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Dynamic displacement tracking in viscoelastic solids by actuation stresses: a one-dimensional analytic example involving shock waves

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Abstract. A one-dimensional (1D) analytic example for dynamic displacement tracking in linear viscoelastic solids is presented. Displacement tracking is achieved by actuation stresses that are produced by eigenstrains. Our 1D example deals with a viscoelastic half-space under the action of a suddenly applied tensile surface traction. The surface traction induces a uni-axial shock wave that travels into the half-space. Our tracking goal is to add to the applied surface traction a transient spatial distribution of actuation stresses such that the total displacement of the viscoelastic half-space coincides with the shock wave produced by the surface traction in a purely elastic half-space. We particularly consider a half-space made of a viscoelastic Maxwell-type material. Analytic solutions to this tracking problem are derived by means of the symbolic computer code MAPLE. The 1D solution presented below exemplifies a formal 3D solution derived earlier by the present authors for linear viscoelastic solids that are described by Boltzmann hereditary laws. In the latter formal solution, no reference was made to shock waves. Our present solution demonstrates its validity also in the presence of singular wave fronts. Moreover, in our example, we show that, as was also indicated in our earlier work, the actuation stress can be split into two parts, one of them producing no stresses, and the other no displacements in two properly enlarged problems.

Key words: displacement tracking; eigenstrains; actuation stresses; linear viscoelasticity; shock waves; symbolic computation.

1. INTRODUCTION

This study is concerned with a special topic in structural displacement control, namely with the tracking, or morphing, of force-induced dynamic displacements in linear viscoelastic structures by means of actuating eigenstrains. In displacement tracking, one seeks a transient spatial distribution of additional eigenstrains, such that the total displacement of the structure coincides with the displacement to be tracked. In the case of zero displacement tracking, one talks about shape control. In the last decades, these questions have emerged in the field of smart or intelligent structures, in which eigenstrains are utilized in a problem-oriented manner for actuation, but also for sensing purposes. In the fundamental relations of mechanics, eigenstrains do appear in the constitutive relations, while applied forces are to be introduced via the relation of the balance of linear momentum. Similarities and differences in the actions of applied forces and actuating eigenstrains form a topic in its own right, which is of particular interest in structural control.

The theory of eigenstrains has a long history. A prominent example is the case of thermal actuation in solids, see e.g. Parkus [1], and Nowacki [2] for thermally induced dynamic displacements and stresses. The notions eigenstrain, incompatible deformation and distortion are closely related. Virtual distortions recently have been used successfully in structural control and smart technologies, see Holnicki-Szulc and Gierlinski [3], and Holnicki-Szulc [4]. Concerning control of smart structures, piezoelectric eigenstrains are frequently utilized in practice, since their origin, the electric field, can be initiated almost immediately and accurately, in contrast to the case of deformations stemming from temperature changes. For fundamentals of the piezoelectric effect in the dynamics of solids, see e.g. Nowacki [5]. The present contribution particularly deals with eigenstrain-induced dynamic displacement tracking of linear viscoelastic structures, assuming geometrically linearized conditions, i.e. sufficiently small deformations, to be present. In this linear setting, it appears to be justified to neglect the influence of eigenstrains upon the material parameters in the constitutive relations. However, in the case of comparatively rapid deformations, the coupling between eigenstrains and the rate of strains may become important. This coupling stems from the fact that both quantities do appear in relation to the balance of total energy. Independently of considering such a coupling, eigenstrains do form an extra term in the constitutive relations in the linear theory. In the latter, let the stress to expressed as a function of strain and eigenstrain. We then denote the extra term containing the eigenstrain as an actuation stress. In a linear elastic material, the actuation stress is a linear mapping of the respective source of eigenstrains in the form of a tensorial product, e.g. in the piezoelectric case the mapping of the electric field vector onto the stress via a time-independent third-order tensor of material parameters. The situation is more involved in the viscoelastic case, where the actuation stress in general represents a linear mapping in the form

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of a hereditary integral, i.e. of a convolution in time. See e.g. Fung, Tong, and Chen [6] for a thorough discussion on the appearance of thermally and electrically induced eigenstrains in the constitutive relations of linear elastic and linear viscoelastic solids.

