trees were first introduced by Harary and Palmer [11] and later extended to k-trees (for $k \geq 1$) in [6,8,13,15]. Independently, Dewdney [7] extended the concept of trees and 2-trees to include the more general (j,k)-trees. Beineke, Pippert [1], and Gionfriddo [9] also studied this concept by recursion in terms of k-dimensional complexes with algebraic topological terms. In fact, the concept of (j,k)-trees is the natural extension of k-trees, in the extreme case for $j = k-1$. The aim of this paper is to study and investigate the properties and characterizations of (j,k)-trees, for all j $(0 \leq j \leq k-1)$, in the specialized areas of colourings, in particular, the chromatic polynomials, vertex-partitions, generalized total colourings and the chromatic index.

Now, we begin with the new definition of (j,k)-trees, where j is expressed in terms of k (i.e., $j = k-r$ for any integer r $(1 \leq r \leq k)$ and is defined purely in terms of graph-theoretic terminology, see [13]).

Definition 2.1. Given any two positive integers r and k such that $1 \leq r \leq k$. $(k-r,k)$*-trees* are defined recursively as follows:

1. A complete graph K_{k-r+1} is the smallest $(k-r,k)$-tree.
2. To a $(k-r,k)$-tree H of order p, where $p = k+(i-1)r+1$, $i \geq 0$, add an extra new set of r mutually adjacent vertices by joining each such vertex to all of $(k-r+1)$ mutually adjacent vertices of H, so that the resulting $(k-r,k)$-tree is of order $p+r$.

Note that if $r = 1$, then a $(k-r,k)$-tree is isomorphic to a k-tree. If $r = k$, then a $(k-r,k)$-tree is a $(0,k)$-tree, which is a connected graph with each of its block is isomorphic to K_{k+1}. Figure 1 gives two more examples of $(k-r,k)$-trees for $k = 3$ and $r = 1$ or 2.

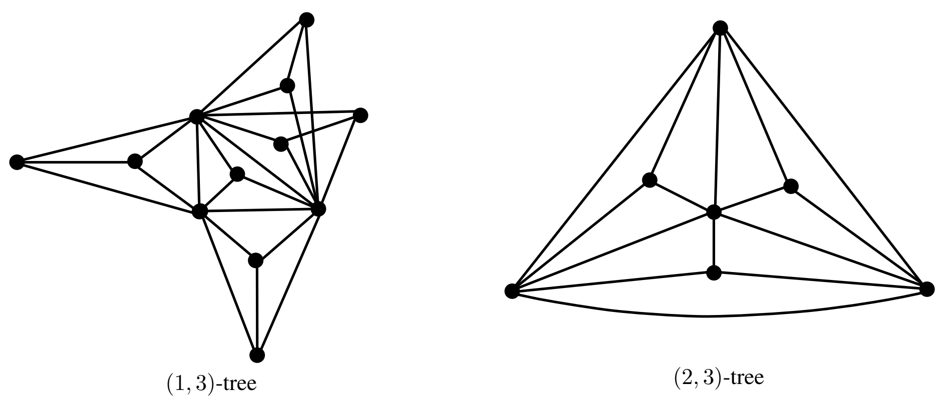

Fig. 1. Examples of a (1,3)-tree and a (2,3)-tree

A subgraph of order m in a graph G is an *m-clique* if it induces a complete subgraph on m vertices. The maximum order of a m-clique of G is a *clique number* of G and is denoted by $\omega(G)$.

Given two graphs G, H and a positive integer ℓ. A graph J is an *ℓ-sum* of G and H if it can be obtained from G and H by identifying the vertices of an ℓ-clique in G with the vertices of an ℓ-clique in H and deleting one edge from each pair of parallel edges. This ℓ-sum of G and H will be denoted by $G \oplus_\ell H$, in short $G \oplus H$, if ℓ is known from the context.

Remark 2.2. Let G be a $(k-r,k)$-tree of order $p = k + (i-1)r + 1$; $i \geq 0$. From Definition 2.1, G can be written as $(k-r+1)$-sum of graphs. Let $G_0 = K_{k-r+1}$, and for $j \geq 1$, let $G_j = G_{j-1} \oplus H_j$, where $H_j = K_{k+1}$. It is clear that G_i is isomorphic to G.

The *Szekeres-Wilf number* $sw(G)$ is defined by $sw(G) = \max\{\delta(H) : H \leq G\}$, where the maximum is taken over all induced subgraphs H of G, and $\delta(H)$ denotes the minimum degree of H.

