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Duality results on mathematical programs with vanishing constraints involving generalized invex functions*

by

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Abstract: In the here presented research, we investigate Wolfe and Mond-Weir duality models applied to a specific category of generalized convex functions known as p-invex functions. We establish various dualities between the primal MPVC and its Wolfe type dual, as well as between the primal MPVC (mathematical program with vanishing constraints) and its Mond-Weir type dual under p- invexity assumptions. To illustrate these theorems, we will include some examples.

Keywords: vanishing constraints, dual models, generalized convexity, constraint qualifications

1. Introduction

Mathematical program with vanishing constraints (MPVC) was first introduced by Achtziger and Kanzow (2008). This is a special class of optimization problems, which is a unified framework for some applications in topology and structural optimization. Subsequent to the works by Achtziger and Kanzow, there were published a couple of collaborative works such as Hoheisel and Kanzow (2007, 2008, 2009), surveying constraint qualifications and optimality conditions for MPVC.

Usually, vanishing constraints violate standard constraint qualifications, like Mangasarian-Fromovitz and linear independence constraint qualifications, but Abadie constraint qualifications are a strong assumption for the MPVC (Achtziger and Kanzow, 2008). In many cases, Guignard constraint qualifications (GCQ) (Achtziger and Kanzow, 2008) can be applied, but checking whether an MPVC satisfies GCQ is not easy and not sufficient to demonstrate a good algorithm convergence result. MPVC is often used in the economic dispatch problems (Jabr, 2012) and the non-linear integer optimal

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control problems (Michael, Kirches and Sager, 2013). Some numerical approaches and theoretical properties regarding MPVC can be found in Achtziger and Kanzow (2008), Dussault, Mounir and Tangi (2019), Hoheisel and Kanzow (2007), Hu et al. (2014, 2017), Kazemi and Kanzi (2018), Tung (2020), as well as Tung and Tam (2021).

Duality theory is important in optimization, see Antczak and Singh (2009), Bot and Grad (2010), Chinchuluun, Yuan and Pardalos (2007), Joshi (2021), and Saglam and Mahmoudov (2022), for instance, because the lower bound on the objective function value of the primal problem is given by weak duality. The classical Wolfe duality was introduced by Wolfe (1961), while the Mond-Weir duality was introduced in Mond and Weir (1981) for differentiable scalar functions.

Over the past decades, the results from the study of the dual problem have been used as a tool to solve various optimization problems in different fields, like variational inequality problems, complex minimax problems, fractional programming problems, semi-infinite programming problems, fractional subset programming problems, minimax fractional programming problems, bi-level infinite and equilibrium programming problems see, e.g., Antczak (2010), Antczak and Singh (2009), Bot and Grad (2010), Joshi (2021a,b), Joshi, Mishra and Kumar (2020), or Mishra and Shukla (2010).

Generalized convex functions, as discussed in various studies (Joshi, 2021a,b; Joshi, Mishra and Kumar, 2020), have been introduced to relax the stringent requirements of convexity in order to derive results pertinent to optimization theory. A notable generalization in this context is the (p,r)-invex function, Antczak (2001). This class of invex functions preserves many properties of convex functions and has proven to be highly useful in a wide range of applications. To the best of our knowledge, there have been no existing results available for the duality models based on $p-$invex functions, Antczak (2001), under mathematical program with vanishing constraints. This gives us a motivation to utilize these concepts to develop duality theorems for the Wolfe and Mond-Weir type duality models in our paper.

This paper is structured as follows: In Section 2 we give some preliminary definitions and results. In Section 3, we investigate duality results under the assumptions of p-invex and strictly p-invex objective function and p-invex constraints. We also derive duality results under the assumption of pseudo-p-invexity and strictly pseudo-p-invex objective function and quasi-p-invex constraints. Section 4 is devoted to the conclusions from the results of this article.

