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RECURRENCE FORMULA, DIFFERENTIAL COMPOUND AND DIFFERENTIAL EQUATIONS FOR HERMITE POLYNOMIALS

Abstract

Introduction and aim: The paper presents a recurrence formula, some differential compounds and differential equation for Hermite polynomials. The aim of the discussion was to give some proofs of presented dependences.
Material and methods: Selected material based on a recurrence formula, some differential compounds and differential equation has been obtained from the right literature. In presented proofs of theorems was used a deduction method.
Results: Has been shown some proof of the theorem of the generating function for Hermite polynomials. It has been done the proof of recurrence formula between Hermite polynomials, some proof of differential compound and two differential equations for Hermite polynomials.
Conclusion: The derivative of Hermite polynomial expressed by Hermite polynomials can be determined from the equation $H'_n(z) = 2nH_{n-1}(z)$ for n = 1, 2, 3,… .

Keywords: Hermite polynomials, recurrence formula, differential compound, differential equations.

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ZWIĄZEK REKURENCYJNY, ZALEŻNOŚĆ RÓŻNICZKOWA I RÓWNANIA RÓŻNICZKOWE DLA WIELOMIANÓW HERMITE'A

Streszczenie

Wstęp i cel: *W pracy przedstawiono związek rekurencyjny, zależności różniczkowe i równanie różniczkowe dla wielomianów Hermite'a. Celem rozważań było przeprowadzenie dowodów omawianych własności.*
Materiał i metody: *Materiał stanowiły wybrane zależności rekurencyjne i równanie różniczkowe uzyskane z literatury przedmiotu. W przeprowadzonych dowodach zastosowano metodę dedukcji.*
Wyniki: *Pokazano dowód twierdzenia o funkcji tworzącej dla wielomianów Hermite'a. Przeprowadzono dowód związku rekurencyjnego między wielomianami Hermite'a, zależności różniczkowej oraz dwóch równań różniczkowych dla wielomianów Hermita.*
Wniosek: *Pochodną wielomianu Hermite'a wyrażoną przez wielomiany Hermite'a można określić z równania* $H'_n(z) = 2nH_{n-1}(z)$ *dla n = 1,2,3… .*

Słowa kluczowe: *Wielomiany Hermie'a, związek rekurencyjny, zależność różniczkowa, równania różniczkowe.*

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1. Hermite polynomials

Definition 1.

Hermite polynomials $H_n(z)$ for complex variable z have the following form [1]-[10]:

$$H_n(z) = (-1)^n \exp(z^2)\frac{d^n}{dz^n}\exp(-z^2) \quad \text{dla} \quad n = 0,1,2,\ldots . \tag{1}$$

Hermite polynomials calculated directly from definition (1) provide the system of functions:

$$H_0(z) = 1, \tag{2}$$

$$H_1(z) = 2z, \tag{3}$$

$$H_2(z) = 4z^2 - 2, \tag{4}$$

$$H_3(z) = 8z^3 - 12z, \tag{5}$$

$$H_4(z) = 16z^4 - 48z^2 - 12, \tag{6}$$

....

$$H_{n-1}(z) = (-1)^{n-1}\exp(z^2)\frac{d^{n-1}}{dz^{n-1}}\exp(-z^2), \tag{7}$$

$$H_n(z) = (-1)^n \exp(z^2)\frac{d^n}{dz^n}(-z^2), \tag{8}$$

....

Theorem 1. *(Generating function)*

Function $$w(z,t) = \exp(2zt - t^2) \tag{9}$$

is the generating function for Hermite polynomials, i.e. there is an expanding in the series [4]:

$$w(z,t) = \exp(2zt - t^2) \equiv \sum_{n=0}^{\infty}\frac{H_n(z)}{n!}t^n \quad \text{dla} \quad |t| < \infty. \tag{10}$$

Proof:

Note that the function w=w(z,t) considered as a function of the complex variable t is an integer function, which can be expanded into a Taylor series, where $t_0=0$:

$$w(z,t) = \exp(2zt - t^2) =$$

$$= w(z,0) + \frac{\frac{\partial}{\partial t}w(z,0)}{1!}t + \frac{\frac{\partial^2}{\partial t^2}w(z,0)}{2!}t^2 + \ldots + \frac{\frac{\partial^{n-1}}{\partial t^{n-1}}w(z,0)}{(n-1)!}t^{n-1} + \frac{\frac{\partial^n}{\partial t^n}w(z,0)}{n!}t^n + \ldots = \tag{11}$$

$$= \sum_{n=0}^{\infty}\frac{\frac{\partial^n}{\partial t^n}w(z,0)}{n!}t^n = \sum_{n=0}^{\infty}\frac{\frac{\partial^n}{\partial t^n}w(z,t)\Big|_{t=0}}{n!}t^n,$$

where $|t| < \infty$.

