Opuscula Math. 33, no. 2 (2013), 223–235
http://dx.doi.org/10.7494/OpMath.2013.33.2.223 Opuscula Mathematica

# SOME GENERALIZED METHOD FOR CONSTRUCTING NONSEPARABLE COMPACTLY SUPPORTED WAVELETS IN $L^2(\mathbb{R}^2)$

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*Communicated by Palle E.T. Jorgensen*

**Abstract.** In this paper we show the construction of nonseparable compactly supported bivariate wavelets. We deal with the dilation matrix $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ and some three-row coefficient mask; that is a scaling function that satisfies a dilation equation with scaling coefficients which can be contained in the set $\{c_n\}_{n\in\mathcal{S}}$, where $\mathcal{S} = S_1 \times \{0,1,2\}$, $S_1 \subset \mathbb{N}$, $\sharp S_1 < \infty$.

**Keywords:** compactly supported wavelet, compactly supported scaling function, multiresolution analysis, dilation matrix, orthonormality, accuracy.

**Mathematics Subject Classification:** 42C40.

## 1. INTRODUCTION

The theory of compactly supported wavelets in $L^2(\mathbb{R})$ is known and was well-developed mainly by Daubechies [7,8]. Such wavelets were applied in various branches of science and can be obtained by using constructive methods. This theory was generalized to higher dimensions, in particular to $L^2(\mathbb{R}^2)$ case. Simple examples of bivariate wavelets with compact support involve tensor products of compactly supported wavelets and scaling functions, constructed in an $L^2(\mathbb{R})$ space. Of course this method produces so called separable wavelet bases which are not interesting from a theoretical point of view. This is because properties of such basis functions mostly arise from features of univariate wavelets. Nevertheless separable wavelets are a very useful tool in signal analysis. For example the separable transform is easy to implement by using two one-dimensional wavelet transforms separately, that is we can adapt the Mallat algorithm from the one-dimensional case. Since there are many algorithms in this field based on wavelet transform (coding algorithms), it is significant to show advantages and defects of applying separable and nonseparable wavelets, that is, wavelets (wavelet bases) which are not separable.

Nonseparable compactly supported wavelets have been studied by many authors. The construction of such wavelets in $L^2(\mathbb{R}^2)$ was given by Cohen and Daubechies [6]. More examples can be found in [9], where Gröchenig and Madych showed how to obtain wavelet bases of Haar-type in $L^2(\mathbb{R}^n)$ using multiresolution analysis with a scaling function which is a characteristic function of a special compact set. Interesting constructions were given also by Ayache in [1, 2], where the author demonstrated some classes of multi-dimensional filter banks generating nonseparable, compactly supported wavelet bases of arbitrarily high regularity. Also "polyphase components" methods lead to nonseparable wavelets with compact support, which was shown for the two- and three-dimensional case by Kovačević and Vetterli [10]. Belogay and Wang [3] constructed nonseparable compactly supported wavelets with arbitrarily high accuracy using two-row coefficient masks. The result was obtained in a two-dimensional case for every dilation matrix $A$ with $q = |detA| = 2$. In particular this construction can be applied to the quincunx matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ which has been used in applications concerning signal processing. It is known that separable wavelets show some defects and one of them is unpleasant image decomposition which reveals similarity in the same direction. Since nonseparable wavelets lead to more isotropic analysis we hope, that they become much useful. Mentioned quincunx matrix $A$ plays an important role here. First of all the equality $q = 2$ means that we deal only with one wavelet which is nonseparable and there are two filters (the image is split into two subsets in a one-level image decomposition). As it was said, to show the predominance of nonseparable wavelets it is necessary to describe the difference between results of algorithms of image compression and reconstruction. There are several papers which give such a comparison. In [17] four different types of nonseparable wavelets associated with quincunx matrix and dilation matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ were tested on different images by using a nonseparable wavelet transform and compared with a separable wavelet by applying a tensor product wavelet transform based on univariate wavelets (CDF53, DB97). As it was shown, in still image compression the nonseparable wavelets reveal better ascendant performance provided that they have the same number of vanishing moments as a tensor product of univariate wavelets. Another comparison was stated in [15] where authors tested different types of coding algorithms i.e. SPIHT and binary tree coding algorithms using two-dimensional separable and nonseparable wavelet transforms. Similarly to [17] computation were done via quincunx sampling (quincunx lifting shame was applied). Results show that compared with the SPIHT algorithm based on the separable wavelet transform, application of the nonseparable wavelet transform with the binary tree coding algorithm increases the quality of reconstructed images. Valuable features of nonseparable wavelets associated with quincunx dilation matrix were demonstrated in [12], where in addition to construction of filter banks, the authors gave a new method for merging panchromatic and multi-spectral (MS) images. An important fact is that this algorithm based on a nonseparable wavelet frame transform creates the fused MS image which preserves better spectral information and higher spatial resolution than the MS image created by the fusion method based on a discrete wavelet frame transform and improved intensity hue-saturation mergers. Since $q = 2$, the image is split only into two subsets in each level decomposition and as a result, this method requires only the half of the amount of computation used in the fused method based on a discrete wavelet transform with dilation matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. It is worth mentioning that the theory of wavelets in $L^2(\mathbb{R}^n)$ has been extended to $L^2(\mathbb{R}^n)^p$ space. For example in [5] generalized definitions of fundamental notions were given, that is the definition of vector-valued multiresolution analysis, orthogonal vector-valued wavelets etc. Authors presented the construction of vector-valued wavelet packet bases of $L^2(\mathbb{R}^n)^p$ and show several examples of such systems. The results were obtained for every integer dilation $a \in \mathbb{N}$, $a \geq 2$. More general dilations and criterions concerning compactly supported vector-valued scaling functions can be found in earlier works like [11].

