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FREE VIBRATIONS OF UNIFORM TIMOSHENKO BEAMS ON PASTERNAK FOUNDATION USING COUPLED DISPLACEMENT FIELD METHOD

Complex structures used in various engineering applications are made up of simple structural members like beams, plates and shells. The fundamental frequency is absolutely essential in determining the response of these structural elements subjected to the dynamic loads. However, for short beams, one has to consider the effect of shear deformation and rotary inertia in order to evaluate their fundamental linear frequencies. In this paper, the authors developed a Coupled Displacement Field method where the number of undetermined coefficients $2n$ existing in the classical Rayleigh-Ritz method are reduced to n, which significantly simplifies the procedure to obtain the analytical solution. This is accomplished by using a coupling equation derived from the static equilibrium of the shear flexible structural element. In this paper, the free vibration behaviour in terms of slenderness ratio and foundation parameters have been derived for the most practically used shear flexible uniform Timoshenko Hinged-Hinged, Clamped-Clamped beams resting on Pasternak foundation. The findings obtained by the present Coupled Displacement Field Method are compared with the existing literature wherever possible and the agreement is good.

Nomenclature

A	– area of cross section	ρ	– mass density of the material of the beam
E	– Young's modulus	ω	– radian frequency
G	– shear modulus	$\bar{K}_w$	– Winkler foundation parameter
I	– area moment of inertia	$\bar{K}_p$	– Pasternak foundation parameter
k	– shear correction factor	K_w	– Winkler stiffness
L	– length of the beam	K_p	– shear layer stiffness
r	– radius of gyration	β	– slenderness ratio
T	– kinetic energy	ν	– Poisson ratio
U	– strain energy		

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1. Introduction

The Winkler foundation model is considered as an infinitely closely spaced springs put together. In actual situation this model neglects the shear interaction between the adjacent springs which can not represent the characteristics of many practical foundations. The Pasternak foundation model is an alternative model for the Winkler foundation in which the shear interaction between the springs is considered to increase the flexibility of the foundation model. The Winkler foundation model is a simple and single-parametric foundation model while the Pasternak foundation model is a two-parametric foundation model which represents the foundation parameter characteristics more accurately for practical purpose. The model proposed by Pasternak assumes the existence of shear interaction between the spring elements by connecting those elements to a layer of incompressible vertical elements. C.F. Lü, C.W. Lim and W.A. Yao [1] presented a simplistic method which describes the stresses and displacements of beams on a Pasternak elastic foundation based on classical two-dimensional elasticity theory. C. Franciosi and A. Masi [2] did a finite element free vibration analysis of beams on two-parameter elastic foundation using exact shape functions. Ivo Caliò & Annalisa Greco [3] analysed axially-loaded Timoshenko beams on elastic foundation through dynamic stiffness matrix method. Joon Kyu Lee et al. [4] studied the free vibrations of prismatic beams resting on Pasternak foundation by giving special attention to the bending-twisting of the beams by deriving governing differential equations and solving them by using the combination of Runge-Kutta and Regula-Falsi methods. M.A. De Rosa [5] worked on free vibrations of Timoshenko beams resting on two parametric elastic foundation by taking two variants of the equation of motion in which the second foundation parameter is a function of the total rotation of the beam or a function of the rotation due to bending. M.A. De Rosa and M.J. Maurizi [6] found out the influence of concentrated masses and the Pasternak soil on the free vibrations of Euler beams. M. Karkon and H. Karkon [7] introduced an element formulation for free vibration analysis of Timoshenko beam on Pasternak elastic foundation using finite element method. Meera Saheb. K. et al. [8] studied free vibration analysis of Timoshenko beams using Coupled Displacement Field method for uniform Timoshenko beams for different beam boundary conditions. Mohamed Taha Hassan and Mohamed Nassar [9] studied the static and dynamic behaviour of a Timoshenko beam subjected to a static axial compressive load and a dynamic lateral load resting on a two parameter foundation using Adomian Decomposition Method (ADM) in which the natural frequencies for free vibration and beam response in forced vibrations are calculated. Nguten Dinh Kien [10] presented finite-element formulation for investigating the free vibration of uniform Timoshenko beams resting on a Winkler-type elastic foundation and pre-stressing by an axial force. P. Obara [11] executed vibration and stability analysis of uniform beams supported on two parameter elastic foundation. S.Y. Lee, Y.H. Kuo, and F.Y. Lin [12] investigated the influence of the Winkler elastic foundation modulus, slenderness ratio and elastically restrained boundary conditions on the critical load of Timoshenko beams subjected to an end follower force. T.M. Wang and J.E. Stefens [13] worked on both free and forced vibrations of continuous Timoshenko beams on Winkler-Pasternak foundations in which the general dynamic slope-deflection equations include the combined effects of rotary inertia and shear deformation. T.M. Wang and L.W. Gagnon [14] considered the dynamic analysis of continuous Timoshenko beams on Winkler-Pasternak foundations by means of the general dynamic slope-deflection equations. T. Yokoyama [15] offered a finite element technique for determining the vibration characteristics of a uniform Timoshenko beam-column supported on a two-parameter elastic foundation in which the beam-column is discretized into a number of simple elements with four degrees of freedom at each node. T. Yokoyama [16] studied the parametric instability behaviour of a Timoshenko beam resting on an elastic foundation of the Winkler type by the finite element technique.

