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Vibration Analysis of a Thick Ring Interacting with the Disk Treated as an Elastic Foundation

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Abstract

In this study the in plane flexural vibration of a thick ring interacting with Winkler foundation is analysed on the basis of the analytical and numerical method. The effect of rotary inertia and shear deformation is included. The normal frequencies and natural mode shapes of the system vibration are determined. Achieved results are discussed and compared with an experimental data. FE models are formulated by using ANSYS code.

Keywords: in–plane vibration, Timoshenko's theory, thick ring with foundation

1. Introduction

The problems of in–plane flexural vibration of circular rings with wheel–plate as an elastic foundation find application in several practical problems [6]. The fundamental circular rings vibration theory is presented in [5]. In the article [6] authors analyse free vibration of a ring gear by using thin ring theory. Free vibration of Timoshenko beam attached to linear elastic foundation are investigated in the paper [1]. The introductory studies related to the systems of the rings with wheel–plate as the elastic foundation are conducted in [3, 4]. In paper [2] the special three–parameter elastic foundation is proposed. In above paper the free in–plane flexural vibration of a circular ring with wheel–plate as a special three–parameter elastic foundation is analyzed using the classical thick ring theory, and the finite element (FE) technique. The procedure of determining the substitute mass density of a ring with massless foundation is presented. Obtained results of calculation are discussed and compared with experimental data. Experimental investigation are conducted by using two objects with the arbitrary chosen geometry.

2. Theoretical formulation

The mechanical model of the system under study consists of circular ring with wheel–plate as a special three–parameter, linear, elastic foundation. It is assumed that ring is homogeneous, perfectly elastic and it has rectangular, and constant cross–sectional area. It is additionally assumed that the centerline of the ring has radius R and an element of the ring, fixed by angle θ, displaces in the radial and circumferential direction, respectively (see Fig. 1). The small displacements in these directions are denoted as $u(\theta,t)$ and $w(\theta,t)$, respectively, and t is time. According with the theory, discussed in [2], the foundation is represented by the special three–parameter Winkler model. The coefficients k_f, k_p and k_S represent the radial and the tangential stiffness per length unit, and the ring cross-section angle rotation stiffness modulus, respectively.

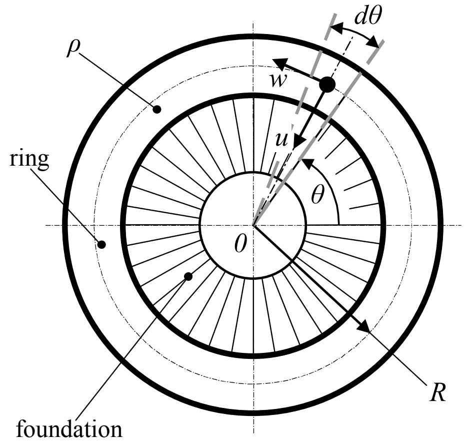

Figure 1. Vibrating system under study

Making use of the classical theory of vibrating thick rings [5], the partial differential equations of motion for the free in–plane flexural vibration can be combined into an only one equation in terms of the radial deflection $u(\theta,t)$ as