Let us transform the expression $\frac{\partial^n w(z,t)}{\partial t^n}$ under the sign of the sum (11).

Therefore:

$$\begin{aligned}
&\left.\frac{\partial^n w(z,t)}{\partial t^n}\right|_{t=0} = \\
&= \frac{\partial^n}{\partial t^n}\left[\exp(2zt - t^2\right]\Big|_{t=0} = \\
&= \frac{\partial^n}{\partial t^n}\left[\exp(z^2 - z^2 + 2zt - t^2)\right]\Big|_{t=0} = \\
&= \frac{\partial^n}{\partial t^n}\left[\exp(z^2)\exp[-(z^2 + 2zt - t^2)\right]\Big|_{t=0} = \\
&= \exp(z^2)\frac{\partial^n}{\partial t^n}\left[\exp[-(z-t)^2\right]\Big|_{t=0} .
\end{aligned} \tag{12}$$

Given the above expression into the expansion of the function w=w (z,x) in a Taylor series and taking into account the fact that

$$u \equiv z - t \tag{13}$$

and derivatives will have the alternately various signs, have more:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{\left.\frac{\partial^n}{\partial t^n} w(z,t)\right|_{t=0}}{n!} t^n = \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n \exp(z^2)\left.\frac{\partial^n}{\partial t^n}\exp(-u^2)\right|_{u=z}}{n!} t^n = \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n \exp(z^2)\frac{\partial^n}{\partial t^n}\exp(-z^2)}{n!} t^n = \\
&= \sum_{n=0}^{\infty}\frac{H_n(z)}{n!} t^n ,
\end{aligned} \tag{14}$$

where$|\, t \,| < \infty$.

Therefore, it has been shown that

$$w(z,t) = \exp(2zt - t^2) \quad \text{dla} \quad |\, t \,| < \infty, \tag{15}$$

which completes the proof of the theorem 1. ■

2. Recurrence formula for Hermite polynomials

Theorem 2. *(Recurrence formula for Hermite polynomials)*

If $H_{n-1}(z)$, $H_n(z)$ and $H_{n+1}(z)$ are Hermite polynomials, then [4]:

$$H_{n+1}(z) - 2zH_n(z) + 2nH_{n-1}(z) = 0 \quad \text{for} \quad n = 1, 2, \ldots. \tag{16}$$

Proof:

Taking into account the following identity:

$$\frac{\partial w}{\partial t} - 2(z - t)w = 0 \tag{17}$$

and fact, that the series (10) can be differentiated term by term we continue:

$$\frac{\partial}{\partial t}\left[\sum_{n=0}^{\infty}\frac{H_n(z)}{n!}t^n\right] - (2z - 2t)\sum_{n=0}^{\infty}\frac{H_n(z)}{n!}t^n = 0. \tag{18}$$

Expanding the series in the firmulae (18) we obtained:

$$\begin{gathered}\frac{\partial}{\partial t}\left[\frac{H_0(z)}{0!}t^0 + \frac{H_1(z)}{1!}t^1 + \frac{H_2(z)}{2!}t^2 + \ldots + \frac{H_n(z)}{n!}t^n\right] + \\ + \sum_{n=0}^{\infty}\frac{2tH_n(z)}{n!}t^n + \sum_{n=0}^{\infty}\frac{-2zH_n(z)}{n!}t^n = 0.\end{gathered} \tag{19}$$

From which it follows that:

$$\sum_{n=0}^{\infty}\left[\frac{nH_n(z)}{n!t} + \frac{2tH_n(z)}{n!} - \frac{2zH_n(z)}{n!}\right]t^n = 0. \tag{20}$$

Equating the coefficients of the power t^n to zero we get further that:

$$\frac{nH_n(z)}{n!t} + \frac{2tH_n(z)}{n!} - \frac{2zH_n(z)}{n!}t^n = 0. \tag{21}$$

Which implies further that:

$$\frac{(n+1)H_{n+1}(z)}{(n+1)!t} + \frac{2nH_{n-1}(z)}{(n-1)!n} - \frac{2zH_n(z)}{n!} = 0 \tag{22}$$

for n = 1, 2, … .