W.Q .Chen, C.F. Lü and Z.G. Bian [17] used a mixed method which consists of state space method and differential quadrature method to find the free vibrations of beams resting on a Pasternak elastic foundation. The aim of the present paper is to develop a procedure to reduce the number of undetermined coefficients from $2n$ to n in a general sense and for a single term approximation, from two to one [8]. This simplifies the solution procedure enormously by almost a factor of two for Timoshenko beams. This is achieved based on the concept of the Coupled Displacement Field (CDF) method, which was successfully applied by the authors [8]. To explain the ease of the present method, the expressions for fundamental frequency parameter values are obtained for uniform shear flexible Timoshenko beams with Hinged-Hinged, Clamped-Clamped beam boundary conditions.

2. Coupled Displacement Field method (CDF)

2.1. Coupling equation

From the kinematics of a shear flexible beam theory (based on the Timoshenko beam theory)

$$\bar{u}(x,z) = z\theta \tag{1}$$

$$\bar{w}(x,z) = w(x,z) \tag{2}$$

where $\bar{w}$ is the transverse displacements at any point of the beam $\bar{u}$ is the axial displacement at any point of the beam and z is the distance of the any point from the neutral axis, θ is the total rotation anywhere on the beam axis, w is the transverse displacement and x, z are the independent spatial variables. The axial and shear strains are given by

$$\varepsilon_x = z\frac{\partial\theta}{\partial x} \tag{3}$$

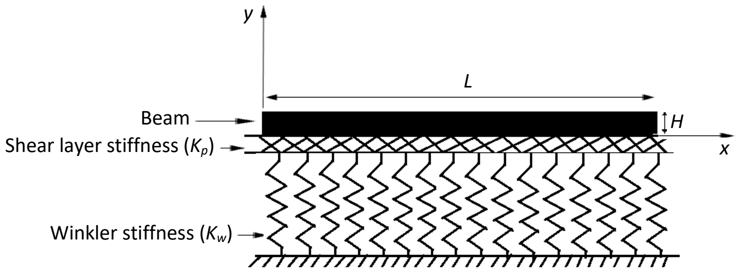

Fig. 1. Uniform Timoshenko beam resting on Pasternak foundation

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \theta \tag{4}$$

The strain energy is represented by U and the work done is represented by W, and the expressions are given by

$$U = \frac{EI}{2} \int_0^L \left(\frac{d\theta}{dx}\right)^2 dx + \frac{kGA}{2} \int_0^L \left(\frac{\partial w}{\partial x} + \theta\right)^2 dx \tag{5}$$

$$W = \int_0^L p(x) w(x) dx \tag{6}$$

GA is the shear rigidity and EI is the flexural rigidity, k is the shear correction factor (= 5/6), $p(x)$ is the static lateral load per unit length acting on the beam, I is the area moment of inertia, E is the Young's modulus, G is the shear modulus, A is the area of cross section, x is the axial coordinate and L is the length of the beam. Applying the principle of minimization of total potential energy as

$$\delta(U - W) = 0 \tag{7}$$

the following static equilibrium equations can be obtained

$$kGA\left(\frac{d^2w}{dx^2} + \frac{d\theta}{dx}\right) + p = 0 \tag{8}$$

$$EI\frac{d^2\theta}{dx^2} - kGA\left(\frac{dw}{dx} + \theta\right) = 0 \tag{9}$$

where θ represents the total rotation and w is transverse displacement.