In any case, the indispensable step in the solution of displacement tracking problems is to find a suitable transient spatial distribution of the actuation stress, which enforces the total of the displacements due to the imposed forces and due to the eigenstrains to coincide with the displacement field to be tracked. The present contribution is concerned with this step; the problem of determining the necessary eigenstrains once the corresponding actuation stress distribution has been found is not treated here. This second step is trivial in case of an uncoupled linear elastic behaviour, or when the hereditary behaviour in the viscoelastic actuation stress can be disregarded approximately. Otherwise, more involved studies would be necessary, which are left to future investigations. Here, as a contribution to the linear theories of elasticity and viscoelasticity, we concentrate on the relations between the actions of imposed forces and eigenstrains by referring to actuation stresses only. The present authors and their co-workers have contributed to this problem in a series of papers on shape control and displacement tracking, however mostly for linear elastic materials with eigenstrains. Various solution techniques have been developed by our group, such as utilizing the Neumann integral technique, a method which is suitable also for treating the case of tracking small displacements superimposed upon large predeformations [7]; see the literature cited in the latter reference for some previous contributions of our group on shape control and displacement tracking in linear elastic solids and structures. A theory for displacement tracking by eigenstrains in three-dimensional (3D) linear viscoelastic bodies that are described by Boltzmann hereditary laws has been presented by the present authors in [8]. The latter formulation makes use of uniqueness theorems of the linear theory of viscoelasticity in the absence of eigenstrains, see Leitman and Fisher [9] for a comprehensive article.

The goal of our present paper is to exemplify our earlier findings in [8], which was a purely theoretical study, by means of an analytic (closed form) one-dimensional (1D) example: we study the uni-axial deformation of a viscoelastic half-space under the action of a suddenly applied tensile surface traction. The surface traction induces a uni-axial shock wave that propagates into the half-space. The tracking goal in our example is to add to the applied surface traction a transient spatial distribution of actuation stresses such that the total displacement of the viscoelastic half-space coincides with the shock wave produced by the surface traction in a purely elastic half-space. We particularly consider a half-space made of a viscoelastic Maxwell-type material. Analytic solutions to this special tracking problem are derived by means of the symbolic computer code MAPLE [10], using the release of 2017. Our present analytic solution demonstrates the validity of the displacement tracking formulation in [8] and also in the presence of singular wave fronts, which originally were not addressed in [8]. Moreover, we exemplarily demonstrate that, as was also indicated in our earlier work [8], the actuation stress can be split into two parts, one of them producing no stresses, and the other no displacements in properly enlarged problems.

2. 1D EXAMPLE PROBLEM

Consider a half-space with axial coordinate $x \geq 0$ in a state of uni-axial deformation. The axial displacement is denoted as $u_x = u$, and the axial normal Cauchy stress component is abbreviated by $\sigma_{xx} = \sigma$. Subsequently, we consider the linear theory of viscoelasticity. We restrict to the case of a vanishing initial history up to, but not including time $t = 0$

$$t < 0: \quad u = 0, \quad \dot{u} = 0. \tag{1}$$

A superimposed dot denotes the derivative with respect to time t. Using the Boltzmann hereditary integral formulation, see Leitman and Fisher [9], the constitutive relation for σ in the present uni-axial case can then be written as

$$\sigma = Y_0 \varepsilon(t) + \int_0^t Y'(s)\varepsilon(t-s)\,\mathrm{d}s - \sigma_A\,, \tag{2}$$

in which $Y_0 = Y(t=0)$ stands for an effective initial Young's modulus, and $Y(t)$ is a time-dependent relaxation function. The abbreviation $Y'(s) = \partial Y/\partial s$ is used. The linearized axial strain is abbreviated by $\varepsilon_{xx} = \varepsilon$, where

$$\varepsilon = u_{,x}\,, \tag{3}$$

the spatial derivative being indicated by $()_{,x} = \partial()/\partial x$. In equation (2) we have included an actuation stress σ_A, which represents the influence of imposed eigenstrains; for a short discussion, see the Introduction given above. As mentioned, eigenstrains may be coupled to the displacement, but we will assume that any influence of the eigenstrains upon the relaxation function Y can be neglected. This is an assumption, which appears to be often reasonable in the linear context. In the case of a Maxwell viscoelastic material, which will be considered subsequently, we have

$$Y(t) = Y_0 e^{-t/\tau}. \tag{4}$$

The relaxation time is denoted by $\tau = (Y_0\mu)^{-1}$, see Leitman and Fisher [9], who used the parameter $\nu = \mu^{-1}$. The viscosity parameter μ vanishes in the case of purely elastic behaviour. Equations (1)–(4) accompany the axial relation of the balance of linear momentum, which, in the absence of imposed body forces, reads

$$\sigma_{,x} = \rho \ddot{u}. \tag{5}$$

Mass density is written as ρ. In the following, we consider a jump in the surface traction, which is applied normally to the free surface. This gives the following boundary condition