The purpose of this paper is to give a generalized method for constructing nonseparable compactly supported wavelets in $L^2(\mathbb{R}^2)$, which allows new types of coefficient masks from those presented in [3]. We focus on the dilation matrix $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ and the standard multiresolution analysis [13, 16]. Although the main result concerns this particular dilation matrix the method presented in [3] shows that it may be possible to give an extension of obtained results to the quincunx matrix or even any $2 \times 2$ dilation matrix with $q = 2$. Since wavelet coefficients are determined by scaling coefficients [14], this construction gives us a class of filter banks which consist of only a low-pass filter and a high-pass filter. Likewise in cited works, the application of wavelet transforms based on presented filters could improve recent results or give valuable algorithms e.g. reduces the amount of operations necessary when using dilation $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. The technique applied to produce two-dimensional filter banks is different from the approach proposed in [10, 12, 15], where methods of polyphase factorization, block central symmetric orthogonal matrices, DFT and IDFT filtering were used. First we extend the orthonormality condition to cases of generalized coefficient masks. Then we parametrize the class of polynomials satisfying this condition and construct low-pass filters. The key is to solve the system of nonlinear functional equations in one complex variable $z \in \mathbb{C}$ by parametrizing solutions with three parameters, where two of them are algebraic polynomials and third is a Laurent polynomial. To ensure that obtained scaling function is orthonormal we need to check additionally Cohen's criterion [4]. It is done in Section 3, where we use the general construction from Section 2 to produce the specific coefficient masks generating nonseparable compactly supported wavelets. Theorem 2.6 which was stated in Section 2 represents the main conclusion and is some generalization of results from [3].

## 2. THE CONSTRUCTION OF NONSEPARABLE COMPACTLY SUPPORTED WAVELETS

We assume, that $\varphi \in L^2(\mathbb{R}^2)$ is a real valued function which satisfies the so called dilation equation with an expanding matrix $A \in \mathcal{M}_{2\times 2}(\mathbb{Z})$, $|detA| = 2$, that is,

$$\varphi(x) = 2 \sum_{n \in \mathbb{Z}^2} c_n \varphi(Ax - n), \tag{2.1}$$

where $c_n \in \mathbb{R}$, $c_n = 0$ for almost every $n \in \mathbb{Z}^2$ and $\sum_{n\in\mathbb{Z}^2} c_n = 1$. We say that $c_n$ are the scaling coefficients of $\varphi$. They define the trigonometric polynomial $m(\xi) = \sum_{n\in\mathbb{Z}^2} c_n e^{-i\langle \xi,n\rangle}$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ such that the Fourier transform of $\varphi$ satisfies equation

$$\widehat{\varphi}(\xi) = m(B^{-1}\xi)\widehat{\varphi}(B^{-1}\xi),$$

where $B = A^T$. Then the polynomial $m(\xi)$ defines coefficient mask $M(z,w)$, $(z,w) \in \mathbb{C}^2$ by the equation $m(\xi_1,\xi_2) = M(e^{-i\xi_1}, e^{-i\xi_2})$. Clearly we may write $m(0) = M(1,1) = 1$ and $M(z,w) = \sum_{(m,n)\in\mathbb{Z}^2} c_{(m,n)} z^m w^n$.

If we work with a multiresolution analysis, we say that $\varphi$ is a scaling function which additionally gives us (by definition of MRA) orthonormality of the set $\{\varphi(x-n)\}_{n\in\mathbb{Z}^2}$. For our purpose we consider the dilation matrix $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ and a three-row coefficient mask $M(z,w)$ of the following form:

$$M(z,w) = A(z) + B(z)w + G(z)w^2, \quad (z,w) \in \mathbb{C}^2, \tag{2.2}$$

where $A(z)$, $B(z)$, $G(z)$ are polynomials of one complex variable. As we know [4,16], if $\varphi$ is a scaling function then the orthonormality condition is satisfied, that is, for all $k \in \mathbb{Z}^2$ we have equality

$$2\sum_{n\in\mathbb{Z}^2} c_n c_{n+Ak} = \delta_{0,k}, \tag{2.3}$$

where $\delta_{0,k}$ is the Kronecker delta. Additionally this condition is equivalent to the identity $\mathcal{P}_M(z,w)+\mathcal{P}_M(-z,w) = 1$, $(z,w) \in \mathbb{C}^2$, $zw \neq 0$, where $M$ is the three-row coefficient mask and autocorrelation $\mathcal{P}_W(z_1,z_2)$, $z_1z_2 \neq 0$ of some polynomial $W(z_1,z_2)$, $(z_1,z_2) \in \mathbb{C}^2$ with real coefficients is of the form $\mathcal{P}_W(z_1,z_2) = W(z_1,z_2)W(z_1^{-1},z_2^{-1})$. By "orthonormal coefficient mask" we mean that condition (2.3) is satisfied. These assumptions and facts lead us to the first conclusion.