$$\begin{aligned}
&\frac{\partial^6 u}{\partial\theta^6}+\left(2-k_f\frac{R^2}{kAG}-k_S\frac{R^2}{EI_1}\right)\frac{\partial^4 u}{\partial\theta^4}+\left(1+k_p\frac{R^2}{kAG}-2k_S\frac{R^2}{EI_1}+k_f k_S\frac{R^4}{kAGEI_1}+\right.\\
&\left.+k_f\frac{R^4}{EI_1}\right)\frac{\partial^2 u}{\partial\theta^2}-\left(k_S\frac{R^2}{EI_1}+k_p k_S\frac{R^4}{kAGEI_1}+k_p\frac{R^4}{EI_1}\right)u-\left(\frac{\rho R^2}{E}+\frac{\rho R^2}{kG}\right)\frac{\partial^6 u}{\partial\theta^4\partial t^2}+\\
&+\frac{\rho^2R^4}{kGE}\frac{\partial^6 u}{\partial\theta^2\partial t^4}+\left(-2\frac{\rho R^2}{E}+k_f\frac{\rho R^4}{kAGE}+\frac{\rho R^2}{kG}+k_S\frac{\rho R^4}{kGEI_1}+\frac{\rho AR^4}{EI_1}\right)\frac{\partial^4 u}{\partial\theta^2\partial t^2}+\\
&-\frac{\rho^2R^4}{kGE}\frac{\partial^4 u}{\partial t^4}-\left(\frac{\rho R^2}{E}+k_p\frac{\rho R^4}{kAGE}+k_S\frac{\rho R^4}{kGEI_1}+\frac{\rho AR^4}{EI_1}\right)\frac{\partial^2 u}{\partial t^2}=0
\end{aligned}\tag{1}$$

where E denotes Young's modulus of elasticity, G is the Kirhoff modulus, I_1 is the area moment of inertia of the rim cross section, ρ is the mass density, A is the cross section area, k is the shear correction factor. The general solution of equation (1) is assumed to be harmonic, i.e.

$$u(\theta,t)=U(\theta)\mathrm{e}^{\mathrm{i}\omega t}\tag{2}$$

where ω is the natural frequency and $\mathrm{i}=\sqrt{-1}$ is the imaginary unit. Substituting solution (2) into equation (1) gives the following expression

$$\begin{gathered}\frac{\mathrm{d}^6U}{\mathrm{d}\theta^6}+\left(2-b_0k_f-k_S\frac{a_0}{R^2}\right)\frac{\mathrm{d}^4U}{\mathrm{d}\theta^4}+\left(1+b_0k_p-2k_S\frac{a_0}{R^2}+k_fk_S\frac{h_0}{\rho I_1}+k_fa_0\right)\frac{\mathrm{d}^2U}{\mathrm{d}\theta^2}+\\-\left(k_S\frac{a_0}{R^2}+k_pk_S\frac{h_0}{\rho I_1}+k_pa_0\right)U+(c_0+d_0)\omega^2\frac{\mathrm{d}^4U}{\mathrm{d}\theta^4}+c_0d_0\omega^4\frac{\mathrm{d}^2U}{\mathrm{d}\theta^2}-c_0d_0\omega^4U+\\-\left(-2c_0+k_fh_0+d_0+k_S\frac{h_0A}{I_1}+\rho Aa_0\right)\omega^2\frac{\mathrm{d}^2U}{\mathrm{d}\theta^2}+\left(c_0+k_ph_0+k_S\frac{h_0A}{I_1}+\rho Aa_0\right)\omega^2U=0\end{gathered}\tag{3}$$

where

$$a_0=\frac{R^4}{EI_1},\quad b_0=\frac{R^2}{kAG},\quad c_0=\frac{\rho R^2}{E},\quad d_0=\frac{\rho R^2}{kG},\quad h_0=\frac{\rho R^4}{kEAG}\tag{4}$$

The solution of equation (3) is assumed in the form

$$U(\theta)=\sum_{j=1}^{3}C_{jn}\sin\left(n\theta+\varphi_{jn}\right),\quad n=2,3,\ldots\tag{5}$$

where C_{jn} and φ_{jn} are constants. When equation (5) is substituted into equation (3), it yields the following frequency equation.

$$\begin{gathered}-c_0d_0\left(n^2+1\right)\omega_n^4+\left[(c_0+d_0)n^4+\left(-2c_0+k_fh_0+d_0+k_S\frac{h_0A}{I_1}+\rho Aa_0\right)n^2+\right.\\\left.+\left(c_0+k_ph_0+k_S\frac{h_0A}{I_1}+\rho Aa_0\right)\right]\omega_n^2-n^6+\left(2-b_0k_f-k_S\frac{a_0}{R^2}\right)n^4+\\-\left(1+b_0k_p-2k_S\frac{a_0}{R^2}+k_fk_S\frac{h_0}{\rho I_1}+k_fa_0\right)n^2-\left(k_S\frac{a_0}{R^2}+k_pk_S\frac{h_0}{\rho I_1}+k_pa_0\right)=0\end{gathered}\tag{6}$$