2. Preliminaries and definitions

Mathematical program with vanishing constraints (MPVC) is structured as follows:

$$\begin{aligned} \min \quad & f(a) \\ \text{subject to } & g_\alpha(a) \leqslant 0,\ \alpha = 1,2,...,v, \\ & h_\gamma(a) = 0,\ \gamma = 1,2,...,q, \\ & \mathscr{L}_\alpha(a) \geqslant 0,\ \alpha = 1,2,...,l, \\ & Q_\alpha(a)\mathscr{L}_\alpha(a) \leqslant 0,\ \alpha = 1,2,...,l, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R},\ g : \mathbb{R}^n \to \mathbb{R}^v,\ h : \mathbb{R}^n \to \mathbb{R}^q,\ Q, \mathscr{L} : \mathbb{R}^n \to \mathbb{R}^l$ are all continuously differentiable functions. In the present paper, X will denote the feasible set of MPVC and will be defined as

$$\begin{aligned} X := \{a \in \mathbb{R}^n : & g_\alpha(a) \leqslant 0,\ \alpha = 1,2,...,v, \\ & h_\gamma(a) = 0,\ \gamma = 1,2,...,q, \\ & \mathscr{L}_\alpha(a) \geqslant 0,\ \alpha = 1,2,...,l, \\ & Q_\alpha(a)\mathscr{L}_\alpha(a) \leqslant 0,\ \alpha = 1,2,...,l\}. \end{aligned}$$

A point $b^* \in X$ is referred to as a local minimum of the MPVC, if and only if there exists an open ball $B(b^*, \delta)$ around b^* with radius $\delta > 0$, such that

$$f(b^*) \leq f(a),\ \forall a \in X \cap B(b^*, \delta).$$

and if $f(b^*) \leq f(a), \forall a \in X$, then the point $b^* \in X$ is said to be a global minimum of the MPVC.

Let $b^* \in X$ be any feasible point of the MPVC; then, the index sets are defined as follows:

$$\begin{aligned} & \tau_g(b^*) = \{\alpha | g_\alpha(b^*) = 0\}, \\ & \tau_h(b^*) = \{1,2,\ldots,q\}, \\ & \tau_+(b^*) = \{\alpha | \mathscr{L}_\alpha(b^*) > 0\}, \\ & \tau_0(b^*) = \{\alpha | \mathscr{L}_\alpha(b^*) = 0\}, \\ & \tau_{+0}(b^*) = \{\alpha | \mathscr{L}_\alpha(b^*) > 0, Q_\alpha(b^*) = 0\}, \\ & \tau_{+-}(b^*) = \{\alpha | \mathscr{L}_\alpha(b^*) > 0, Q_\alpha(b^*) < 0\}, \\ & \tau_{0+}(b^*) = \{\alpha | \mathscr{L}_\alpha(b^*) = 0, Q_\alpha(b^*) > 0\}, \\ & \tau_{00}(b^*) = \{\alpha | \mathscr{L}_\alpha(b^*) = 0, Q_\alpha(b^*) = 0\}, \\ & \tau_{0-}(b^*) = \{\alpha | \mathscr{L}_\alpha(b^*) = 0, Q_\alpha(b^*) < 0\}. \end{aligned}$$

Now, we provide Lagrangian function and gradient of Lagrangian function:

$$\Gamma(y,\delta,\omega,\beta^{\mathscr{L}},\beta^{Q})$$
$$= f(y)+\sum_{\alpha=1}^{v}\delta_\alpha g_\alpha(y)+\sum_{\gamma=1}^{q}\omega_\gamma h_\gamma(y)-\sum_{\alpha=1}^{l}\beta_\alpha^{\mathscr{L}}\mathscr{L}_\alpha(y)+\sum_{\alpha=1}^{l}\beta_\alpha^{Q}Q_\alpha(y)$$

and

$$\nabla\Gamma(y,\delta,\omega,\beta^{\mathscr{L}},\beta^{Q})$$
$$= \nabla f(y)+\sum_{\alpha=1}^{v}\delta_\alpha \nabla g_\alpha(y)+\sum_{\gamma=1}^{q}\omega_\gamma \nabla h_\gamma(y)-\sum_{\alpha=1}^{l}\beta_\alpha^{\mathscr{L}}\nabla\mathscr{L}_\alpha(y)+\sum_{\alpha=1}^{l}\beta_\alpha^{Q}\nabla Q_\alpha(y).$$