**Fact 2.1.** *The three-row coefficient mask $M$ is orthonormal if and only if the following equations are satisfied:*

$$\mathcal{P}_A(z) + \mathcal{P}_A(-z) + \mathcal{P}_B(z) + \mathcal{P}_B(-z) + \mathcal{P}_G(z) + \mathcal{P}_G(-z) = 1, \tag{2.4}$$

$$A(z^{-1})G(z) + A(-z^{-1})G(-z) = 0, \tag{2.5}$$

$$A(z^{-1})B(z) + B(z^{-1})G(z) + A(-z^{-1})B(-z) + B(-z^{-1})G(-z) = 0, \tag{2.6}$$

*where for a univariate polynomial $W(z)$, $z \in \mathbb{C}$, we define*

$$\mathcal{P}_W(z) := W(z)W(z^{-1}), \quad z \neq 0.$$

The next step is to replace (2.4)-(2.6) by an equivalent condition on $A$, $B$, $G$. Of course the equality (2.5) and condition $G(0) \neq 0$ imply that $A$ is a polynomial of odd degree. The next proposition is an observation from [3] and can be adapted to our case.

**Proposition 2.2** ([3]). *Let $G(0) \neq 0$. Polynomials $A$, $G$ satisfy the condition* (2.5) *if and only if there exist an odd integer $\nu \geq \deg A$ and polynomials $s$, $q$, $l$ with real coefficients such that $z^\nu A(z^{-1}) = s(z^2)l(z)$, $G(z) = q(z^2)l(-z)$, where* $\gcd(l(z), l(-z)) = 1$.

Thus we must show that the condition (2.6) also could be described by some generalized version of the previous proposition. Indeed, we can formulate the following statement.

**Lemma 2.3.** *Let $G(0) \neq 0$. Then polynomials $A$, $B$, $G$ satisfy conditions* (2.5), (2.6) *if and only if there exist an odd integer $\nu \geq \deg A$, $\nu \geq \deg B$, polynomials $s$, $q$, $l$ with real coefficients and a Laurent polynomial $\widetilde{s}$ with real coefficients such that:*

$$\text{(i)}\quad z^{\nu}A(z^{-1}) = s(z^2)l(z), \qquad \text{(ii)}\quad G(z) = q(z^2)l(-z),$$
$$\text{(iii)}\quad B(z) = \frac{s(z^{-2})\widetilde{s}(z^2)l(-z) - z^{\nu}q(z^2)\widetilde{s}(z^{-2})l(z^{-1})}{\mathcal{P}_s(z^2) + \mathcal{P}_q(z^2)},$$

*where* $\gcd(l(z), l(-z)) = 1$.

*Proof.* If polynomials $A$, $B$, $G$ satisfy (2.5), (2.6) and $G(0) \neq 0$, then Proposition 2.2 leads us to the following equalities:

$$1)\ z^{\nu}A(z^{-1}) = s(z^2)l(z), \qquad 2)\ G(z) = q(z^2)l(-z),$$

where $\nu \geq \deg A$, $\nu \geq \deg B$ is an odd integer, $s$, $q$, $l$ are polynomials with real coefficients and $\gcd(l(z), l(-z)) = 1$.

Applying equality 1) and 2) to the equation (2.6) we obtain

$$l(z)P_1(z) = l(-z)P_2(z), \tag{2.7}$$

where

$$P_1(z) := z^{-\nu}s(z^2)B(z) + B(-z^{-1})q(z^2),$$
$$P_2(z) := z^{-\nu}s(z^2)B(-z) - B(z^{-1})q(z^2).$$

Observe that $P_1(-z) + P_2(z) = 0$. By virtue of (2.7), we may also write $[z^{\nu}P_1(z)]/l(-z) = K(z)$, $[z^{\nu}P_2(z)]/l(z) = W(z)$, where $K$, $W$ are polynomials with real coefficients. Thus we can say that $K$, $W$ are some Laurent polynomials. Additionally the equation (2.7) leads us to the conclusion $K(z) = W(z)$. Moreover, the following equalities are satisfied:

$$l(z)W(z) - l(z)K(-z) = z^{\nu}P_2(z) + z^{\nu}P_1(-z) = z^{\nu}[P_1(-z) + P_2(z)] = 0.$$