A graph G is *n-degenerate* for $n \geq 0$, if $sw(G) \leq n$. The following proposition is immediate from Remark 2.2.

Proposition 2.3. *Every $(k-r,k)$-tree G of order $p \geq k+1$, has the Szekeres-Wilf number $sw(G) = k$, the size $|E(G)| = \frac{1}{2}\left(p(2k-r+1) - (k-r+1)(k+1)\right)$ and the clique number $\omega(G) = k+1$.*

A vertex v of a graph G is a *simplicial vertex* if all the vertices adjacent to v in G are mutually adjacent. The following simple characterization of $(k-r,k)$-trees is immediate from Remark 2.2.

Proposition 2.4. *Let G be a graph of order $p \geq k+r+1$; $k \geq r \geq 1$. Then G is a $(k-r,k)$-tree if and only if G has r simplicial vertices $u_1, u_2, \ldots, u_r$, each of degree k such that their union induces an r-clique in G and the subgraph $G - \{u_1, u_2, \ldots, u_r\}$ is a $(k-r,k)$-tree.*

The *vertex-partition number* of a graph G, denoted $\rho_n(G)$; $n \geq 0$, is the minimum number of sets into which $V(G)$ can be partitioned, so that each set induces an n-degenerate subgraph of G.

In [12], Lick and White obtained the following upper bound on $\rho_n(G)$ for any graph G,

$$\rho_n(G) \leq 1 + \left\lfloor \frac{sw(G)}{n+1} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer $\leq x$.

From Proposition 2.3, we have the exact value of the vertex-partition number of a $(k-r,k)$-tree and it is interesting to note that this parameter does not depend on r for any admissible integers n and k.

Proposition 2.5. *Let G be a $(k-r,k)$-tree of order $\geq k+1$. Then*

$$\rho_n(G) = 1 + \left\lfloor \frac{k}{n+1} \right\rfloor.$$

Proof. Our proof starts with the observation that if H is an induced subgraph of G, then $\rho_n(H) \leq \rho_n(G)$. Since $\omega(G) = k+1$, i.e., the complete graph K_{k+1} is an induced subgraph of G, we have $\rho_n(K_{k+1}) = 1 + \left\lfloor \frac{k}{n+1} \right\rfloor \leq \rho_n(G)$. The upper bound of Lick and White, and the fact that $sw(G) = k$ imply $\rho_n(G) \leq 1 + \left\lfloor \frac{k}{n+1} \right\rfloor$. □

3. GENERALIZED TOTAL COLOURING

Definition 3.1. Let $C = \{1, 2, \ldots, n\}$, $\mathcal{O}$ denotes the class of edgeless graphs, and $\mathcal{D}_1$, the class of 1-degenerate graphs, i.e., forests. Then a function $c : V \cup E \to C$ is a *total* $(\mathcal{O}, \mathcal{D}_1)$*-colouring* of G if the following three conditions hold :

1. $G[\{c^{-1}(j)\} \cap V] \in \mathcal{O}$ for all $j \in C$,
2. $G[\{c^{-1}(j)\} \cap E] \in \mathcal{D}_1$ for all $j \in C$,
3. $c(v) \neq c(e) \neq c(u)$ for every edge $e = vu$ of G, i.e., the incident elements of G are coloured differently.

The minimum number of colours needed in a total $(\mathcal{O}, \mathcal{D}_1)$-colouring of G is the *total* $(\mathcal{O}, \mathcal{D}_1)$*-chromatic number* and is denoted by $\chi''_{\mathcal{O},\mathcal{D}_1}(G)$.

An *acyclic* n*-colouring* of a graph G is a proper vertex n-colouring of G satisfying the additional requirement that the subgraph induced by the union of every pair of colour classes is acyclic. The minimum n such that a graph G has an acyclic n-colouring is the *acyclic chromatic number* of G and is denoted by $\chi_a(G)$.

Theorem 3.2 ([4])**.** *If a graph G has an acyclic k-colouring, then G has a total $(\mathcal{O}, \mathcal{D}_1)$-colouring with k colours when k is odd and with $k+1$ colours when k is even.*

Theorem 3.3. *Let G be a $(k-r, k)$-tree of order $\geq k+1$. Then there is a total $(\mathcal{O}, \mathcal{D}_1)$-colouring of G with $k+2$ colours such that for any k, only colours $1, 2, \ldots, k+1$ are used to colour the vertices and edges of G if k is even, while the colours $1, 2, \ldots, k+2$ are used to colour the edges of G if k is odd.*

Proof. Let us construct a $(k-r, k)$-tree G of order $\geq k+1$, as described in Remark 2.2, by using the graphs $H_1, H_2, \ldots, H_i$ in this order, with each H_j being isomorphic to K_{k+1}.