For $a\in X$, we define the following index sets:

$$\begin{aligned}
&\tau_g^+(a)=\{\alpha\in\{1,2,...,v\}|\delta_\alpha>0\},\\
&\tau_h^+(a)=\{\gamma\in\tau_h(a)|\omega_\gamma>0\},\\
&\tau_h^-(a)=\{\gamma\in\tau_h(a)|\omega_\gamma<0\},\\
&\tau_{0+}^+(a)=\{\alpha\in\tau_{0+}(a)|\beta_\alpha^{\mathscr{L}}>0\},\\
&\tau_{0+}^-(a)=\{\alpha\in\tau_{0+}(a)|\beta_\alpha^{\mathscr{L}}<0\},\\
&\tau_{00}^+(a)=\{\alpha\in\tau_{00}(a)|\beta_\alpha^{\mathscr{L}}>0\},\\
&\tau_{+0}^+(a)=\{\alpha\in\tau_{+0}(a)|\beta_\alpha^{\mathscr{L}}>0\},\\
&\tau_{+-}^+(a)=\{\alpha\in\tau_{+-}(a)|\beta_\alpha^{\mathscr{L}}>0\},\\
&\tau_{0-}^+(a)=\{\alpha\in\tau_{0-}(a)|\beta_\alpha^{\mathscr{L}}>0\},\\
&\tau_{+0}^{++}(a)=\{\alpha\in\tau_{+0}(a)|\beta_\alpha^{Q}>0\},\\
&\tau_{+-}^{++}(a)=\{\alpha\in\tau_{+-}(a)|\beta_\alpha^{Q}>0\}.
\end{aligned}\tag{1}$$

DEFINITION 1 (ACHTZIGER AND KANZOW, 2008) *Let $b^*\in X$ be a feasible point of the MPVC. The Abadie constraint qualification (ACQ), is said to hold at b^*, if $T(b^*)=\Gamma(b^*)$, where*

$$T(b^*)=\left\{A\in\mathbb{R}^n:\exists\{x^k\}\subseteq X,\exists\{t_k\}\downarrow 0, x^k\to b^* \text{ and } \frac{x^k-b^*}{t_k}\to A\right\}$$

is known to be a standard tangent cone of MPVC at b^*, and

$$\begin{aligned}
\Gamma(b^*)=\{A\in\mathbb{R}^n:&\nabla g_\alpha(b^*)^T A\le 0,\ \alpha\in\tau_g(b^*),\\
&\nabla h_\gamma(b^*)^T A=0,\ \gamma=1,2,...,q,\\
&\nabla\mathscr{L}_\alpha(b^*)^T A=0,\ \alpha\in\tau_{0+}(b^*),\\
&\nabla\mathscr{L}_\alpha(b^*)^T A\ge 0,\ \alpha\in\tau_{00}(b^*)\cup\tau_{0-}(b^*),\\
&\nabla Q_\alpha(b^*)^T A\le 0,\ \alpha\in\tau_{+0}(b^*)\}
\end{aligned}$$

is called corresponding linearized cone of the MPVC at b^*.

DEFINITION 2 (ACHTZIGER AND KANZOW, 2008) *Let* $b^* \in X$ *be a feasible point of the MPVC. The VC-ACQ (vanishing constraints-Abadie constraint qualification) is said to hold at* b^*, *iff* $\Gamma^{VC}(b^*) \subseteq T(b^*)$, *where*

$$\begin{aligned}\Gamma^{VC}(b^*) = \{A \in \mathbb{R}^n : &\nabla g_\alpha(b^*)^T A \leq 0,\ \alpha \in \tau_g(b^*),\\ &\nabla h_\gamma(b^*)^T A = 0,\ \gamma = 1,2,...,q,\\ &\nabla \mathscr{L}_\alpha(b^*)^T A = 0,\ \alpha \in \tau_{0+}(b^*),\\ &\nabla \mathscr{L}_\alpha(b^*)^T A \geq 0,\ \alpha \in \tau_{00}(b^*) \cup \tau_{0-}(b^*),\\ &\nabla Q_\alpha(b^*)^T A \leq 0,\ \alpha \in \tau_{00}(b^*) \cup \tau_{+0}(b^*)\}\end{aligned}$$

represents the corresponding VC-linearized cone of the MPVC at b^*.

THEOREM 1 (ACHTZIGER AND KANZOW, 2008) *Let* $b^* \in X$ *be a local minimum of the MPVC such that VC-ACQ holds at* b^*. *Then,* $\exists$ *Lagrange multipliers* $\delta_\alpha \in \mathbb{R}(\alpha = 1,2,...,v), \omega_\gamma \in \mathbb{R}(\gamma \in \tau_h), \beta_\alpha^{\mathscr{L}}, \beta_\alpha^Q \in \mathbb{R}(\alpha = 1,2,...,l)$, *such that*