Equations (8) and (9) are coupling equations and can be used for getting the solution for the static analysis of the shear-deformable beams. By observing the equation (8) carefully one can see that, it depends on the load term p and the equation (9) is independent of the load term p. Hence, the equation (9) is used to combine the total rotation θ and the transverse displacement w, such that the two unknown coefficients problem becomes a single unknown coefficient problem and the final linear free vibration problem becomes very simpler to solve.

The concept of Coupled Displacement Field method is explained in detail. In Coupled Displacement Field method (CDF), using the single term admissible function for total rotation, the function for transverse displacement one can derive using the coupling equation (9). The assumed admissible function for total rotation satisfies all the applicable boundary conditions such as kinematic, natural boundary conditions and the symmetric condition. For Hinged-Hinged uniform Timoshenko beam, the admissible function for (m^{th} mode of vibration) the total rotation is assumed as

$$\theta = a\frac{m\pi}{L}\cos\frac{m\pi x}{L}, \tag{10}$$

where m is the mode number, a is the central lateral displacement of the beam which is also the maximum lateral displacement.

Rewriting equation (9)

$$\frac{dw}{dx} = -\theta + \frac{EI}{kGA}\frac{d^2\theta}{dx^2}. \tag{11}$$

First and second differentiation of equation (10) with respect to x is given as follows

$$\frac{d\theta}{dx} = -a\left(\frac{m\pi}{L}\right)^2\sin\frac{m\pi x}{L}, \tag{12}$$

$$\frac{d^2\theta}{dx^2} = -a\left(\frac{m\pi}{L}\right)^3\cos\frac{m\pi x}{L}. \tag{13}$$

By substituting the value of (10), (13) into equation (11), and after simplification the following expression can be obtained

$$\frac{dw}{dx} = -a\frac{m\pi}{L}\left(1 + \frac{EI}{kGA}\frac{m^2\pi^2}{L^2}\right)\cos\frac{m\pi x}{L} \tag{14}$$

By integrating the above equation, the lateral displacement can be obtained as

$$w = -a\left[1 + \left(\frac{m\pi}{L}\right)^2\gamma\right]\sin m\pi\zeta \tag{15}$$

where $\gamma = \dfrac{EI}{kGA}$ and $\zeta = \dfrac{x}{L}$.

It is observed from equation (10) and equation (15) that the θ distribution and the transverse displacement w, contains the same unknown coefficient a and equation (15) satisfies all the kinematic, natural and symmetric beam boundary conditions

$$w(0) = w(L) = \left.\frac{dw}{dx}\right|_{x=\frac{L}{2}} = 0. \tag{16}$$

2.2. Linear free vibrations

Linear free vibrations can be studied, once the Coupled Displacement Field for the lateral displacement w, for an assumed θ distribution is evaluated, using the principle of conservation of total energy at any instant of time, neglecting damping, which states that total energy is constant when the structural member is vibrating

$$U + T = \text{const.}$$

The expression for strain energy stored in the beam due to bending and transverse shear force is given as

$$U_E = \frac{EI}{2}\int_0^L \left(\frac{d\theta}{dx}\right)^2 dx + \frac{kGA}{2}\int_0^L \left(\frac{dw}{dx}+\theta\right)^2 dx \tag{17}$$

Substituting the equations (10), (12) and (14) into the above equation and after simplification

$$U_E = \frac{EI}{2}\frac{a^2m^4\pi^4}{L^4}\left[\left(\frac{L}{2}-\frac{L\sin(2\pi m)}{4\pi m}\right)+\frac{EI}{kGA}\left(\frac{m\pi}{L}\right)^2\left(\frac{L}{2}+\frac{L\sin(2\pi m)}{4\pi m}\right)\right] \tag{18}$$

The term $\sin(2\pi m) = 0$ for $m = 1, 2, 3, 4, 5 \ldots$ so the above equation becomes (for 1st mode)

$$U_E = \frac{EI}{4}\frac{a^2\pi^4}{L^3}\left[1+\frac{EI}{kGA}\left(\frac{\pi}{L}\right)^2\right] \tag{19}$$