Thus we have relations $K(z) = W(z) = K(-z)$, that is $W(z) = K(z) = \widetilde{s}(z^2)$, where $\widetilde{s}$ is a Laurent polynomial with real coefficients. Now we can rewrite the equation (2.7) in an equivalent form

$$z^{-\nu}s(z^2)B(-z) - B(z^{-1})q(z^2) = \widetilde{s}(z^2)l(z)z^{-\nu},$$

where $\widetilde{s}$ is some Laurent polynomial with real coefficients. Multiplying both sides of the above equation by $z^{\nu}s(z^{-2})$ we obtain:

$$\begin{aligned} s(z^{-2})s(z^2)B(-z) &= z^{\nu}s(z^{-2})B(z^{-1})q(z^2) + s(z^{-2})\widetilde{s}(z^2)l(z) = \\ &= z^{\nu}[\widetilde{s}(z^{-2})l(-z^{-1}) - z^{-\nu}B(-z)q(z^{-2})]q(z^2) + \\ &\quad + s(z^{-2})\widetilde{s}(z^2)l(z). \end{aligned}$$

Therefore polynomial $B(z)$ satisfies equation

$$[s(z^2)s(z^{-2}) + q(z^2)q(z^{-2})]B(-z) = s(z^{-2})\widetilde{s}(z^2)l(z) + z^\nu q(z^2)\widetilde{s}(z^{-2})l(-z^{-1}).$$

The previous relation determines polynomial $B(z)$, that is,

$$B(z) = \frac{s(z^{-2})\widetilde{s}(z^2)l(-z) - z^\nu q(z^2)\widetilde{s}(z^{-2})l(z^{-1})}{\mathcal{P}_s(z^2) + \mathcal{P}_q(z^2)},$$

where $s$, $q$, $l$ are polynomials with real coefficients, $\widetilde{s}$ is a Laurent polynomial with real coefficients and $\nu \geq \deg A$, $\nu \geq \deg B$ is an odd integer.

It is easy to see that if polynomials $A$, $B$, $G$ are described by (i), (ii), (iii) with an odd integer $\nu \geq \deg A$, $\nu \geq \deg B$, then equalities (2.5) and (2.6) are satisfied. □

Observe that Lemma 2.3 gives us a full parametrization of function $A$, $B$, $G$ which satisfy equations (2.5) and (2.6). In this case we can use polynomials $s$, $q$, $l$ and $\widetilde{s}$ as parameters. Obviously $A$, $B$, $G$ have to be polynomials, thus the parametrization must be chosen properly. Via (i), (ii) it is easy to do that for $A$ and $G$, but (iii) needs to be considered separately. For that purpose we take the parametrization:

$$\widetilde{s}(z^2) := T(z^2)p(z^2), \quad T(z^2) := \mathcal{P}_s(z^2) + \mathcal{P}_q(z^2),$$

where $s$, $q$, $p$ are polynomials with real coefficients. We have the equality $T(z^{-2}) = T(z^2)$, thus $B(z)$ takes the following form:

$$B(z) = s(z^{-2})p(z^2)l(-z) - z^\nu q(z^2)p(z^{-2})l(z^{-1}).$$

This shows that fixing $s$, $q$, $l$ we are able to choose polynomial $p$ and an odd integer $\nu \geq \deg A$, $\nu \geq \deg B$ such that the above formula defines a polynomial. Clearly $\nu$ can be arbitrarily large thus in front of the arbitrariness of $s$, $q$, $l$, relations (i) and (ii) also may define various polynomials $A$, $G$. This brings us to the next conclusion.

**Corollary 2.4.** *Assuming that $s$, $q$, $l$, $p$ are polynomials with real coefficients, the solution of equations* (2.5), (2.6) *can be described as follows:*

$$\begin{aligned} &\text{(i)} \quad z^\nu A(z^{-1}) = s(z^2)l(z), \qquad \text{(ii)} \quad G(z) = q(z^2)l(-z), \\ &\text{(iii)} \quad B(z) = s(z^{-2})p(z^2)l(-z) - z^\nu q(z^2)p(z^{-2})l(z^{-1}), \end{aligned}$$

*where $s$, $q$, $l$, $p$ and an odd integer $\nu \geq \deg A$, $\nu \geq \deg B$ are chosen such that the above formulas define polynomials.*

Since our main result is related to the scaling function $\varphi$, we have to assume that polynomials $A$, $B$, $G$ in Lemma 2.3 satisfy the additionally condition (2.4). More precisely, for that purpose we take the parametrization given in Corollary 2.4. The below lemma gives us a description of such polynomials.

**Lemma 2.5.** *Let $G(0) \neq 0$. If polynomials $A$, $B$, $G$ are given by* (i), (ii) *and* (iii) *in Corollary 2.4, then the equation* (2.4) *is satisfied if and only if the following conditions hold:*

$$\mathcal{P}_s(z^2) = b - \mathcal{P}_q(z^2), \tag{2.8}$$

$$\mathcal{P}_l(z) + \mathcal{P}_l(-z) = \frac{1}{(1+c^2)b}, \tag{2.9}$$

*where $b, c \in \mathbb{R}$, $b > 0$, $c \neq 0$ and $p(z^2) = cz^{2k}$, $k \in \mathbb{N}$.*

*Proof.* Formulas (i), (ii), (iii) in Corollary 2.4 imply that polynomials $A$, $B$, $G$ satisfy equations:

1) $\mathcal{P}_A(z) + \mathcal{P}_A(-z) = \mathcal{P}_s(z^2)[\mathcal{P}_l(z) + \mathcal{P}_l(-z)]$,
2) $\mathcal{P}_B(z) + \mathcal{P}_B(-z) = \mathcal{P}_p(z^2)[\mathcal{P}_s(z^2) + \mathcal{P}_q(z^2)][\mathcal{P}_l(z) + \mathcal{P}_l(-z)]$,
3) $\mathcal{P}_G(z) + \mathcal{P}_G(-z) = \mathcal{P}_q(z^2)[\mathcal{P}_l(z) + \mathcal{P}_l(-z)]$.