$$\nabla\Gamma(b^*, \delta, \omega, \beta^{\mathscr{L}}, \beta^Q) = 0 \tag{2}$$

and

$$\begin{aligned}&h_\gamma(b^*) = 0\ (\gamma \in \tau_h(b^*)),\\ &\delta_\alpha \geq 0,\ g_\alpha(b^*) \leq 0,\\ &\delta_\alpha g_\alpha(b^*) = 0\ (\alpha = 1,2,...,v),\\ &\beta_\alpha^{\mathscr{L}} = 0\ (\alpha \in \tau_+(b^*)),\\ &\beta_\alpha^{\mathscr{L}} \geq 0\ (\alpha \in \tau_{00}(b^*) \cup \tau_{0-}(b^*)),\\ &\beta_\alpha^{\mathscr{L}} \text{ is free } (\alpha \in \tau_{0+}(b^*)),\\ &\beta_\alpha^Q = 0\ (\alpha \in \tau_{0+}(b^*) \cup \tau_{0-}(b^*) \cup \tau_{+-}(b^*)),\\ &\beta_\alpha^Q \geq 0\ (\alpha \in \tau_{00}(b^*) \cup \tau_{+0}(b^*)).\end{aligned} \tag{3}$$

Note: Throughout the paper "with respect to" will be denoted by "w.r.t.".

The following definition of p-invex function is taken from [Antczak (2001); Definition 1, Case 3].

DEFINITION 3 *Let* $S \subseteq \mathbb{R}^n$ *be any nonempty set and let* $f : \mathbb{R}^n \to \mathbb{R}$ *be continuously differentiable function. Then,* f *is said to be p-invex at* $b^* \in S$ *w.r.t. the kernel function* $\xi : S \times S \to \mathbb{R}^n$ *on* S, *if for any* $a \in S$, *we get*

$$f(a) - f(b^*) \geq \frac{1}{p}\langle \nabla f(b^*), e^{p\xi(a,b^*)} - 1\rangle, p \neq 0.$$

If $p = 0$, the definition reduces to

$$f(a) - f(b^*) \geq \langle \nabla f(b^*), \xi(a, b^*) \rangle.$$

Based on the definition of p-invex function, we are introducing the definition of pseudo-p-invex and quasi-p-invex functions.

DEFINITION 4 *Let $S \neq \phi \subseteq \mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. Then, f is said to be pseudo-p-invex at $b^* \in S$ w.r.t. the kernel function $\xi : S \times S \to \mathbb{R}^n$ on S, if for any $a \in S$, we get*

$$\frac{1}{p} \langle \nabla f(b^*), e^{p\xi(a,b^*)} - 1 \rangle \geq 0 \implies f(a) \geq f(b^*), p \neq 0.$$

If $p = 0$, then the definition reduces to

$$\langle \nabla f(b^*), \xi(a, b^*) \rangle \geq 0 \implies f(a) \geq f(b^*).$$

DEFINITION 5 *Let $S \subseteq \mathbb{R}^n$ be any nonempty set and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. Then, f is said to be quasi-p-invex at $b^* \in S$ w.r.t. the kernel function $\xi : S \times S \to \mathbb{R}^n$ on S, if for any $a \in S$, we get*

$$f(a) \leq f(b^*) \implies \frac{1}{p} \langle \nabla f(b^*), e^{p\xi(a,b^*)} - 1 \rangle \leq 0, p \neq 0.$$

If $p = 0$, then the definition reduces to

$$f(a) \leq f(b^*) \implies \langle \nabla f(b^*), \xi(a, b^*) \rangle \leq 0.$$

Now, we provide the duality models and establish various duality theorems. These duality models are taken from Mishra, Singh and Laha (2016) and Hu, Wang and Chen (2020).

3. Duality models

This section gives two duality models, namely the Wolfe type dual model and the Mond-Weir type dual model, respectively.

3.1. Wolfe type dual model

For $a \in X$, VC-WD(a) denotes the Wolfe type dual of the MPVC, which is as follows:

$$\begin{aligned} &\max \quad \Gamma(y, \delta, \omega, \beta^{\mathscr{L}}, \beta^{Q}) \\ &\text{subject to } \nabla \Gamma(y, \delta, \omega, \beta^{\mathscr{L}}, \beta^{Q}) = 0, \\ &\qquad \delta_\alpha \geq 0, \ \forall \alpha = 1, 2, ..., v, \\ &\qquad \beta^{Q}_\alpha = \kappa_\alpha \mathscr{L}_\alpha(a), \ \kappa_\alpha \geq 0, \ \forall \alpha = 1, 2, ..., l, \\ &\qquad \beta^{\mathscr{L}}_\alpha = \rho_\alpha - \kappa_\alpha Q_\alpha(a), \ \rho_\alpha \geq 0, \ \forall \alpha = 1, 2, ..., l. \end{aligned} \tag{4}$$