Multiplying both sides of the equation (22) by expression n! we get:

$$H_{n+1}(z) + 2nH_{n-1}(z) - 2zH_n(z) = 0 \tag{23}$$

for n = 1, 2, … , which completes the proof of the theorem 2. ■

3. Differential compound for Hermite polynomials

Let us derive some compounds of Hermite polynomials, which presents the possibility to express the derivative of Hermite polynomial by Hermite polynomial.

Theorem 3. *(Expressing the derivative of Hermite polynomial by Hermite polynomial)*

If $H_n(z)$ and $H_{n-1}(z)$ are Hermite polynomials, then [4]:

$$\frac{dH_n(z)}{dz} = 2nH_{n-1}(z) \tag{24}$$

for n = 1, 2, … .

Proof:

To present the proof of theorem 3 we use the following identity:

$$\frac{\partial w}{\partial z} - 2tw = 0 \tag{25}$$

and a series in the form (10). Then we have:

$$\frac{\partial}{\partial z}\left[\sum_{n=0}^{\infty} \frac{H_n(z)}{n!} t^n\right] - 2t\sum_{n=0}^{\infty} \frac{H_n(z)}{n!} t^n = 0. \tag{26}$$

After rewriting series we have further:

$$\frac{\partial}{\partial z}\left[\frac{H_0(z)}{0!} t^0 + \frac{H_1(z)}{1!} t^1 + \frac{H_2(z)}{2!} t^2 + \ldots + \frac{H_n(z)}{n!} t^n\right] - \sum_{n=0}^{\infty} \frac{2tH_n(z)}{n!} t^n = 0. \tag{27}$$

Whence it follows that:

$$\left[\frac{\frac{\partial}{\partial z}H_0(z)}{0!} t^0 + \frac{\frac{\partial}{\partial z}H_1(z)}{1!} t^1 + \frac{\frac{\partial}{\partial z}H_2(z)}{2!} t^2 + \ldots + \frac{\frac{\partial}{\partial z}H_n(z)}{n!} t^n\right] - \sum_{n=0}^{\infty} \frac{2tH_n(z)}{n!} t^n = 0. \tag{28}$$

Therefore:

$$\sum_{n=0}^{\infty}\left[\frac{\frac{\partial H_n(z)}{\partial z}}{n!} - \frac{2tH_n(z)}{n!}\right] t^n = 0. \tag{29}$$

Equating the coefficient of the power t^n to zero we get further that:

$$\frac{\frac{\partial H_n(z)}{\partial z}}{n!} - \frac{2H_{n-1}(z)}{(n-1)!} = 0. \tag{30}$$

for n = 1, 2, ….

Hence it follows that:

$$\frac{\frac{\partial H_n(z)}{\partial z}}{(n-1)!n} - \frac{2H_{n-1}(z)}{(n-1)!} = 0. \qquad (31)$$

Multiplying both sides of equation (30) by n! we get:

$$\frac{\partial H_n(z)}{\partial z} = 2nH_{n-1}(z) \qquad (32)$$

for n = 1, 2, …. , which completes the proof of the theorem 3. ■

4. Differential equations for Hermite polynomials

Theorem 4. *(Differential equation of the second order for Hermite polynomials) [4]*

If $H_n(z)$ are Hermite polynomials, then:

$$\frac{d^2H_n(z)}{dz^2} - 2z\frac{dH_n(z)}{dz} + 2nH_n(z) = 0 \quad \text{for} \quad n = 0,1,2,\dots. \qquad (33)$$

Proof:

From the obtained compounds (16) and (24) we calculate the polynomial $H_{n-1}(z)$ we obtain:

$$H_{n-1}(z) = \frac{1}{2n}\left[2zH_n(z) - H_{n+1}(z)\right] \qquad (34)$$

and

$$H_{n-1}(z) = \frac{1}{2n}\frac{dH_n(z)}{dz}. \qquad (35)$$

Comparing sides of the equation (34) and (35) we get further:

$$\frac{dH_n(z)}{dz} = 2zH_n(z) - H_{n+1}(z). \qquad (36)$$

Therefore

$$\frac{dH_n(z)}{dz} - 2zH_n(z) + H_{n+1}(z) = 0. \qquad (37)$$

The above equation (37) we differentiate both sides with respect to variable z and get:

$$\frac{d^2H_n(z)}{dz^2} - 2H_n(z) - 2z\frac{dH_n(z)}{dz} + \frac{dH_{n+1}(z)}{dz} = 0. \qquad (38)$$