Equation (6) is a quadratic equation in ω_n^2 and hence two frequency values are associated with each value of n. The smaller value of ω_n^2 corresponds to the flexural mode, and the higher value corresponds to the thickness–shear mode. In equation (6) n must be an integer with a value greater than 1.

3. The finite element models

In this section the FE models of the system under consideration are formulated to discretize the continuous model given by the equation (1). To find the eigenpairs (eigenvalue, eigenvector) related to the natural frequencies and natural mode shapes of the ring with elastic foundation, the block Lanczos method is employed [5]. The essential problem of this section is prepared the FE model of the system with proper value of the ring substitute mass density ρ_z and massless elastic foundation, respectively. Two objects are considered. Analysed systems have the geometry as it is shown in Figure 2. For each object, the FE model is realized as follows. The ring part is modeled as the solid body and the foundation part is modeled as the massles solid body. The ten node tetrahedral element (solid187) with three degrees of freedom in each node is used to solve the problem. For each case, the proper value of the ring substitute mass density ρ_z is selected during calculations to minimise the frequency error defined by [2, 3]

$$\varepsilon_n = \left(\omega_n^f - \omega_n^c\right) / \omega_n^c \cdot 100\,\% \tag{7}$$

where ω_n^f is the natural frequencies of the model and ω_n^c is the the natural frequencies of the object, respectively.

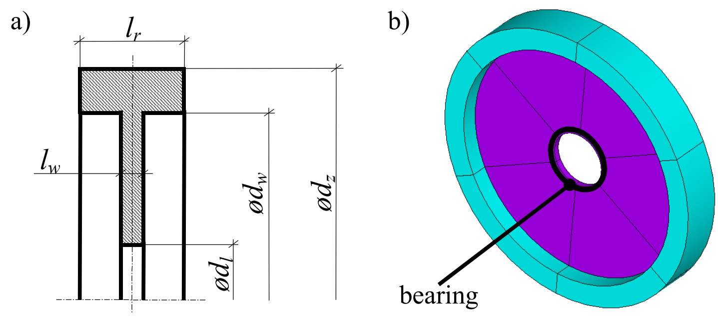

Figure 2. (a) geometrical dimensions, (b) model of the system

The prepared models include 97404 (for the first object) and 143760 (for the second object) solid elements, respectively.

4. Numerical analysis

Numerical analysis results of the circular ring with wheel–plate free vibration are obtained using the models suggested earlier. For all results presented here, the first seven natural frequencies and mode shapes are discussed.

Table 1. Parameters characterizing the systems of rings with foundation

No. of object	d_z [m]	d_w [m]	h [m]	ρ [kg/m^3]	R [m]	d_1 [m]	l_r [m]	l_w [m]	E [Pa]	ν
1	0.191	0.159	0.016	$7.85 \cdot 10^3$	0.0875	0.02	0.008	0.002	$2.1 \cdot 10^{11}$	0.28
2	0.203	0.147	0.028	$7.85 \cdot 10^3$						

Table 1 displays the parameters characterizing the objects under investigation. In this table, h is the depth of the ring; ν is the Poisson ratio and the rest of geometrical dimensions are defined as shown in Figure 2. At first the computations are conducted to evaluate the ring substitute mass density ρ_z of the FE models related to the corresponding objects. Satisfactory results are obtained for the following values of ρ_z. So, for the FE model related to the first object $\rho_z = 9.8\cdot10^3$ [kg/m^3] and for the FE model referred to the second object $\rho_z = 9.17\cdot10^3$ [kg/m^3]. For both cases, the same values of ρ_z are included in the analytical solutions. Moreover the proper values of stiffness modulus k_f, k_p and k_S in the corresponding analytical models are selected during numerical simulations. The results of calculation of the natural frequencies are shown in Table 2.