Strain energy stored due to two parameter foundation can be calculated as

$$U_F = \frac{K_w}{2}\int_0^L w^2 dx + \frac{K_p}{2}\int_0^L \left(\frac{dw}{dx}\right)^2 dx \tag{20}$$

where K_w is the Winkler foundation stiffness and K_p is the Pasternak foundation stiffness. Substituting the equations (14) and (15) into the above equation and after simplification

$$U_F = \frac{EI}{2L^4}a^2\left(1+\frac{EI}{kGA}\frac{m^2\pi^2}{L^2}\right)^2\left[\bar{K}_w\left(\frac{L}{2}-\frac{L\sin(2\pi m)}{4\pi m}\right)+ m^2\pi^2\bar{K}_p\left(\frac{L}{2}+\frac{L\sin(2\pi m)}{4\pi m}\right)\right] \tag{21}$$

The term $\sin(2\pi m) = 0$ for $m = 1, 2, 3, 4, 5 \ldots$ so the above equation becomes (for 1st mode)

$$U_F = \frac{EI}{4L^3}a^2\left(1+\frac{EI}{kGA}\frac{\pi^2}{L^2}\right)^2\left[\bar{K}_w+\pi^2\bar{K}_p\right] \tag{22}$$

$\bar{K}_w$ is the Winkler stiffness foundation parameter $\left(=\dfrac{K_w L^4}{EI}\right)$ and $\bar{K}_p$ is Pasternak stiffness foundation parameter $\left(=\dfrac{K_p L^2}{EI}\right)$, the total strain energy stored in the beam due to deformation and foundation can be calculated as

$$U = U_E + U_F. \tag{23}$$

Substituting equation (19) and (22) into the above equation and after simplification

$$U=\frac{EIa^2}{4L^3}\left[(m\pi)^4\left(1+\frac{3.12m^2\pi^2}{\beta^2}\right)+\left(1+\frac{3.12m^2\pi^2}{\beta^2}\right)^2\left(\bar{K}_w+\bar{K}_p(m\pi)^2\right)\right] \tag{24}$$

The expression for kinetic energy is

$$T=\frac{\rho A\omega^2}{2}\int_0^L w^2dx+\frac{\rho I\omega^2}{2}\int_0^L \theta^2dx \tag{25}$$

Substituting equations (10) and (15) in the above equation and after simplification

$$T=\frac{\rho A\omega^2 a^2}{2}\left[\left(1+\frac{3.12m^2\pi^2}{\beta^2}\right)^2\left(\frac{L}{2}-\frac{L\sin(2\pi m)}{4\pi m}\right)+\right.$$
$$\left.\frac{m^2\pi^2}{\beta^2}\left(\frac{L}{2}+\frac{L\sin(2\pi m)}{4\pi m}\right)\right] \tag{26}$$

For first mode

$$T=\frac{\rho A\omega^2 a^2 L}{4}\left[\left(1+\frac{3.12\pi^2}{\beta^2}\right)^2+\frac{\pi^2}{\beta^2}\right] \tag{27}$$

by applying the Lagrangian

$$\frac{\partial(U-T)}{\partial a}=0. \tag{28}$$

After simplification, the non-dimensional frequency parameter is obtained and is given as

$$\lambda=\pi^4\left[\frac{m^4\left(1+\dfrac{3.12\pi^2m^2}{\beta^2}\right)+\left(1+\dfrac{3.12m^2\pi^2}{\beta^2}\right)^2\dfrac{1}{\pi^2}\left(\dfrac{\bar{K}_w}{\pi^2}+\bar{K}_p m^2\right)}{\left(1+\dfrac{3.12m^2\pi^2}{\beta^2}\right)^2+\dfrac{m^2\pi^2}{\beta^2}}\right] \tag{29}$$

where $\lambda=\dfrac{\rho A\omega^2 L^4}{EI}$, $\beta=\dfrac{L}{r}$ is the slenderness ratio.

The above expression is used to calculate the frequency parameter values for any number of modes ($m = 1, 2, 3, 4, 5, 6 \dots n$ modes).

If $m = 1$, then it becomes fundamental frequency parameter i.e

$$\lambda = \pi^4 \left[\frac{\left(1 + \frac{3.12\pi^2}{\beta^2}\right) + \left(1 + \frac{3.12\pi^2}{\beta^2}\right)^2 \frac{1}{\pi^2} \left(\frac{\bar{K}_w}{\pi^2} + \bar{K}_p\right)}{\left(1 + \frac{3.12\pi^2}{\beta^2}\right)^2 + \frac{\pi^2}{\beta^2}} \right]$$

The fundamental frequency parameter values for various slenderness ratio and foundation parameters are shown in Table 3 and compared with the existing literature. The frequency parameter values for Hinged-Hinged uniform Timoshenko beam resting on Pasternak foundation at second mode is also calculated by putting the value of $m = 2$ in equation (29) and these values are shown in Table 4.