Using the compound (33) we have:

$$\frac{dH_{n+1}(z)}{dz} = 2(n+1)H_n(z). \qquad (39)$$

Therefore:

$$\frac{d^2H_n(z)}{dz^2} - 2H_n(z) - 2z\frac{dH_n(z)}{dz} + 2nH_n(z) + 2H_n(z) = 0. \qquad (40)$$

After entering the following substitution $p \equiv H_n(z)$ in the equation (40), and after the reduction we obtain finally

$$\frac{d^2p}{dz^2} - 2z\frac{dp}{dz} + 2np = 0 \tag{41}$$

for n = 0,1, 2, … , which completes the proof of the theorem 4. ■

Theorem 5. *(Differential equation of the second order with Hermite polynomials)*

If $H_n(z)$ is Hermite polynomial, then [4]:

$$\frac{d^2m}{dz^2} + (2n+1-z^2)m = 0, \tag{42}$$

where

$$m(z) \equiv H_n(z)\cdot\exp\left(-\frac{z^2}{2}\right). \tag{43}$$

for n = 0, 1, 2, … .

Proof:

We introduce for simplicity of writing the following determination:

$$v(z) = \exp\left(-\frac{z^2}{2}\right). \tag{44}$$

Let us determine, at first, the second derivative of the function (43):

$$\begin{aligned}
\frac{d^2m(z)}{dz^2} &= \frac{d^2}{dz^2}\left[v(z)\cdot H_n(z)\right] = \frac{d}{dz}\left[-zv(z)H_n(z) + v(z)\frac{dH_n(z)}{dz}\right] = \\
&= -v(z)H_n(z) + z^2v(z)H_n(z) - zv(z)\frac{dH_n(z)}{dz} - zv(z)\frac{dH_n(z)}{dz} + v(z)\frac{d^2H_n(z)}{dz^2} = \\
&= -v(z)H_n(z) + z^2v(z)H_n(z) - 2zv(z)\frac{dH_n(z)}{dz} + v(z)\frac{d^2H_n(z)}{dz^2}.
\end{aligned} \tag{45}$$

Therefore:

$$\frac{d^2m(z)}{dz^2} = -v(z)H_n(z) + z^2v(z)H_n(z) - 2zv(z)\frac{dH_n(z)}{dz} + v(z)\frac{d^2H_n(z)}{dz^2}. \tag{46}$$

The equation (33) is multiplied both sides by function v(z), then we obtain:

$$v(z)\frac{d^2H_n(z)}{dz^2} - 2zv(z)\frac{dH_n(z)}{dz} + 2nv(z)H_n(z) = 0. \tag{47}$$

To the thus formed equation (47) for its left side we add and subtract the expressions $z^2v(z)H_n(z)$ and $v(z)H_n(z)$, then we have:

$$v(z)\frac{d^2H_n(z)}{dz^2}-2zv(z)\frac{dH_n(z)}{dz}+2nv(z)H_n(z)+ \\ +z^2v(z)H_n(z)-z^2v(z)H_n(z)+v(z)H_n(z)-v(z)H_n(z)=0. \tag{48}$$

Using the commutative law and connectivity law of addition we get further get:

$$\left[-v(z)H_n(z)++z^2v(z)H_n(z)-2zv(z)\frac{dH_n(z)}{dz}+v(z)\frac{d^2H_n(z)}{dz^2}\right]+ \\ +2nv(z)H_n(z)+v(z)H_n(z)-z^2v(z)H_n(z)=0. \tag{49}$$

By virtue of the formulae (46) and (43) we finally obtain:

$$\frac{d^2m(z)}{dz^2}+2n\cdot m(z)+m(z)-z^2m(z)=0. \tag{50}$$

Therefore:

$$\frac{d^2m(z)}{dz^2}+(2n+1-z^2)\cdot m(z)=0, \tag{51}$$

where the function m=m(z) is defined by the formula (43) for n = 0, 1, 2, … . This fact completes the proof of the theorem 5. ■

5. Conclusion

A derivative of Hermite polynomial expressed by the Hermie polynomials can be determined from the equation $H'_n(z) = 2nH_{n-1}(z)$ for n = 1, 2, … .

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