Putting 1), 2) and 3) together we observe that (2.4) is equivalent to the following equation

$$[1 + \mathcal{P}_p(z^2)][\mathcal{P}_s(z^2) + \mathcal{P}_q(z^2)][\mathcal{P}_l(z) + \mathcal{P}_l(-z)] = 1.$$

It shows that each expression $1+\mathcal{P}_p(z^2)$, $\mathcal{P}_s(z^2)+\mathcal{P}_q(z^2)$ must be a constant. Therefore we conclude that $p(z^2) = cz^{2k}$ and polynomials $s$, $q$, $l$ are given by (2.8), (2.9), where $k \in \mathbb{N}$ and $b, c \in \mathbb{R}$, $b > 0$, $c \neq 0$. □

In our case the general theory of wavelets shows that to construct a compactly supported scaling function $\varphi$ of some multiresolution analysis and its associated wavelet $\psi$ with compact support, we need to construct an orthonormal coefficient mask $M$ such, that the trigonometric polynomial $m(\xi_1, \xi_2) = M(e^{-i\xi_1}, e^{-i\xi_2})$ satisfies Cohen's criterion, that is, for all $j \geq 1$:

$$m(B^{-j}\xi) \neq 0 \text{ for all } \xi \in K, \tag{2.10}$$

where $B = A^T$ and $K \subset \mathbb{R}^2$ is a compact fundamental domain of the lattice $2\pi\mathbb{Z}^2$ [4, 16]. Therefore we are ready to formulate the below theorem.

**Theorem 2.6.** *Let $A$, $B$, $G$ be polynomials defined as follows:*

$$\text{(i)} \quad z^\nu A(z^{-1}) = s(z^2)l(z), \qquad \text{(ii)} \quad G(z) = q(z^2)l(-z),$$
$$\text{(iii)} \quad B(z) = s(z^{-2})p(z^2)l(-z) - z^\nu q(z^2)p(z^{-2})l(z^{-1}),$$

*where $p(z^2) = cz^{2k}$, $c \in \mathbb{R}$, $c \neq 0$, $k \in \mathbb{N}$ and $s$, $q$, $l$ are polynomials with real coefficients which satisfy equations:*

$$\mathcal{P}_s(z^2) = b - \mathcal{P}_q(z^2),$$
$$\mathcal{P}_l(z) + \mathcal{P}_l(-z) = \frac{1}{(1+c^2)b},$$

*with $b \in \mathbb{R}$, $b > 0$. If $m(\xi_1, \xi_2) = M(e^{-i\xi_1}, e^{-i\xi_2})$ satisfies Cohen's criterion* (2.10)*, where $M$ is a three-row coefficient mask given by* (2.2)*, then $\varphi$ is a compactly supported scaling function of some multiresolution analysis and its associated wavelet $\psi$ has a compact support.*

We must note that we deal with a nonseparable coefficient mask $M$, thus using the same argumentation as in [3] the previous theorem gives us a method for constructing nonseparable compactly supported wavelets.

## 3. EXAMPLES OF NONSEPARABLE COEFFICIENT MASKS

The next step is to find such parametrization of polynomials $A$, $B$, $G$ from Theorem 2.6, such that Cohen's criterion is satisfied. In our case it is sufficient to show that polynomial $m(\xi)$ satisfies the following condition:

$$m(\xi) \neq 0 \quad \text{for all} \quad \xi \in K_1 = \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big] \times \Big[-\pi, \pi\Big]. \tag{3.1}$$

Therefore consider such a trigonometric polynomial $m(\xi)$, $\xi \in \mathbb{R}^2$, where $l(z)$ is a constant and assume that $m(\xi_1, \xi_2) = 0$ for some $(\xi_1, \xi_2) \in K_1$. The previous theorem shows that $m(\xi)$ can be expressed by

$$m(\xi_1, \xi_2) = l(1)\big[s(e^{i2\xi_1})\alpha(\xi_1, \xi_2) + q(e^{-i2\xi_1})\beta(\xi_1, \xi_2)\big], \tag{3.2}$$

where $l(z) = l(1) = [2(1 + c^2)b]^{-\frac{1}{2}}$, $b$, $c \in \mathbb{R}$, $b > 0$, $c \neq 0$ and trigonometric polynomials $\alpha$, $\beta$ are defined as follows:

$$\alpha(\xi_1, \xi_2) := e^{-i\nu\xi_1} + ce^{-i(2k\xi_1 + \xi_2)}, \quad \beta(\xi_1, \xi_2) := e^{-i2\xi_2} - ce^{-i(\nu\xi_1 - 2k\xi_1 + \xi_2)}.$$