$$x = 0: \quad \sigma = \sigma_0 \mathrm{H}(t). \tag{6}$$

The Heaviside jump function is abbreviated by H, and σ_0 is a constant. The boundary condition in equation (6) is discontinuous in time, giving rise to a singular surface of order one, which travels into the inner of the half-space. At a singular surface of order one, the first derivatives of the displacement may suffer discontinuities, i.e. jumps, across the singular surface. Such a displacement field is also denoted as a shock wave. The jumps at the shock front must satisfy certain relations. The latter represent consequences of the balance of linear momentum and thus are independent from the constitutive relations at hand, see Gurtin [11] for a thorough discussion. In the present 1D case, the jump relation for a shock wave can be written in a scalar form as

$$x_1: \quad [\![\vartheta]\!] = 0, \tag{7}$$

where the modified stress field ϑ is

$$\vartheta = \sigma + \rho c \dot{u}. \tag{8}$$

The instantaneous place of the shock front is denoted as x_1. Jumps are indicated by a double rectangular bracket, meaning the difference between the values of some entity immediately before and behind the propagating shock front. The speed of propagation of this singular surface is denoted as c. For a linear viscoelastic material, c corresponds to the elastic speed of propagation, $c = (Y_0 \rho^{-1})^{1/2}$, see Leitman and Fisher [9]. Since the shock front in our case travels into the virgin material, see equation (1), the relation of jump in equation (7) can be replaced by the requirement

$$x_1: \quad \vartheta = 0, \tag{9}$$

which we will use for visualizing the fulfilment of equation (7) later on. In the present study on displacement tracking by actuation stresses, we will restrict to fields σ_A that are associated with a shock front located at x_1 only. Equations (1)–(9) represent the viscoelastic framework under consideration. Due to the linearity of this problem, we split the total solution into a part due to the applied surface traction σ_0, indicated by a subscript t, and a part due to the additionally imposed actuation stress σ_A, with this part being indicated by the subscript a

$$u = u_t + u_a, \quad \sigma = \sigma_t + \sigma_a. \tag{10}$$

3. DISPLACEMENT TRACKING PROBLEM

In displacement tracking, also denoted as structural morphing, one intends to apply an actuation stress σ_A such that the total displacement u is equal to some admissible field z, despite applied forces being present. In our 1D problem, we thus require that there is

$$u = u_t + u_a = z, \tag{11}$$

despite the presence of the imposed surface traction σ_0. In order to be admissible, z cannot be chosen freely but must satisfy certain kinematic conditions, depending on the problem at hand. In our 1D half-space case, z should vanish, when x tends to infinity. Wave fronts present in z, if there are any, should not travel faster into the half-space, as this is physically possible. Also, we will require that z represents a 1D deformation having a vanishing initial history, likewise to u. Particularly, we seek an actuation stress σ_A, such that

$$u = z = u_{\mathrm{tel}}, \tag{12}$$

where u_{tel} represents the displacement that would be produced by the surface traction σ_0, when the latter is applied to a purely elastic half-space, i.e. in the case of $\mu = 0$. Obviously, u_{tel} does satisfy the kinematic requirements for z mentioned above. With equation (12), we here ask for actuation stresses that act additionally upon a viscoelastic solid in order to produce a purely elastic solution, as if the half-space would behave elastically. The complementary problem, namely to find additional actuation stresses acting upon an elastic solid, such that an inelastic solution is generated, has been treated by the senior author and co-workers before, where numerical computation schemes were developed, particularly for elasto-viscoplastic problems, see [12] and the literature cited there. In the present linear viscoelastic displacement tracking framework, a general 3D formulation has been derived by the present authors earlier in [8], from which conditions can be derived for the 1D actuation stress distribution σ_A that must be applied to the half-space in addition to σ_0, such that equations (11) and (12) are satisfied. In the present paper, the 3D formulations stated in [8] reduce to the following 1D quasi-static boundary value problem

$$\sigma_{A,x} - \left(Y_0 z_{,xx}(t) + \int_0^t Y'(s) z_{,xx}(t-s)\,\mathrm{d}s - \rho \ddot{z} \right) = 0, \tag{13}$$

$$x = 0: \quad \sigma_A = -\sigma_0 \mathrm{H}(t) + Y_0 z_{,x}(t) + \int_0^t Y'(s) z_{,x}(t-s)\,\mathrm{d}s. \tag{14}$$

We will demonstrate the validity of equations (13) and (14) by providing analytic solutions to our example problem next.