Let $G_1 = H_1$, and for $j = 1, \ldots, i-1$, let $G_{j+1} = G_j \oplus H_{j+1}$.

To colour the vertices of G, we apply the greedy algorithm to colour the vertices in the order in which they are added in the construction of G. First, properly colour the vertices of G_1 in any order (this is possible since G_1 being isomorphic to K_{k+1}). If all the vertices of G_j have been coloured for some j; $1 \leq j \leq i-1$, then G_{j+1} has r uncoloured vertices and they can also be coloured greedily in any order. It now follows that G is $(k+1)$-colourable. Since $\omega(G) = k+1$, it follows that $\chi(G) = k+1$.

Since G has a tree-like structure (by its construction) as mentioned above, every two colour classes induce an acyclic subgraph in G. This property can also be deduced in an inductive way from the fact that, if G_j has this property, then G_{j+1} also has so. Thus, the acyclic chromatic number of G satisfies $\chi_a(G) = k+1$. The result then follows by Theorem 3.2. □

4. THE CHROMATIC POLYNOMIAL OF $(k-r,k)$-TREES

Two graphs are said to be *chromatically equivalent* if they have the same chromatic polynomial. In [15], Skupień proved that $(k-1,k)$-trees (i.e., k-trees) of the same order are chromatically equivalent. Moreover in [8], Dmitriev obtained a stronger result by showing that there exists no graph chromatically equivalent to a $(k-1,k)$-tree not being a $(k-1,k)$-tree. Broader results are obtained independently in [3,6] from which Dmitriev's result follows.

The main purpose of this section is to show that any two $(k-r,k)$-trees of the same order are chromatically equivalent. But if $r \neq 1$, then for any $(k-r,k)$-tree G, there exists a chromatically equivalent graph H, not being a $(k-r,k)$-tree; in other words, Dmitriev's result cannot be extended further for $(k-r,k)$-trees when $r \neq 1$.

A simplicial vertex of degree k in a $(k-r,k)$-tree G is called an *endvertex* of G. We obtain the main results of this paper. Note that $(x)_n$ denotes $[\binom{x}{n} n!]$.

Theorem 4.1. *Let G be a $(k-r,k)$-tree of order p, where $p = k+ir+1$, $i \geq 0$ and $k \geq r \geq 1$. Then*

$$f(G,t) = (t)_{k+1}\ [(t-k+r-1)_r]^i.$$

Proof. Let $Q_r^k(p)$ denote a $(k-r,k)$-tree of order $p = k+ir+1$ for $i \geq 0$. We proceed by induction on p. The result is obvious for $p = k+1$.

Assume that the chromatic polynomial of all $(k-r,k)$-trees of order $p-r$ is given by $(t)_{k+1}[(t-k+r-1)_r]^{i-1}$. In view of Proposition 2.3 and Proposition 2.4, $Q_r^k(p)$ contains a $(k+1)$-clique induced by the union of endvertices $u_1, u_2, \ldots, u_r$, and $v_1, v_2, \ldots, v_{k-r+1}$ vertices in $Q_r^k(p-r)$. By the induction hypothesis, we have

$$[Q_r^k(p) - \{u_1, u_2, \ldots, u_r\}] = Q_r^k(p-r)$$

and

$$f(Q_r^k(p-r), t) = (t)_{k+1}[(t-k+r-1)_r]^{i-1}. \tag{4.1}$$

In a colouring of $Q_r^k(p)$ with t colours, the vertex u_1 can be assigned any colour different from that assigned to $v_1, v_2, \ldots, v_{k-r+1}$, so that u_1 may be coloured in any of the $(t-k+r-1)$ ways and next, the vertex u_2 can be assigned any colour different from that assigned to $v_1, v_2, \ldots, v_{k-r+1}$ and u_1, so that u_2 may be coloured in $(t-k+r-2)$ ways. Continuing this process until there remains no u_js, and ultimately, the last vertex u_r is assigned any colour different from that assigned to $v_1, v_2, \ldots, v_{k-r+1}, u_1, u_2, \ldots, u_{r-2}$, and u_{r-1}. Hence, u_r may be coloured in any of the $(t-k)$ ways. Thus, we have

$$\begin{aligned} f(Q_r^k(p), t) &= (t-k+r-1)(t-k+r-2) \times \ldots \times (t-k) f(Q_r^k(p-r), t) \\ &= (t)_{k+1}[(t-k+r-1)_r]^i \quad \text{from (4.1).} \end{aligned}$$