Since $m(\xi_1, \xi_2) = 0$ by assumption, for some $(\xi_1, \xi_2) \in K_1$, the below equality holds

$$\big|s(e^{i2\xi_1})\big|^2 \big|\alpha(\xi_1, \xi_2)\big|^2 = \big|q(e^{i2\xi_1})\big|^2 \big|\beta(\xi_1, \xi_2)\big|^2. \tag{3.3}$$

By simple computation, we obtain

$$\begin{aligned} \big|\alpha(\xi_1, \xi_2)\big|^2 &= 1 + c^2 + 2c\cos[(\nu - 2k)\xi_1 - \xi_2], \\ \big|\beta(\xi_1, \xi_2)\big|^2 &= 1 + c^2 - 2c\cos[(\nu - 2k)\xi_1 - \xi_2]. \end{aligned}$$

Then putting $z = e^{i\xi_1}$ in (2.8) we get the equation $\big|s(e^{i2\xi_1})\big|^2 + \big|q(e^{i2\xi_1})\big|^2 = b$. Moreover, under an extra assumption $c \notin \{-1, 1\}$ inequalities $\big|\alpha(\xi_1, \xi_2)\big| > 0$, $\big|\beta(\xi_1, \xi_2)\big| > 0$ hold, thus (3.3) leads us to the conclusion that

$$\big|s(e^{i2\xi_1})\big|^2 = b\bigg(\frac{1}{2} - \frac{c}{1 + c^2}\cos[(\nu - 2k)\xi_1 - \xi_2]\bigg), \tag{3.4}$$

where $(\xi_1, \xi_2) \in K_1$ and $c \in \mathbb{R} \setminus \{-1, 0, 1\}$.

These considerations bring us to the next statement.

**Corollary 3.1.** *Let $A$, $B$, $G$ be polynomials defined by* (i), (ii), (iii) *of Theorem 2.6, where* $l(z) = [2(1+c^2)b]^{-\frac{1}{2}}$, $p(z^2) = cz^{2k}$, $b > 0$, $c \in \mathbb{R} \setminus \{-1, 0, 1\}$, $k \in \mathbb{N}$, *and* $s$, $q$ *satisfy equation* (2.8). *If* $m(\xi_1, \xi_2) = 0$ *for some* $(\xi_1, \xi_2) \in K_1$, *then the equality* (3.4) *holds.*

Still we are going to determine values $s(1)$, $q(1)$ such that the condition $m(0,0) = 1$ holds. Since the equation (2.8) is satisfied we need to solve the following system:

$$\begin{cases} (1+c)s(1) + (1-c)q(1) = \sqrt{2(1+c^2)b}, \\ s^2(1) + q^2(1) = b. \end{cases}$$

The solution of that system is unique, thus by simple computation we obtain $s(1) = \frac{(1+c)\sqrt{b}}{\sqrt{2(1+c^2)}}$, $q(1) = \frac{(1-c)\sqrt{b}}{\sqrt{2(1+c^2)}}$. To show an example of polynomial $q$ we may take $q(z^2) = \frac{z^2-1}{n} + q(1)$, $n \in \mathbb{N}$. Observe that taking $n$ large enough and $c \in (0,1)$, the polynomial $s(z)$ can by defined by the relation (2.8) for $|z| = 1$. Indeed, we have

$$\begin{aligned} \left|s(e^{i2\xi_1})\right|^2 &= \frac{2}{n^2}[1 - nq(1)]\cos 2\xi_1 + b - \frac{1}{n^2} - \left[q(1) - \frac{1}{n}\right]^2 = \\ &= \frac{4}{n^2}\left[\frac{n(1-c)\sqrt{b}}{\sqrt{2(1+c^2)}} - 1\right]\sin^2\xi_1 + b\left(\frac{1}{2} + \frac{c}{1+c^2}\right). \end{aligned} \tag{3.5}$$

Now we need to show that $s$, $q$ are chosen properly, that is polynomial $m(\xi)$ defined by (3.2) satisfies (3.1). Corollary 3.1 implies that it is sufficient to check the condition (3.1) only for $\xi = (\xi_1, \xi_1) \in K_1$ which satisfy the equality (3.4). Therefore define the polynomial $K(\xi_1, \xi_2)$ as follows:

$$K(\xi_1, \xi_2) := \left|s(e^{i2\xi_1})\right|^2 - b\left(\frac{1}{2} - \frac{c}{1+c^2}\cos[(\nu - 2k)\xi_1 - \xi_2]\right),$$

where $b > 0$, $c \in (0,1)$. As we see,

$$K(\xi_1, \xi_2) = \frac{bc}{1+c^2}(1 + \cos[(\nu - 2k)\xi_1 - \xi_2]) + \frac{4}{n^2}\left[\frac{n(1-c)\sqrt{b}}{\sqrt{2(1+c^2)}} - 1\right]\sin^2\xi_1.$$