In the reference [17], the thickness to length ratio, i.e., $\frac{H}{L}$ is being converted in to the slenderness ratio $\beta = \frac{L}{r}$, which we know

$$I = Ar^2 \tag{30}$$

and for rectangular section of thickness H and width b

$$I = \frac{bH^3}{12}. \tag{31}$$

By solving equations (30) and (31) the value of r becomes $r = \frac{H}{\sqrt{12}}$.

The same procedure is followed as discussed in the above section, for calculating fundamental frequency parameter for the given Clamped-Clamped beam boundary condition resting on Pasternak foundation.

3. Numerical results and discussion

The concept of Coupled Displacement Field method is demonstrated to determine the linear non-dimensional fundamental frequency parameter values of uniform Timoshenko beams resting on Pasternak foundation with the two most practically used such as Hinged-Hinged, Clamped-Clamped beam boundary conditions.

The beams considered here are with axially immovable ends. Suitable single term trigonometric/algebraic admissible function is used to represent the total rotation (θ) in the Coupled Displacement Field method. The corresponding coupled lateral displacement (w) is derived using the coupling equation. Numerical results (the non-dimensional fundamental frequency parameter) are obtained in terms of foundation parameters and slenderness ratios. To assess the accuracy of the results, the present results obtained from the Coupled Displacement Field method are compared with the existing literature.

Table 1 shows the expressions, such as total rotation (trigonometric function), derived transverse displacement and linear non dimensional fundamental frequency parameter for Clamped-Clamped beam boundary condition resting on Pasternak foundation. Table 2 shows the expressions for total rotation (algebraic function), transverse displacement and frequency parameter for Clamped-Clamed uniform Timoshenko for second mode.

Table 1.

Expressions for total rotation, derived transverse displacement and fundamental frequency parameter for Clamped-Clamped beam boundary condition resting on Pasternak foundation

Parameters	Expression for
Total rotation (θ)	$\theta = a\frac{2\pi}{L}\sin\frac{2\pi x}{L}$
Transverse displacement (w)	$w = a\left[1+\left(\frac{2\pi}{L}\right)^2\frac{EI}{kGA}\right]\left(\cos\frac{2\pi x}{L}-1\right)$
Fundamental frequency parameter	$\lambda = \dfrac{\left(501.69045+\frac{63950.0376}{\beta^2}\right)\left[1+\frac{3}{16}\frac{\bar{K}_w}{\pi^4}\gamma+\frac{\bar{K}_P}{4}\gamma\right]}{\left[1+\frac{285.8238}{\beta^2}+\frac{15171.48039}{\beta^4}\right]}$ where $\gamma = \left(1+\frac{123.1726}{\beta^2}\right)$

Table 2.

Expressions for total rotation, derived transverse displacement and frequency parameter for Clamped-Clamped beam boundary condition at second mode for uniform Timoshenko beams resting on Pasternak foundation

Parameters	Expression for
Total rotation (θ)	$\theta = aL^4(-\zeta+6\zeta^2-10\zeta^3+5\zeta^4)$
Transverse displacement (w)	$w = aL^5\left[\left(\frac{\zeta^2}{2}-2\zeta^3+\frac{10}{4}\zeta^4-\zeta^5\right)+\frac{\gamma}{L^2}(12\zeta-30\zeta^2+20\zeta^3)\right]$
Frequency parameter (2^{nd} mode)	$\lambda = \dfrac{\pi^2\left[\left(0.01447+\frac{7.5869}{\beta^2}\right)+\frac{\bar{K}_w}{\pi^2}\left(0.000036+\frac{11.1244}{\beta^4}+\frac{0.0113}{\beta^2}\right)\right]}{\left[0.000036+\frac{11.1244}{\beta^4}+\frac{0.0113}{\beta^2}\right]} + \dfrac{\pi^2\left[\frac{\bar{K}_P}{\pi^2}\left(0.00158+\frac{233.62}{\beta^4}+\frac{0.8913}{\beta^2}\right)\right]}{\left[0.000036+\frac{11.1244}{\beta^4}+\frac{0.0113}{\beta^2}\right]}$

Table 3.
($\lambda^{1/4}$) Fundamental frequency parameter values of Hinged-Hinged Timoshenko beam resting on Pasternak foundation