□

Theorem 4.2. *For any $(k-r,k)$-tree G of order $p \geq k+r+1$; $k \geq r \geq 2$, there exists a graph H not being a $(k-r,k)$-tree, which is chromatically equivalent to G.*

Proof. Let G be a $(k-r,k)$-tree of order $p = k + ir + 1$; $2 \leq r \leq k$ and $i \geq 1$. By Remark 2.2, G has at least two $(k+1)$-cliques, in which some contain r endvertices. Let H be a graph obtained from G by removing a fixed endvertex u, and adding a new vertex u', joining it to any k vertices from one of these $(k+1)$-cliques, to which u does not belong. Certainly, this resulting graph H is triangulated, and not isomorphic to any $(k-r,k)$-tree G. It is well-known that every triangulated graph (not necessarily being a $(k-r,k)$-tree), say F, has the chromatic polynomial of the form: $t^{n_0}(t-1)^{n_1} \times \ldots \times (t-s)^{n_s}$, where $n_j \neq 0$ and each $n_j : 0 \leq j \leq s$, is the multiplicity of degree j of a simplicial vertex in a perfect vertex elimination order for F. By this fact, and by the above mentioned construction, both G and H are triangulated and they have the same n_js $(0 \leq j \leq k)$. Hence, G and H are chromatically equivalent. □

Remark 4.3. We give an example which illustrates Theorem 4.2. Consider the $(1,3)$-tree G of order 6 and the graph H as shown in Figure 2 (its construction from G is as indicated in the proof of Theorem 4.2). The chromatic polynomials of both G and H can be easily computed and are the same polynomial as follows:

$$f(G,t) = f(H,t) = t(t-1)(t-2)^2(t-3)^2 = (t)_4(t-2)_2.$$

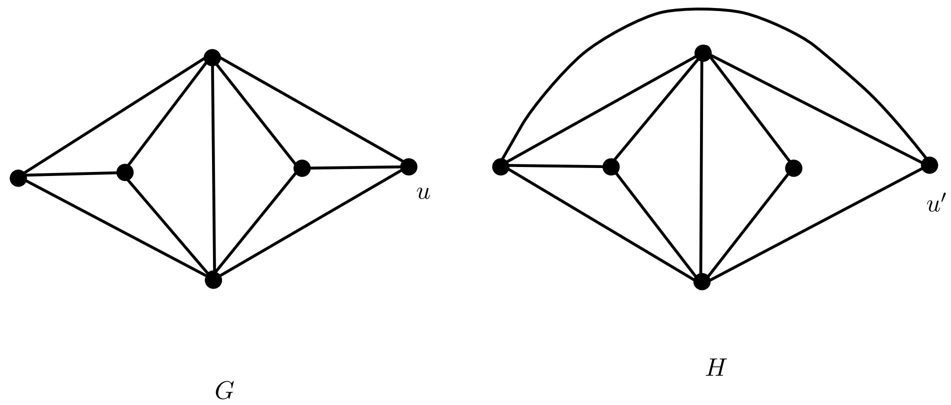

Fig. 2. Graphs G and H considered in Remark 4.3

5. CHROMATIC INDEX

Let $\chi'(G)$ denote the *chromatic index of* G, i.e., the least number of colours required to colour the edges of G in such way that any two adjacent edges have different colours. Vizing [16] showed that $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$. Graphs G for which $\Delta(G) = \chi'(G)$ holds are of *Class* 1 and otherwise they are of *Class* 2.

Proposition 5.1 ([17])**.** *If $\Delta(G) \geq 2sw(G)$, then G is of Class 1.*

Proposition 5.2 ([14])**.** *If G is a graph of order $p = 2s$ and $\Delta(G) = 2s-1$, then G is of Class 1. Let G be a graph of order $p = 2s+1$ and $\Delta(G) = 2s$. Then G is of Class 2 if and only if G has at least $2s^2+1$ edges.*