Since $n \in \mathbb{N}$ is large enough the inequalities $\frac{bc}{1+c^2} > 0$, $\frac{n(1-c)\sqrt{b}}{\sqrt{2(1+c^2)}} - 1 > 0$ hold. This shows that if $K(\xi_1, \xi_2) = 0$ for some $\xi = (\xi_1, \xi_1) \in K_1$, then we have $\cos[(\nu - 2k)\xi_1 - \xi_2] = -1$, $\sin^2\xi_1 = 0$ which gives us immediately $\xi_1 = 0$ and $\xi_2 = \pi + 2j\pi$, $j \in \mathbb{Z}$. Applying (3.2) for $\xi_1 = 0$, $\xi_2 = \pi + 2j\pi$ we obtain

$$m(0, \pi + 2j\pi) = m(0, \pi) = l(1)[(1-c)s(1) + (1+c)q(1)] = \frac{1-c^2}{1+c^2} \neq 0,$$

where $c \in (0,1)$. As we mentioned before, the last inequality implies that Cohen's criterion is satisfied.

The next theorem is a summary of the above construction.

**Theorem 3.2.** *Let $A$, $B$, $G$ be polynomials defined as follows:*

$$\text{(i)}\quad z^\nu A(z^{-1}) = s(z^2)l(z), \qquad \text{(ii)}\quad G(z) = q(z^2)l(-z),$$
$$\text{(iii)}\quad B(z) = s(z^{-2})p(z^2)l(-z) - z^\nu q(z^2)p(z^{-2})l(z^{-1}),$$

*where $p(z^2) = cz^{2k}$, $l(z) = [2(1+c^2)b]^{-\frac{1}{2}}$, $q(z^2) = \frac{z^2-1}{n} + \frac{(1-c)\sqrt{b}}{\sqrt{2(1+c^2)}}$, $k \in \mathbb{N}$, $b, c \in \mathbb{R}$, $b > 0$, $c \in (0,1)$ and $n \in \mathbb{N}$ is large enough, that is the equation*

$$|s(z^2)|^2 = b - |q(z^2)|^2 \geq 0, \ \ |z| = 1,$$

*defines some polynomial $s$ with $s(1) = \frac{(1+c)\sqrt{b}}{\sqrt{2(1+c^2)}}$. Then the polynomial $m(\xi_1, \xi_2) = M(e^{-i\xi_1}, e^{-i\xi_2})$, where $M$ is the three-row coefficient mask given by (2.2), generates the nonseparable compactly supported wavelet $\psi$ associated with a multiresolution analysis for the dilation matrix $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$.*

Now we are going to give an explicit examples of low-pass filters generating nonseparable compactly supported wavelets. For that purpose we set $n = 1$ and $b = 2(1+c^2)$ in Theorem 3.2. In this case it is easy to solve the equation

$$\mathcal{P}_s(z^2) = b - \mathcal{P}_q(z^2),$$

where $q(z^2) = z^2 - c$. Namely the polynomial $s$ in the above theorem can be chosen in two ways, $s_1(z^2) = z^2 + c$ and $s_2(z^2) = cz^2 + 1$. To get some parametrized family of low-pass filters $m$ from Theorem 3.2 assume the possibility that $s(z^2) = z^2 + c$. Since $l(z) = \frac{1}{2(1+c^2)}$ we can compute explicitly coefficients of $m$. We consider the example, where $\nu = 3$ and $k = 2$. This follows that nonzero coefficients $c_{(m,n)}$ take the form:

$$c_{(1,0)} = \frac{1}{2(1+c^2)},\ c_{(3,0)} = \frac{c}{2(1+c^2)},\ c_{(1,1)} = -\frac{c}{2(1+c^2)},\ c_{(-1,1)} = \frac{c^2}{2(1+c^2)},$$
$$c_{(2,1)} = \frac{c}{2(1+c^2)},\ c_{(4,1)} = \frac{c^2}{2(1+c^2)},\ c_{(2,2)} = \frac{1}{2(1+c^2)},\ c_{(0,2)} = -\frac{c}{2(1+c^2)},$$

where $c \in (0,1)$. The consequence of previous assumptions is, that $B$ contains the negative power of $z$. Nevertheless this example shows that Theorem 3.2 is valid in situations when $A$, $B$ or $G$ is a Laurent polynomial. The plot of the corresponding scaling function $\varphi$ for $c = 0.5$ was depicted in Figure 1.

It is known that in this case the support of every scaling function is contained in some compact set $Q = \{x \in \mathbb{R}^2 : \ x = \sum_{j=1}^{\infty} A^{-j} s_j \ \text{ for } \ s_j \in S\}$, where $S = \{(m,n) \in \mathbb{Z}^2 : \ c_{(m,n)} \neq 0\}$. Obviously $S$ is independent of $c$, that is $S = \{(1,0),\ (3,0),\ (1,1),\ (-1,1),\ (2,1),\ (4,1),\ (2,2),\ (0,2)\}$ which gives us the relation $\operatorname{supp} \varphi \subset Q$ for every $c \in (0,1)$ and $Q$ is unique.