Let $S_w(a) \subseteq \mathbb{R}^n \times \mathbb{R}^v \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$ denote the feasible set, i.e.,

$$\begin{aligned} S_w(a) = \{(y,\delta,\omega,\beta^{\mathscr{L}},\beta^Q,\rho,\kappa) : \nabla\Gamma(y,\delta,\omega,\beta^{\mathscr{L}},\beta^Q) = 0, \\ \delta_\alpha \geq 0,\ \forall\alpha = 1,2,...,v, \\ \beta^Q_\alpha = \kappa_\alpha \mathscr{L}_\alpha(a),\ \kappa_\alpha \geq 0,\ \alpha = 1,2,...,l, \\ \beta^{\mathscr{L}}_\alpha = \rho_\alpha - \kappa_\alpha Q_\alpha(a),\ \rho_\alpha \geq 0,\ \alpha = 1,2,...,l\}. \end{aligned} \tag{5}$$

We denote the projection of the set $S_w(a)$ on $\mathbb{R}^n$, as follows

$$prS_w(a) = \{y \in \mathbb{R}^n : (y,\delta,\omega,\beta^{\mathscr{L}},\beta^Q,\rho,\kappa) \in S_w(a)\}.$$

Another dual problem, which is denoted VC-WD, and which is independent of the MPVC, is defined as follows:

$$\begin{aligned} &\max \Gamma(y,\delta,\omega,\beta^{\mathscr{L}},\beta^Q) \\ &\quad s.t.(y,\delta,\omega,\beta^{\mathscr{L}},\beta^Q,\rho,\kappa) \in \cap_{a\in X} S_w(a). \end{aligned} \tag{6}$$

The set $S_w = \cap_{a\in X} S_w(a)$ represents the set of all feasible points of the VC-WD and prS_w represents the projection of the set S_w on $\mathbb{R}^n$.

Now we provide the weak duality theorem.

THEOREM 2 *Let $a \in X$ be a feasible point for the MPVC and $(y,\delta,\omega,\beta^{\mathscr{L}},\beta^Q,\rho,\kappa) \in S_w$ be a feasible point for the VC-WD. If one of the following conditions holds:*

(1) $\Gamma(.,\delta,\omega,\beta^{\mathscr{L}},\beta^Q)$ is p-invex at $y \in X \cup prS_w$ w.r.t. the kernel function ξ,

(2) $f,\ g_\alpha(\alpha \in \tau_g^+(a)),\ h_\gamma(\gamma \in \tau_h^+(a)),\ -h_\gamma(\gamma \in \tau_h^-(a)),$
$-\mathscr{L}_\alpha(\alpha \in \tau_{+0}(a) \cup \tau_{+-}(a) \cup \tau_{00}(a) \cup \tau_{0-}(a) \cup \tau_{0+}^+(a)),\ -\mathscr{L}_\alpha(\alpha \in \tau_{0+}^-(a)),$
$-Q_\alpha(\alpha \in \tau_{0+}(a)),\ Q_\alpha(\alpha \in \tau_{00}(a) \cup \tau_{+0}(a) \cup \tau_{0-}(a) \cup \tau_{+-}(a))$
are p-invex at $y \in X \cup prS_w$ for the same real number $p \neq 0$
and w.r.t. the common kernel function ξ;

Then $f(a) \geq \Gamma(y,\delta,\omega,\beta^{\mathscr{L}},\beta^Q)$.