Lemma 5.3. *Let G be a $(k-r,k)$-tree, $k \geq 2$, of order p and has a spanning star. Then G is of Class 2 if and only if p is odd and*

$$p \leq \varphi(k,r) = k+1+\frac{1}{2}(\sqrt{(r-1)^2+4(k-2)}-(r-1)).$$

Proof. From Proposition 5.2, G with a spanning star is of Class 2 if and only if G has an odd order $p = 2s+1$ and at least $2s^2+1$ edges. Hence,

$$\begin{aligned} |E(G)| &= \frac{1}{2}\left(p(2k-r+1)-(k-r+1)(k+1)\right) \\ &= \frac{1}{2}\left((2s+1)(2k-r+1)-(k-r+1)(k+1)\right) \geq 2s^2+1. \end{aligned}$$

This implies $4s^2-2s(2k-r+1)+(k^2-rk+2) \leq 0$.
From this we have,

$$p \leq k+1+\tfrac{1}{2}(\sqrt{(r-1)^2+4(k-2)}-(r-1)).$$

□

Conjecture 5.4. *Let G be a $(k-r,k)$-tree, $k \geq 2$, of order $p \geq k+1$. Then G is of Class 2 if and only if $p \leq \varphi(k,r) = k+1+\frac{1}{2}(\sqrt{(r-1)^2+4(k-2)}-(r-1))$ and p is odd.*

Lemma 5.5. *Every $(k-r,k)$-tree G of order p has a spanning star if and only if $k = tr+a$, $t \geq 2$, $0 \leq a \leq r-1$ and the following two conditions hold.*

1. *$p \leq 2k+1-a$, if $r \leq \frac{k}{2}$.*
2. *$p \leq k+1+r$, if $r > \frac{k}{2}$.*

Proof. (1) Let $r \leq \frac{k}{2}$. Define $(k-r,k)$-trees G_j of order $p_j = (k-r+1)+jr$, $j \geq 0$, inductively. Let $\ell = n-r+1$, $G_0 = K_{n-r+1}$, with $V(G_0) = Y_0$, $G_1 = G_0 \oplus_\ell K_{k+1}$ with $V(G_1) = X_1 \cup Y_1$, where $Y_1 = Y_0$, $|X_1| = r$ and $|V(G_1)| = p_1 = (k-r+1)+r$.

Let $G_2 = G_1 \oplus_\ell K_{k+1}$ with $V(K_{k+1}) = X_2 \cup Y_2$, $X_2 \cap V(G_1) = \emptyset$, $Y_2 = Y_1$. Obviously, $|X_2| = r$ and $|V(G_2)| = p_2 = (k-r+1)+2r$. It is clear that G_2 has a spanning star, and G_2 is a unique, up to isomorphism, $(k-r,k)$-tree of order $k+1+r$.

For $j \geq 3$, let G_j be defined as follows: $G_j = G_{j-1} \oplus_\ell K_{k+1}$, where $V(K_{k+1}) = X_j \cup Y_j$, $|X_j| = r$, $X_j \cap V(G_{j-1}) = \emptyset$. Obviously, $|V(G_j)| = p_j = (k-r+1)+jr$.

Let n be the smallest number such that G_n does not contain a spanning star. Consider a $(k-r,k)$-tree G_3 which, by Remark 2.2, is obtained from G_2 and is described above. By the construction of G_2, if $x_1 \in X_1$ and $x_2 \in X_2$, then $x_1x_2 \notin E(G_2)$. Thus, without loss of generality, we can assume that $Y_3 \subseteq X_2 \cup Y_2$. Observe that the center of every spanning star of G_2 is in the set Y_0. To optimize n, we have to assume that $Y_3 = X_2 \cup Z_3$, where $Z_3 \subseteq Y_0$ and $|Z_3| = (k-r+1)-r = k+1-2r$.

Since $r \leq \frac{k}{2}$, $Y_0 \setminus Z_3 \neq \emptyset$. Thus, G_3 has a spanning star and every spanning star has a center in Z_3. Since $k = tr + a$, the graph G_j for $j = t + 1$, has a spanning star, and there is a graph G_{t+2} without a spanning star. Thus, G_{t+1} is of the order $p_{t+1} \leq (k - r + 1) + (t + 1)r = 2k + 1 - a$.

(2) Let $r > \frac{k}{2}$. Since a $(k - r, k)$-tree of order $k + 1 + r$, G_2 as mentioned above, is uniquely constructed, and now we consider the possible graph G_3 from G_2 by $(k - r + 1)$-sum. Since $k - r + 1 \leq r$, there is a $(k - r, k)$-tree of order $k + 1 + 2r$ without a spanning star. □

Theorem 5.6. *Let G be a $(k - r, k)$-tree, $k \geq 2$ of order $p \geq k + 1$ and $k = tr$, $t \geq 2$ (i.e., $r \leq k/2$). Then G is of Class 2 if and only if $p \leq \varphi(k, r)$ and p is odd.*

Proof. Let G be a $(k - r, k)$-tree of order $p \leq \varphi(k, r)$ and p is odd. Then we have $p \leq 2k + 1$. By Lemma 5.5, G has a spanning star, and by Lemma 5.3, G is of Class 2.