To approximate the set $Q$ we can use a sequence $(Q_N)_{N=0,1,\ldots}$ of compact sets defined by the formula

$$Q_{N+1} = \bigcup_{s \in S} A^{-1}(Q_N + s) \ \text{ for } \ N = 0, 1, \ldots,$$

where $Q_0$ can be any compact set in $\mathbb{R}^2$. Since $(Q_N)$ is convergent to $Q$ [13], we conclude that supp $\varphi \subset [-1, 7] \times [-1, 6]$, where $\varphi$ represents any scaling function constructed for $c \in (0, 1)$. It can be seen in Figure 2 which shows the good approximation of set $Q$ obtained by means of the sequence $(Q_N)$. Obviously for a given scaling function $\varphi$ we are able to construct its associated wavelet $\psi$. For that purpose it is sufficient to apply the following equation

$$\psi(x) = \sum_{n \in \mathbb{Z}^2} d_n \varphi(Ax - n),$$

where $d_n = (-1)^{n_1} c_{e-n}$, $n = (n_1, n_2)$, $e = (1, 0)$ [14]. To give at least one example of such wavelet we consider again the case $c = 0.5$. Therefore the scaling function $\varphi$ from Figure 1 has the corresponding wavelet whose plot was shown in Figure 3.

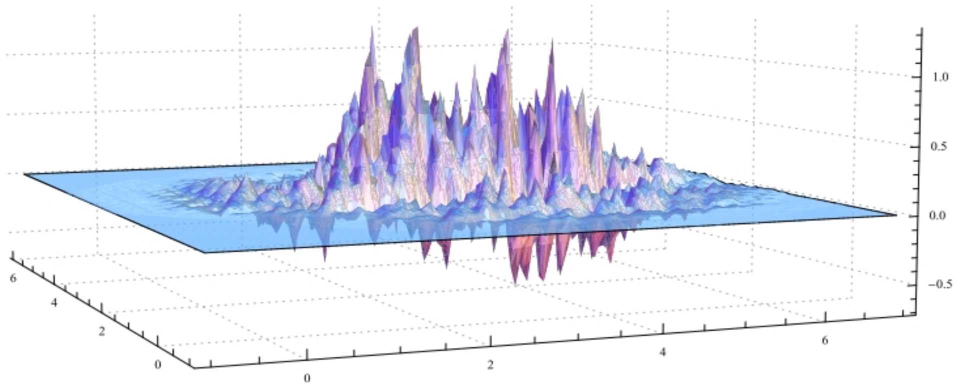


**Fig. 1.** A scaling function constructed by $\nu = 3$, $k = 2$ and $c = 0.5$

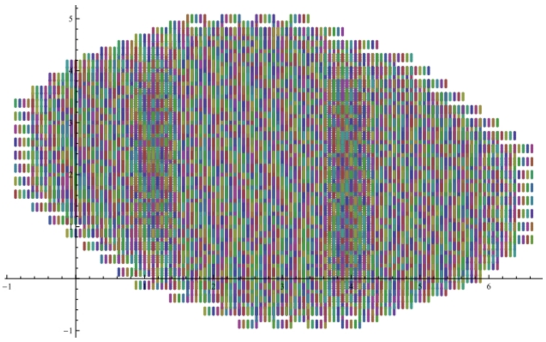

**Fig. 2.** The set $Q_N$ for $S = \{(1,0),\ (3,0),\ (1,1),\ (-1,1),\ (2,1),\ (4,1),\ (2,2),\ (0,2)\}$ and $N = 5$. Computation were done with $Q_0 = \left[-\frac{1}{2}, \frac{1}{2}\right]^2$

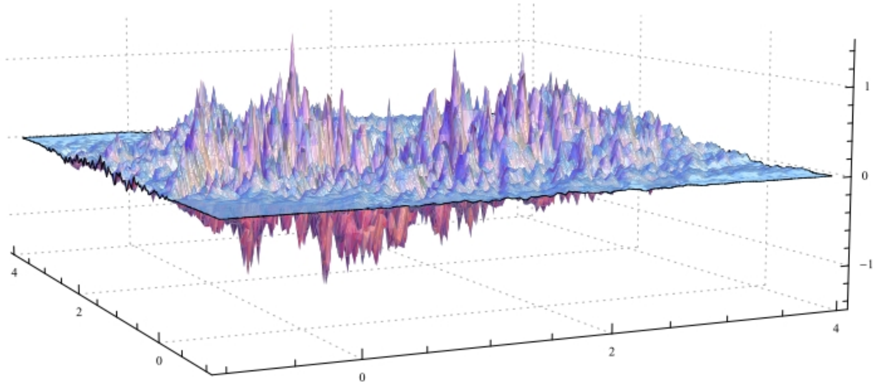


**Fig. 3.** A nonseparable compactly supported wavelet associated with a scaling function $\varphi$ by $\nu = 3$, $k = 2$ and $c = 0.5$

**Acknowledgments**

*The author thanks the referees for their valuable suggestions. The plots were done using* Mathematica. *This work was supported by National Science Centre grant No. N N201 610240 for years 2011–2013.*

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*Received: November 22, 2011.*
*Revised: August 16, 2012.*
*Accepted: September 13, 2012.*