PROOF (1) Let $f(a) < \Gamma(y,\delta,\omega,\beta^{\mathscr{L}},\beta^Q)$, i.e.,

$$f(a) < f(y) + \sum_{\alpha=1}^{v} \delta_\alpha g_\alpha(y) + \sum_{\gamma=1}^{q} \omega_\gamma h_\gamma(y) - \sum_{\alpha=1}^{l} \beta^{\mathscr{L}}_\alpha \mathscr{L}_\alpha(y) + \sum_{\alpha=1}^{l} \beta^Q_\alpha Q_\alpha(y). \tag{7}$$

Since $a \in X$ and using (4), it follows that

$$\begin{aligned}
&g_\alpha(a) < 0, \delta_\alpha \geq 0,\ \alpha \notin \tau_g(a),\\
&g_\alpha(a) = 0, \delta_\alpha \geq 0,\ \alpha \in \tau_g(a),\\
&h_\gamma(a) = 0, \omega_\gamma \in \mathbb{R},\ \gamma \in \tau_h,\\
&-\mathscr{L}_\alpha(a) < 0, \beta_\alpha^{\mathscr{L}} \geq 0,\ \alpha \in \tau_+(a),\\
&-\mathscr{L}_\alpha(a) = 0, \beta_\alpha^{\mathscr{L}} \in \mathbb{R},\ \alpha \in \tau_0(a),\\
&Q_\alpha(a) > 0, \beta_\alpha^{Q} = 0,\ \alpha \in \tau_{0+}(a),\\
&Q_\alpha(a) = 0, \beta_\alpha^{Q} \geq 0,\ \alpha \in \tau_{00}(a) \cup \tau_{+0}(a),\\
&Q_\alpha(a) < 0, \beta_\alpha^{Q} \geq 0,\ \alpha \in \tau_{0-}(a) \cup \tau_{+-}(a),
\end{aligned}$$

that is,

$$\sum_{\alpha=1}^{v} \delta_\alpha g_\alpha(a) + \sum_{\gamma=1}^{q} \omega_\gamma h_\gamma(a) - \sum_{\alpha=1}^{l} \beta_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(a) + \sum_{\alpha=1}^{l} \beta_\alpha^{Q} Q_\alpha(a) \leq 0. \tag{8}$$

By combining (7) and (8), we get

$$\begin{aligned}
f(a) &+ \sum_{\alpha=1}^{v} \delta_\alpha g_\alpha(a) + \sum_{\gamma=1}^{q} \omega_\gamma h_\gamma(a) - \sum_{\alpha=1}^{l} \beta_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(a) + \sum_{\alpha=1}^{l} \beta_\alpha^{Q} Q_\alpha(a)\\
&< f(y) + \sum_{\alpha=1}^{v} \delta_\alpha g_\alpha(y) + \sum_{\gamma=1}^{q} \omega_\gamma h_\gamma(y) - \sum_{\alpha=1}^{l} \beta_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(y) + \sum_{\alpha=1}^{l} \beta_\alpha^{Q} Q_\alpha(y)
\end{aligned} \tag{9}$$

i.e.,

$$\Gamma(a, \delta, \omega, \beta^{\mathscr{L}}, \beta^{Q}) < \Gamma(y, \delta, \omega, \beta^{\mathscr{L}}, \beta^{Q}). \tag{10}$$

By the p-invexity of $\Gamma(., \delta, \omega, \beta^{\mathscr{L}}, \beta^{Q})$ w.r.t. the kernel function ξ, we obtain

$$\Gamma(y, \delta, \omega, \beta^{\mathscr{L}}, \beta^{Q}) + \frac{1}{p}\langle \nabla\, \Gamma(y, \delta, \omega, \beta^{\mathscr{L}}, \beta^{Q}), e^{p\xi(a,y)} - 1\rangle \leq \Gamma(a, \delta, \omega, \beta^{\mathscr{L}}, \beta^{Q}).$$

Using first equation of (4), we obtain

$$\Gamma(a, \delta, \omega, \beta^{\mathscr{L}}, \beta^{Q}) \geq \Gamma(y, \delta, \omega, \beta^{\mathscr{L}}, \beta^{Q}). \tag{11}$$

which contradicts (10) and this is the required proof.

(2) By the p-invexity of $g_\alpha(\alpha \in \tau_g^+(a)), h_\gamma(\gamma \in \tau_h^+(a)), -h_\gamma(\gamma \in \tau_h^-(a))$, $-\mathscr{L}_\alpha(\alpha \in \tau_{+0}(a) \cup \tau_{+-}(a) \cup \tau_{00}(a) \cup \tau_{0-}(a) \cup \tau_{0+}^+(a)), -\mathscr{L}_\alpha(\alpha \in \tau_{0+}^-(a))$,