Let $p > \varphi(k, r)$. Then we have $p > 2k + 1$. If G has a spanning star, then by Lemma 5.3 G is of Class 1. Suppose that G does not have a spanning star. From the proof of Lemma 5.5, it follows that $G_{t+1} \subseteq G$, but $\Delta(G) \geq \Delta(G_{t+1}) = 2k = 2sw(G)$. Thus, by Proposition 5.1, G is of Class 1. □

Let G be a $(k - r, k)$-tree such that every vertex v of G belongs to at most two $(k + 1)$-cliques; that is, $d_G(v) \in \{k, k + r\}$. Then we call G a *simple $(k - r, k)$-tree.*

Theorem 5.7. *Let G be a $(k - r, k)$-tree and $r > \frac{k}{2}$. Then Conjecture 5.4 is true if*

1. *G is not a simple $(k - r, k)$-tree of order $p \geq k + 1 + 2r$ or*
2. *$k = r$ or*
3. *$k + 1 = 2r$.*

Proof. (1) Let $p \leq \varphi(k, r)$ and p is odd. It implies that $\varphi(k, r) < k + r$. Then by Lemma 5.3, G has a spanning star. Obviously in this case, $G = K_{k+1}$ and if G is of odd order, then G is of Class 2.

If $p = k + 1 + r$, then although G has a spanning star, the order of G is $p > \varphi(k, r)$. Hence, by Lemma 5.3, G is of Class 1.

Let $p > k + 1 + r$. Obviously, $p > \varphi(k, r)$. If G has either a spanning star or $\Delta(G) \geq 2k$, then G is of Class 1.

Suppose that G does not contain a spanning star and $\Delta(G) < 2k$. Since G is not a simple $(k - r, k)$-tree, there is a vertex v in G which belongs to at least three $(k + 1)$-cliques. Hence, $\Delta(G) \geq d_G(v) \geq k + 2r > 2k = 2sw(G)$. Thus, G is of Class 1.

(2) Let $k = r$. Then $\Delta(G) = 2k = 2sw(G)$. Thus, G is of Class 1.

(3) Let $k + 1 = 2r$. It easy to see that G has a path structure formed by j $(k + 1)$-cliques. If $j \leq 2$, then G has a spanning star, and by Lemma 5.3, G is of Class 1. If $j \geq 3$, then G does not contain a spanning star. According to Remark 2.2, G can be represented by an ℓ-sum in the following way: $G = H_1 \oplus_\ell H_2 \oplus_\ell \cdots \oplus_\ell H_j$, where $H_i = K_{2r}$, $\ell = r$.

Case 1. Let r be even. Removing all edges which belong to ℓ-cliques and all edges which join endvertices of G (they form a *pendant* clique K_r), we have a bipartite graph G'. Obviously, $\chi'(G') = \Delta(G') = 2r$. Observe that the ℓ-cliques are vertex disjoint and thus edge-disjoint. They are also disjoint with both r-cliques formed by endvertices

of G. Since r is even, to colour edges of these r-cliques, we need $r-1$ new colours. Hence, the edges of G can be coloured with $3r-1=\Delta(G)$ colours. Thus, G is of Class 1.
Case 2. Let r be odd. Colour the edges of all ℓ-cliques and of both pendant cliques K_r with colours $\{1,2,\dots,r\}$. Since r is odd, for any r-colouring of the edges of K_r the missing colours at r vertices are all different and exactly one at each vertex. Thus, we can colour with colours $\{1,2,\dots,r\}$ a perfect matching between H_1 and H_2, H_3 and H_4 and so on. Uncoloured edges of G induce a bipartite graph G'' with $\Delta(G'')=2r-1$. Obviously, the edges of G'' can be coloured with new $2r-1$ colours. Hence, all edges of G are coloured properly with $\Delta(G)=3r-1$ colours. Thus, G is of Class 1. □

For $r=1$, Theorem 5.6 gives the following characterization of Class 2, k-trees.

Theorem 5.8 ([5])**.** *Let G be a k-tree $(k\geq 2)$ of order p, $p\geq k+1$. Then G is of Class* 2 *if and only if $p\leq k+1+\sqrt{k-2}$ and p is odd.*

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