4. SOLUTION OF THE DISPLACEMENT TRACKING PROBLEM

We first compute an analytic solution for the viscoelastic displacements u_t due to the action of the applied surface traction σ_0 alone, i.e. taking actuation stresses to be absent. We consider Maxwell material according to equation (4). To the best knowledge of the authors, analytic solutions to this problem have not been treated in the open literature so far. The problem, however, is complementary to the problem of a prescribed uni-axial surface velocity, see Lee and Kanter [13], who derived analytic solutions by hand for the corresponding stress field using the Laplace transform technique, see e.g. Churchill [14] for a comprehensive discussion of the latter. Lee and Kanter [13] also suggested a non-dimensional formulation, in which the relaxation time τ is hidden, such that the dimensionless stresses are independent of τ. Although this non-dimensional formulation is elegant and useful, we do not use it in the following paper since we seek additional actuation stresses, such that the total viscoelastic displacement does coincide with the elastic one due to the surface traction σ_0 alone. We, therefore, utilize the dimensional relations given in Section 2 above directly. Solutions to the corresponding boundary value problem with vanishing initial history are obtained utilizing the symbolic computer code MAPLE. Some remarks concerning the course of computations appear to be in order. The MAPLE package *PiecewiseTools* is used to express piecewise continuous functions. This is the case here, cf. equation (6); the use of generalized functions implemented in MAPLE, such as *Heaviside* and *Dirac*, allows us to seek a solution without making direct reference to the relation of jump at the shock front, see equation (7), which should be satisfied automatically then. The condition stated in equation (9) for the vanishing of the modified stress field ϑ is used in order to prove this. The problem is transformed into the Laplace domain using the MAPLE package *inttran*s. The solution in the Laplace domain is obtained by the MAPLE package *DETools*, using the command *dsolve*. The command *assume* is used in order to introduce the range of possible values for certain parameters. Care must be given to the fact that the solutions should not become unbounded when the coordinate x tends to infinity. Applying the command *odetest* in order to check the validity of the solution in the Laplace domain is helpful. Transformation to and re-transformation from the Laplace domain is performed using the *inttran*s commands *laplace* and *invlaplace*. It turns out that, at least in our hands, closed-form solutions cannot be obtained by MAPLE completely automatically with these commands, but additional manipulations must be introduced. Particularly, it is necessary to introduce theorems on substitution, convolution, and re-transformation of products of functions, see e.g. Churchill [14], as well as to make identical expansions, in order to be able to utilize proper standard re-transformation rules, such as Equations (92) and (93) in [14]. The latter ones are at the disposal of MAPLE. Eventually, for the Maxwell material, see equation (4), the displacement field due to σ_0 is obtained analytically in the form of the following convolution integral

$$u_t = \frac{\sigma_0}{Y_0} c \int_0^t \left((s-t)\mu Y_0 - 1 \right) \mathrm{H}\left(\frac{sc-x}{c} \right) \times \mathrm{e}^{-\frac{s\mu Y_0}{2}} \mathrm{I}_0 \left(\frac{\sqrt{c^2 s^2 - x^2}\,\mu Y_0}{2c} \right) \mathrm{d}s. \tag{15}$$

This result involves the modified Bessel function of the first kind I_0. The elastic limit, $\mu = 0$, becomes

$$u_{\mathrm{tel}} = -\frac{\sigma_0}{Y_0}(tc-x)\mathrm{H}\left(\frac{tc-x}{c} \right). \tag{16}$$

In our hands, MAPLE did not yield further closed-form simplifications of the convolution integral in equation (15), but the integral can be easily evaluated with the help of the MAPLE package *plots*. The velocity field $\dot{u}_t$ and the stress field σ_t are computed by MAPLE analogous to the analytic procedure sketched above. The velocity is derived directly by differentiation of equation (15); for the stresses, it turns out to be preferable to derive and solve their own boundary value problem, similar to the one that was described by Lee and Kanter [13] for their related problem. In the present case, the following more involved result is obtained

$$\begin{aligned} \sigma_t = \sigma_0 \mathrm{e}^{-\frac{t\mu Y_0}{2}} \Bigg(& \frac{\mu Y_0}{2} \int_0^t \mathrm{I}_0 \left(\frac{\sqrt{c^2(s-t)^2 - x^2}\,\mu Y_0}{2c} \right) \\ & \times \left(\mathrm{I}_0\left(\frac{s\mu Y_0}{2} \right) + \mathrm{I}_1\left(\frac{s\mu Y_0}{2} \right) \right) \mathrm{H}\left(\frac{(t-s)c-x}{c} \right) \mathrm{d}s \\ & - \mathrm{I}_0 \left(\frac{\sqrt{c^2 t^2 - x^2}\,\mu Y_0}{2c} \right) \left(\mathrm{H}(x-tc) - 1 \right) \Bigg). \end{aligned} \tag{17}$$