Foundation parameters		Slenderness ratio (β)													
		17			52			100	200	300	415				
$\bar{K}_w$	$\frac{\bar{K}_p}{\pi^2}$	Present CDF method	ref. [17]	Exact [17]	Present CDF method	ref. [17]	Exact [17]	Present CDF method	Present CDF method	Present CDF method	Present CDF method	ref. [17]	Exact [17]	ref. [6]	ref. [7]
0	0	3.0420	3.0479	3.0480	3.1298	3.1302	3.1302	3.1384	3.1408	3.1412	3.1414	3.1414	3.1414	3.1416	3.1416
	0.5	3.3961	3.3945	3.3946	3.4671	3.4667	3.4667	3.4741	3.4760	3.4764	3.4765	3.4765	3.4766	3.4767	3.4767
	1	3.6649	3.6580	3.6580	3.7274	3.7265	3.7266	3.7337	3.7354	3.7357	3.7358	3.7358	3.7359	3.7360	3.7360
	2.5	4.2379	4.2183	4.2183	4.2897	4.2880	4.2881	4.2950	4.2965	4.2967	4.2968	4.2968	4.2969	4.2970	4.2970
	3	4.3856	–	–	4.4358	–	–	4.4409	4.4423	4.4426	4.4427	–	–	–	–
	3.5	4.5197	–	–	4.5687	–	–	4.5737	4.5751	4.5754	4.5755	–	–	–	–
	4	4.6429	–	–	4.6909	–	–	4.6959	4.6973	4.6975	4.6976	–	–	–	–
	4.5	4.7570	–	–	4.8043	–	–	4.8092	4.8105	4.8108	4.8109	–	–	–	–
	5	4.8634	–	–	4.9101	–	–	4.9150	4.9163	4.9166	4.9167	–	–	–	
10^2	0	3.6776	3.6705	3.6705	3.7398	3.7389	3.7389	3.7460	3.7478	3.7481	3.7482	3.7482	3.7482	3.7483	3.7484
	0.5	3.8956	3.8839	3.8840	3.9528	3.9516	3.9517	3.9586	3.9603	3.9605	3.9606	3.9606	3.9607	3.9608	3.9608
	1	4.0821	4.0663	4.0664	4.1361	4.1347	4.1347	4.1416	4.1432	4.1434	4.1435	4.1435	4.1436	4.1437	4.1437
	2.5	4.5265	4.4991	4.4991	4.5754	4.5734	4.5735	4.5804	4.5819	4.5821	4.5822	4.5822	4.5823	4.5824	4.5824
	3	4.6492	–	–	4.6971	–	–	4.7021	4.7035	4.7038	4.7039	–	–	–	–
	3.5	4.7628	–	–	4.8101	–	–	4.8150	4.8163	4.8166	4.8167	–	–	–	–
	4	4.8689	–	–	4.9156	–	–	4.9204	4.9218	4.9220	4.9221	–	–	–	–
	4.5	4.9684	–	–	5.0147	–	–	5.0195	5.0209	5.0211	5.0212	–	–	–	–
	5	5.0623	–	–	5.1082	–	–	5.1130	5.1144	5.1146	5.1147	–	–	–	–

Numerical results in the form of non-dimensional fundamental frequency parameter for various slenderness ratios, Winkler foundation parameters and Pasternak foundation parameters are given in Tables 3, 4, 5 and 6, respectively for the Timoshenko Hinged-Hinged and Clamped-Clamped beam boundary conditions resting on Pasternak foundation for first and second modes.

Table 4.
($\lambda^{1/4}$) Frequency Parameter values of Hinged-Hinged Timoshenko Beam resting on Pasternak foundation for second mode