Table 2. Results of computation related to the systems

No.	k_f [N/m^2]	k_p [N/m^2]	k_S [N/m]	2	3	4	5	6	7	8
natural frequencies of the considered models ω_n [Hz] (analytical solutions)										
1	$2.65\cdot10^9$	$6\cdot10^7$	$3.6\cdot10^7$	8747	12939	17243	21582	25944	30328	34734
2	$1.2\cdot10^9$	$6\cdot10^7$	$8.85\cdot10^7$	7065	12189	17158	22033	26865	31682	36500
natural frequencies of the considered models ω_n [Hz] (FE solutions)										
1	–	–	–	8903	13296	16796	20277	23931	27806	31898
2	–	–	–	7363.4	11786	15980	20439	25142	30012	34982

In the Figure 3 two mode shapes comes from the FE model of the first object are displayed.

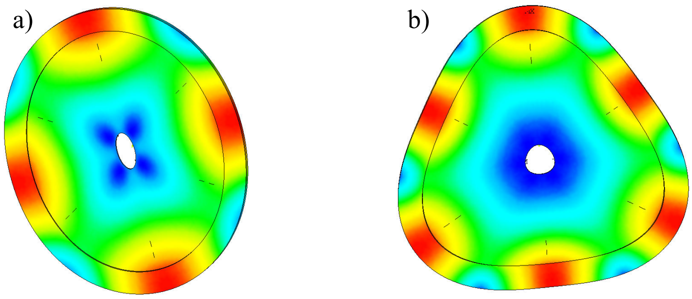

Figure 3. Mode shapes related to the following frequencies:(a) ω_2, (b) ω_3 (FE solution)

5. Experimental verification

In this section the results related to the experimental verification of the considered analytical and numerical models are discussed. LMS measurement environment is used in the experimental investigation. The measuring set consisted of the PCB model 086C03 type modal hammer equipped with a gauging point made of steel, accelerometer PCB model 353B18, LMS SCADA data acquisition system, and SCM-V4E type measuring module supported by LMS Test.Lab software. The experimental investigation is conducted to identify natural frequencies and corresponding mode shapes related to the in–plane flexural vibration of the considered objects. As mentioned earlier, for the measurement experiment, two objects with the geometry shown in Figure 2 and Table 1 are made. The values of the excited natural frequencies are shown in Table 3. These values are compared with the values of natural frequencies from the FE and analytical models, respectively. In the same Table the values of the frequency error related to the discussed models are presented. Achieved results are satisfactory albeit, the best fit is obtained for the analytical model related to the first object.

Table 3. Results of verifications of the systems

n No. of models	2	3	4	5	6	7	8
natural frequencies of the considered objects ω_n [Hz] (experimental data)							
1	8660	12943.8	16802.5	20618.1	25211.9	29550.6	34155.6
2	7207.5	11537.5	16058.8	20933.8	26278.8	31647.5	37106.3
frequency error ε_n [%] (comparison of the analytical solutions with the experimental data)							
1	1.01	-0.04	2.62	4.68	2.9	2.63	1.69
2	-1.98	5.65	6.84	5.25	2.23	0.11	-1.63
frequency error ε_n [%] (comparison of the FE solutions with the experimental data)							
1	2.81	2.72	-0.04	-1.65	-5.08	-5.9	-6.61
2	2.16	2.15	-0.49	-2.36	-4.33	-5.17	-5.73

6. Conclusions

Based on the classical theory of vibrating rings, a comprehensive study of the free in–plane flexural vibration analysis of thick rings with wheel–plate as a three–parameter Winkler elastic foundation is investigated. The separation of variables method is applied to solve the eigenvalue problem. Obtained analytical solutions are compared with the corresponding FE solution results. Presented in the paper theoretical and numerical investigation, are verified successfully during experimental studies.

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