The elastic stress limit becomes

$$\sigma_{\mathrm{tel}} = -\sigma_0 \left(\mathrm{H}(x-tc) - 1 \right). \tag{18}$$

Displacements, stresses, and modified stresses are presented in Figs. 1–5. Considering consistent dimensions, the following artificial input parameters are utilized: $\sigma_0 = 1$, $Y_0 = 1$, $\mu = 1$ and $\rho = 1$; results are depicted for three time-instants: $t = 0.5$ (blue), $t = 1.5$ (black), and $t = 2.5$ (red). Figure 1 shows the displacement u_t, and Fig. 2 the elastic limit u_{tel}. Figure 3 presents the stress σ_t, and Fig. 4 the elastic limit σ_{tel}, while Fig. 5 gives the modified stress ϑ_t. It is seen from Fig. 5 that the relations of

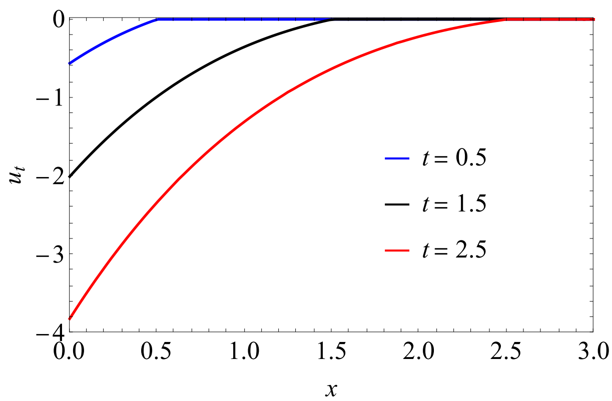

Fig. 1. Displacement due to σ_0 for three time-instants

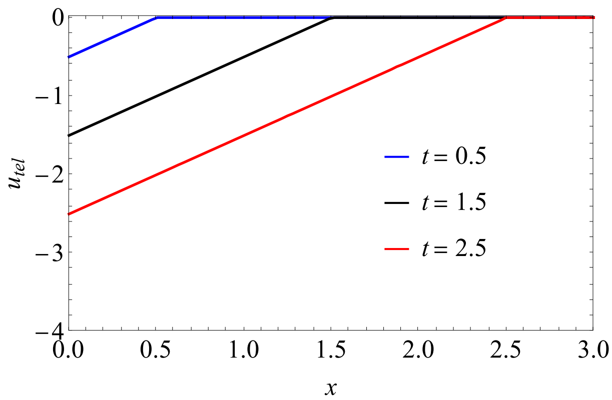

Fig. 2. Elastic displacement due to σ_0 for three time-instants

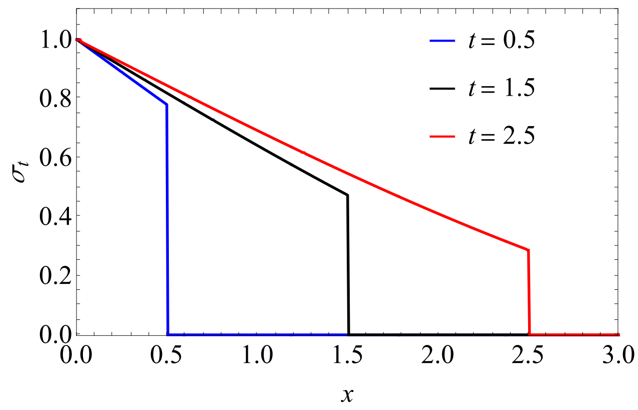

Fig. 3. Stress due to σ_0 for three time-instants

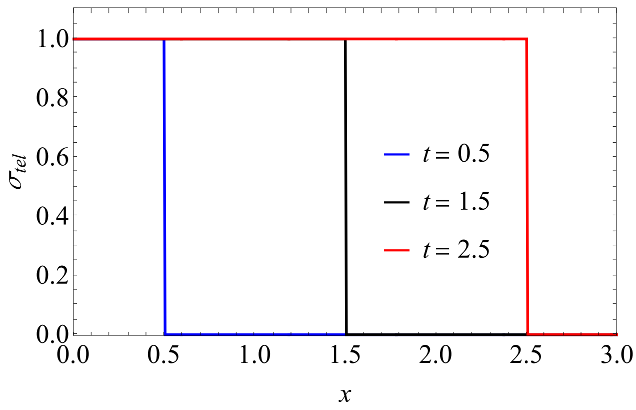

Fig. 4. Elastic stress due to σ_0 for three time-instants

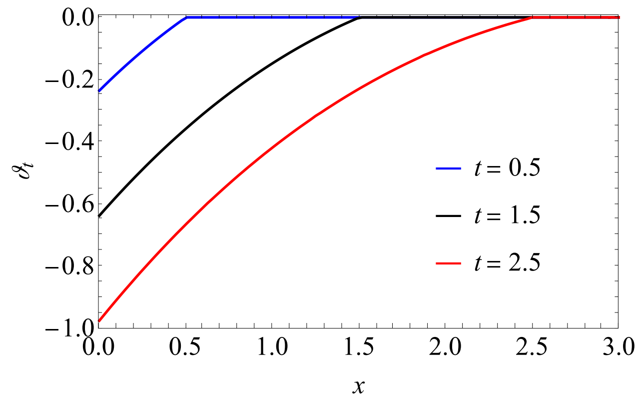

Fig. 5. Modified stress due to σ_0 for three time-instants

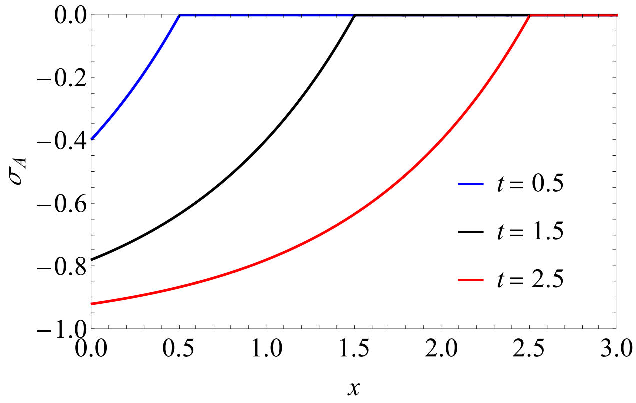

Fig. 6. Actuation stress σ_A for three time-instants

jump at the viscoelastic shock front, cf. equations (7)–(9), are indeed satisfied.

Note that in the viscoelastic case, the absolute values of the displacements at the surface $x = 0$ are larger than in the elastic one, which may motivate the displacement tracking problem under consideration, equation (12). The next step is to compute the solution of the quasi-static boundary value problem stated in equations (13) and (14) for the actuation stress σ_A, having substituted the elastic displacement u_{tel} from equation (16) for the displacement z to be tracked, see equation (12). Symbolic computation yields