Foundation parameters		Slenderness ratio (β)						
		17	52	100	200	300	415	
$\bar{K}_w$	$\frac{\bar{K}_p}{\pi^2}$	Present CDF method	Present CDF method	Present CDF method	Present CDF method	Present CDF method	Present CDF method	ref. [7]
0	0	5.6568	6.1930	6.4471	6.2768	6.2803	6.2817	6.2832
	0.5	5.8937	6.3861	6.6210	6.4649	6.4682	6.4695	6.4709
	1	6.1050	6.5631	7.0739	6.6379	6.6411	6.6423	6.6437
	2.5	6.6339	7.0226	7.2071	7.0889	7.0917	7.0928	7.0940
	3	6.7850	7.1575	7.3334	7.2217	7.2244	7.2255	–
	3.5	6.9266	7.2851	7.4534	7.3476	7.3502	7.3512	–
	4	7.0600	7.4063	7.5679	7.4673	7.4699	7.4709	–
	4.5	7.1863	7.5219	7.6774	7.5815	7.5840	7.5850	–
	5	7.3062	7.6324	6.3571	7.6907	7.6932	7.6942	–
10^2	0	5.7820	6.2943	6.5381	6.3754	6.3788	6.3801	6.3816
	0.5	6.0049	6.4788	6.7052	6.5554	6.5586	6.5599	6.5613
	1	6.2054	6.6487	7.1432	6.7217	6.7248	6.7260	6.7273
	2.5	6.7127	7.0928	7.2727	7.1580	7.1608	7.1618	7.1630
	3	6.8588	7.2238	7.3957	7.2871	7.2898	7.2909	–
	3.5	6.9960	7.3481	7.5128	7.4097	7.4124	7.4134	–
	4	7.1257	7.4663	7.6247	7.5265	7.5291	7.5301	–
	4.5	7.2486	7.5792	7.7319	7.6381	7.6407	7.6416	–
	5	7.3656	7.6873	6.4471	7.7451	7.7475	7.7485	–

It is observed from Table 3 and Table 4 that the fundamental frequency parameter increases with the increase of Pasternak foundation parameter for a given Winkler foundation parameter. It is in general, observed from Table 3 and Table 4 that the fundamental frequency parameter increases with the increase of slenderness ratio for a given Pasternak and Winkler foundation parameter. It is found from Table 3 and Table 4 that the fundamental frequency parameter increases with the increase of Pasternak foundation parameter for a given Winkler foundation parameter and slenderness ratio. It may be noted here that the results obtained using coupled displacement field method for the Timoshenko Hinged-Hinged and Clamped-Clamped beams with and without foundation match very well those of

Table 5.
($\lambda^{1/4}$) Fundamental frequency parameter values of Clamped-Clamped Timoshenko beam resting on Pasternak foundation

Foundation parameters		Slenderness ratio (β)											
		17		52		100	200	300	415				
$\bar{K}_w$	$\frac{\bar{K}_p}{\pi^2}$	Present CDF method	ref. [17]	Present CDF method	ref. [17]	Present CDF method	Present CDF method	Present CDF method	Present CDF method	ref. [17]	ref. [6]	ref. [7]	ref. [2]
0	0	4.2720	4.2634	4.6665	4.6655	4.7142	4.7280	4.7306	4.7316	4.7314	4.7300	4.7300	4.7300
	0.5	4.4508	4.4190	4.8120	4.8038	4.8567	4.8697	4.8721	4.8731	4.8683	4.8680	4.8670	4.8700
	1	4.6104	4.5595	4.9454	4.9302	4.9877	5.0000	5.0023	5.0032	4.9938	4.9940	4.9926	4.9900
	2.5	5.0098	4.9102	5.2917	5.2567	5.3288	5.3397	5.3418	5.3425	5.3195	5.3200	5.3184	5.3200
	3	5.1239	–	5.3933	–	5.4292	5.4398	5.4417	5.4425	–	–	–	–
	3.5	5.2309	–	5.4894	–	5.5243	5.5346	5.5365	5.5372	–	–	–	–
	4	5.3316	–	5.5808	–	5.6147	5.6247	5.6266	5.6273	–	–	–	–
	4.5	5.4270	–	5.6679	–	5.7010	5.7107	5.7126	5.7133	–	–	–	–
	5	5.5176	–	5.7511	–	5.7835	5.7931	5.7949	5.7955	–	–	–	–
10^2	0	4.5390	4.5417	4.8854	4.8926	4.9287	4.9413	4.9437	4.9446	4.9515	4.9500	4.9504	4.9500
	0.5	4.6901	4.6720	5.0132	5.0235	5.0543	5.0663	5.0686	5.0694	5.0718	5.0710	5.0707	5.2300
	1	4.8278	4.7909	5.1319	5.1254	5.1712	5.1827	5.1849	5.1857	5.1834	5.1820	5.1824	5.5400
	2.5	5.1825	5.0974	5.4458	5.4198	5.4812	5.4915	5.4935	5.4942	5.4783	5.4770	5.4773	5.4800
	3	5.2860	–	5.5393	–	5.5737	5.5838	5.5857	5.5864	–	–	–	–
	3.5	5.3837	–	5.6283	–	5.6618	5.6717	5.6735	5.6742	–	–	–	–
	4	5.4764	–	5.7133	–	5.7460	5.7556	5.7574	5.7581	–	–	–	–
	4.5	5.5647	–	5.7946	–	5.8266	5.8361	5.8379	5.8385	–	–	–	–
	5	5.6489	–	5.8726	–	5.9040	5.9133	5.9151	5.9157	–	–	–	–

Table 6.