$$\sigma_A = \sigma_0 \mathrm{H}\left(\frac{tc - x}{c}\right)\left(\mathrm{e}^{\frac{-(tc-x)Y_0\mu}{c}} - 1\right). \tag{19}$$

This analytic result is depicted for the above-stated input parameters and time-instants in Fig. 6.

The displacements u_a produced by the actuation stress σ_A of equation (19), taking a stress-free surface, $\sigma_a = 0$ in $x = 0$, are computed again symbolically by MAPLE. As this should be, cf. equations (11) and (12), it indeed turns that

$$u_a = u_{\text{tel}} - u_t\,, \tag{20}$$

which gives evidence for the validity of the displacement tracking conditions stated in equations (13) and (14), i.e. that the total solution becomes $u = u_t + u_a = z = u_{\text{tel}}$. For the parameters considered in the figures above, the displacements u_a, the corresponding stress field σ_a and the modified stress ϑ_a are shown in Figs. 7–9. Again, the relations of jump turn out to be satisfied. The validity of equations (20) can be easily estimated from Figs. 1, 2 and 7.

Figures 3, 4 and 8 show that the stress field σ_a turns out to obey the relation

$$\sigma_a = \sigma_{\text{tel}} - \sigma_t\,, \tag{21}$$

such that the total stress in the viscoelastic body due to the surface traction σ_0 and the actuation stress σ_A also does coincide with the elastic solution, $\sigma = \sigma_t + \sigma_a = \sigma_{\text{tel}}$. Alternatively to the analytic results obtained by MAPLE, this possibly surprising fact can be justified as follows: The stress σ_{tel} in the elastic half-space due to the surface traction σ_0, see equation (6), obeys the linear constitutive relation $\sigma_{\text{tel}} = Y_0 z_{,x}$ with $z = u_{\text{tel}}$, see equation (12). The relation of the balance of linear momentum, σ_A equation (5), can be written as $Y_0 u_{\text{tel},xx} = \rho \ddot{u}_{\text{tel}}$, since nobody forces are present. It turns out that the solution of equations (13) and (14) thus can be written as

$$\sigma_A = \int_0^t Y'(s) u_{\text{tel},x}(t-s)\,\mathrm{d}s. \tag{22}$$

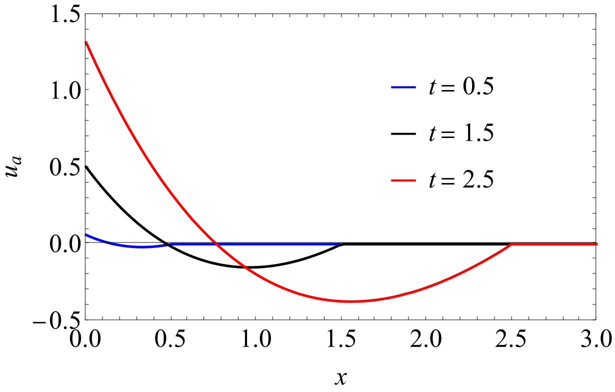

Fig. 7. Displacement due to σ_A for three time-instants

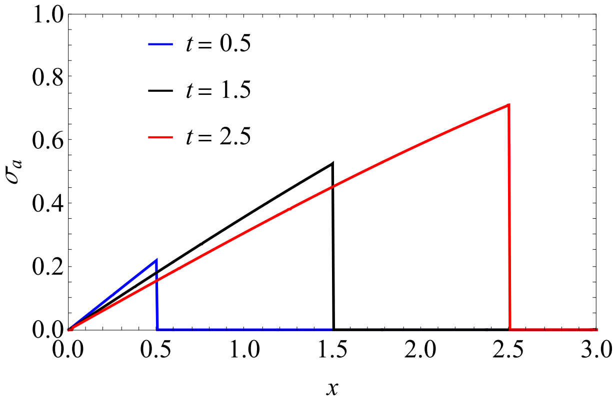

Fig. 8. Stress due to σ_A for three time-instants

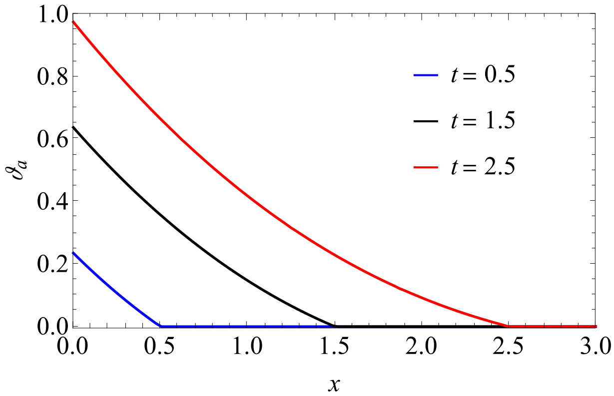

Fig. 9. Modified stress due to σ_A for three time-instants

Substituting the Maxwell relation equation (4) yields equation (19). Hence, using $u = u_{\mathrm{tel}}$ in the constitutive relation equation (2) for the total displacement u, we obtain

$$\sigma = Y_0 u_{\mathrm{tel},x} + \int_0^t Y'(s) u_{\mathrm{tel},x}(t-s)\,\mathrm{d}s - \sigma_A = \sigma_{\mathrm{tel}}\,. \tag{23}$$

In other words, the actuation stress σ_A cancels out in the constitutive relation. Last but not least, Fig. 9 shows that the relations of the jump are again satisfied.

5. REMARKS ON ACTUATION SPLITTING

Above, by presenting an instructive analytic 1D solution, we have demonstrated the validity of 3D formulations for viscoelastic displacement tracking stated in Irschik and Krommer [8]. In the latter 3D formulation, it also was found that actuation stress for viscoelastic displacement tracking can be split into two parts, one producing no stress, and the other producing no displacements in properly enlarged viscoelastic boundary value problems with vanishing initial history. The two boundary value problems are enlarged by additional opposite fictitious body forces $\pm\rho\ddot{z}$, respectively, such that the total of the two solutions satisfies the relations of the original problem. The fictitious body forces each are to be introduced in relation to the balance of linear momentum, equation (5). Transferring the viscoelastic dynamic 3D results of [8] to the present case, the actuation stress σ_{A1}, which is associated with the fictitious body force $-\rho\ddot{z}$, must satisfy the following quasi-static boundary-value problem