($\lambda^{1/4}$) Frequency parameter values of Clamped-Clamped Timoshenko beam resting on Pasternak foundation for second mode

Foundation parameters		Slenderness ratio (β)									
		17	52	100	200	300	415				
$\bar{K}_w$	$\frac{\bar{K}_p}{\pi^2}$	Present CDF method	Present CDF method	Present CDF method	Present CDF method	Present CDF method	Present CDF method	ref. [17]	ref. [6]	ref. [7]	ref. [2]
0	0	5.1754	6.8092	7.5036	7.8128	7.8799	7.9065	7.8533	7.8540	7.8532	7.8500
	0.5	5.2901	6.9029	7.6045	7.9174	7.9853	8.0122	7.9680	7.9680	7.9678	7.9700
	1	5.3977	6.9930	7.7015	8.0181	8.0868	8.1140	8.0777	8.0780	8.0775	8.0800
	2.5	5.6871	7.2441	7.9725	8.2992	8.3702	8.3983	8.3812	8.3800	8.3811	8.3800
	3	5.7744	7.3222	8.0570	8.3869	8.4585	8.4870	–	–	–	–
	3.5	5.8579	7.3980	8.1389	8.4719	8.5442	8.5729	–	–	–	–
	4	5.9380	7.4715	8.2184	8.5544	8.6274	8.6564	–	–	–	–
	4.5	6.0149	7.5428	8.2956	8.6346	8.7082	8.7375	–	–	–	–
	5	6.0890	7.6123	8.3708	8.7126	8.7869	8.8164	–	–	–	–
10^2	0	5.2412	6.8505	7.5496	7.8613	7.9289	7.9558	7.9044	7.9040	7.9043	7.9000
	0.5	5.3518	6.9425	7.6487	7.9641	8.0325	8.0596	8.0169	8.0170	8.0168	8.1600
	1	5.4559	7.0311	7.7441	8.0630	8.1322	8.1597	8.1247	8.1240	8.1245	8.3900
	2.5	5.7369	7.2784	8.0110	8.3398	8.4112	8.4395	8.4234	8.4230	8.4232	8.4200
	3	5.8220	7.3555	8.0942	8.4262	8.4983	8.5269	–	–	–	–
	3.5	5.9036	7.4302	8.1750	8.5100	8.5828	8.6117	–	–	–	–
	4	5.9819	7.5028	8.2535	8.5914	8.6649	8.6940	–	–	–	–
	4.5	6.0572	7.5733	8.3298	8.6706	8.7447	8.7741	–	–	–	–
	5	6.1298	7.6419	8.4041	8.7477	8.8224	8.8521	–	–	–	–

exact values and other open literature. In general the Coupled Displacement Field method is equally applicable for calculating frequencies at higher modes.

Similar trend is also observed as discussed in the above section for the case of uniform Timoshenko Clamped-Clamped beam boundary condition resting on Pasternak foundation. It is in general observed from Table 5 and Table 6 that more frequencies are observed in the case of uniform Timoshenko Clamped-Clamped beam boundary condition when compared to the Hinged-Hinged beam boundary condition. Stiffening effect has been observed in the case of uniform Timoshenko Clamped-Clamped beam boundary condition.

4. Conclusions

It is shown that the present CDF method provides a unified approach for the vibration analysis of Hinged-Hinged, Clamped-Clamped beams resting on the Pasternak foundation, compared to the exact values and other existing open literature. The beams considered here are axially immovable ends. Using the present formulation, the values of fundamental frequency parameter is calculated for a two-parameter elastic foundation such as Winkler and Pasternak one. In this method, because of the use of the coupling equation which couples the total rotation and transverse displacement, the computational efforts for solving the free vibration problem are reduced. Numerical results, in terms of frequency parameter in terms of several slenderness ratios, Winkler foundation parameters and Pasternak foundation parameters for the first mode and second mode, are presented in this paper for the most practically used beam boundary conditions such as Hinged-Hinged, Clamped-Clamped. The accuracy of the results are presented in Tables 3, 4,5 and 6.

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