$$\sigma_{A1,x} + \rho\ddot{z} = 0, \tag{24}$$

$$x = 0: \quad \sigma_{A1} = -\sigma_0 \mathrm{H}(t). \tag{25}$$

The solution is

$$\sigma_{A1} = -\sigma_0 \mathrm{H}\left(\frac{tc - x}{c}\right) = -\sigma_{\mathrm{tel}}\,, \tag{26}$$

see equation (18) above. Following [8], the second part σ_{A2}, which is associated with the fictitious body force $+\rho\ddot{z}$ in the correspondingly enlarged boundary value problem, must obey the relation

$$\sigma_{A2} = Y_0 z_{,x}(t) + \int_0^t Y'(s) z_{,x}(t-s)\,\mathrm{d}s = \sigma_{\mathrm{tel}} + \sigma_A\,, \tag{27}$$

see equations (18) and (19) for the Maxwell case. This demonstrates that $\sigma_{A1} + \sigma_{A2} = \sigma_A$, as it should be. By symbolic computation, it then can be shown that σ_{A1} indeed does not produce stresses, while σ_{A2} does not produce displacements in the correspondingly enlarged problems.

6. CONCLUSIONS

In the framework of the linear theory of viscoelasticity, the present contribution deals with relations between solutions produced by imposed forces and by imposed eigenstrains, the latter being characterized by actuation stresses. Particularly, such relations are studied in connection with the problem of structural displacement tracking, which has become of increasing importance in modern smart materials and structures. In displacement tracking, one seeks a transient spatial distribution of additional actuation stresses, such that the total displacement of the structure does coincide with some desired displacement to be tracked, despite the presence of imposed forces. Our present goal here is to deliver some closed-form analytic 1D solutions, in order to exemplify a 3D displacement tracking strategy presented by us earlier in [8], and to give clear evidence for the appropriateness of this strategy. In these analytic exam- ples, we also exemplarily demonstrate that the solution strategy in [8] remains valid in the presence of singular waves, which were originally not addressed there. For that sake, we treat uni-axial displacements and stresses in a viscoelastic half-space, where we study the case of Maxwell material in greater detail. A tensile surface traction is suddenly applied at the surface of the half-space, giving rise to a uni-axial singular wave of order one, a shock wave. As a special tracking problem in its own right, we ask for additional actuation stresses such that the total displacement of the viscoelastic half-space does coincide with the displacements of a purely elastic half-space due to the applied traction, but in the absence of actuation stresses. We also shortly discuss a splitting technique sketched in [8], in which the actuation stress is split into a portion without stress and a portion without displacements, both portions acting in properly enlarged problems. Analytic expressions for displacement and stresses in our study are computed for Maxwell material via symbolic computation, utilizing the code MAPLE [10]. Some formal problems, which must be overcome for obtaining proper symbolic solutions in the present shock wave situation, are shortly addressed. It is hoped that the present study can stimulate further research on displacement tracking, particularly concerning feasible technological methods for producing eigenstrains that result in the actuation stresses, which are necessary for applying the above results in practice.

The theoretical character of our contribution to displacement tracking of viscoelastic solids and structures in the presence of shock waves, together with the presented 1D examples, aims to serve as a strategic basis for developing related generalizations and practical applications in more complex intelligent or smart structures. As long as continuous vibrations of linear elastic systems are considered, it can be said that smart structures have already gained considerable success and dissemination in engineering practice. It thus seems now to be the time for developing the theoretical basis for considering various complicating effects. This has motivated us to reconsider and exemplify here our older theoretical study [8] on displacement tracking of viscoelastic structures, thereby relaxing the assumption of continuity of fields and demonstrating the applicability of the formulations in [8], also in the presence of shock waves. Displacement tracking of linear elastic solids and structures by eigenstrains in the presence of singular waves was considered in [15] before. As an outlook to future research that is required to facilitate more complex practical applications, additional generalizations of the fundamental relations mainly should concern geometrical and physical nonlinearities. From space limitations, we here mention only a few recent works of our own group on the modelling of nonlinear smart structures: small vibrations superimposed upon pre-deformed smart structures were treated in [7] as a first step; generalizations concerning finite deformations, nonlinear electromechanical coupling, and hysteretic material behaviour were studied, e.g. in [16] and [17]. We additionally mention the forthcoming Special Issue [18] on advanced nonlinear modelling and numerical methods for smart materials and structures, which contains various recent contributions of other groups also. It is hoped that the present paper can stimulate the development of displacement tracking algorithms for advanced problems, such as those treated in [16–18].

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