$-Q_\alpha(\alpha \in \tau_{0+}(a)), Q_\alpha(\alpha \in \tau_{00}(a) \cup \tau_{+0}(a) \cup \tau_{0-}(a) \cup \tau_{+-}(a))$, at $y \in X \cup prS_w$, $a \in X, (y, \delta, \omega, \beta^{\mathscr{L}}, \beta^Q, \rho, \kappa) \in S_w$, w.r.t. the common kernel function ξ, we get

$$
\begin{aligned}
& g_\alpha(y) + \frac{1}{p}\langle \nabla g_\alpha(y), e^{p\xi(a,y)} - 1\rangle \leq g_\alpha(a) \leq 0,\ \delta_\alpha > 0,\ \alpha \in \tau_g^+(a), \\
& h_\gamma(y) + \frac{1}{p}\langle \nabla h_\gamma(y), e^{p\xi(a,y)} - 1\rangle \leq h_\gamma(a) = 0,\ \omega_\gamma > 0,\ \gamma \in \tau_h^+(a), \\
& h_\gamma(y) + \frac{1}{p}\langle \nabla h_\gamma(y), e^{p\xi(a,y)} - 1\rangle \geq h_\gamma(a) = 0,\ \omega_\gamma < 0,\ \gamma \in \tau_h^-(a), \\
& -\mathscr{L}_\alpha(y) - \frac{1}{p}\langle \nabla \mathscr{L}_\alpha(y), e^{p\xi(a,y)} - 1\rangle \leq -\mathscr{L}_\alpha(a) \leq 0,\ \beta_\alpha^{\mathscr{L}} \geq 0,\ \alpha \in \tau_{+0}(a) \\
& \qquad\qquad \cup \tau_{+-}(a) \cup \tau_{00}(a) \cup \tau_{0-}(a) \cup \tau_{0+}^+(a), \\
& -\mathscr{L}_\alpha(y) - \frac{1}{p}\langle \nabla \mathscr{L}_\alpha(y), e^{p\xi(a,y)} - 1\rangle \leq -\mathscr{L}_\alpha(a) = 0,\ \beta_\alpha^{\mathscr{L}} < 0,\ \alpha \in \tau_{0+}^-(a), \\
& Q_\alpha(y) + \frac{1}{p}\langle \nabla Q_\alpha(y), e^{p\xi(a,y)} - 1\rangle \geq Q_\alpha(a) > 0,\ \beta_\alpha^Q = 0,\ \alpha \in \tau_{0+}(a), \\
& Q_\alpha(y) + \frac{1}{p}\langle \nabla Q_\alpha(y), e^{p\xi(a,y)} - 1\rangle \leq Q_\alpha(a) = 0,\ \beta_\alpha^Q \geq 0,\ \alpha \in \tau_{+0}(a) \cup \tau_{00}(a), \\
& Q_\alpha(y) + \frac{1}{p}\langle \nabla Q_\alpha(y), e^{p\xi(a,y)} - 1\rangle \leq Q_\alpha(a) < 0,\ \beta_\alpha^Q \geq 0,\ \alpha \in \tau_{0-}(a) \cup \tau_{+-}(a),
\end{aligned}
$$

which shows that

$$
\begin{aligned}
& \sum_{\alpha=1}^{v} \delta_\alpha g_\alpha(y) + \sum_{\gamma=1}^{q} \omega_\gamma h_\gamma(y) - \sum_{\alpha=1}^{l} \beta_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(y) + \sum_{\alpha=1}^{l} \beta_\alpha^Q Q_\alpha(y) \\
+ & \frac{1}{p}\left\langle \sum_{\alpha=1}^{v} \delta_\alpha \nabla g_\alpha(y) + \sum_{\gamma=1}^{q} \omega_\gamma \nabla h_\gamma(y) - \sum_{\alpha=1}^{l} \beta_\alpha^{\mathscr{L}} \nabla \mathscr{L}_\alpha(y) + \sum_{\alpha=1}^{l} \beta_\alpha^Q \nabla Q_\alpha(y), e^{p\xi(a,y)} - 1 \right\rangle \\
& \leq 0. \qquad (12)
\end{aligned}
$$

Using p-invexity of f at $y \in X \cup prS_w$, w.r.t. the kernel function ξ, we get

$$
f(y) + \frac{1}{p}\langle \nabla f(y), e^{p\xi(a,y)} - 1\rangle \leq f(a). \qquad (13)
$$

By combining (12) and (13), we get

$$
\Gamma(y, \delta, \omega, \beta^{\mathscr{L}}, \beta^Q) + \frac{1}{p}\langle \nabla\, \Gamma(y, \delta, \omega, \beta^{\mathscr{L}}, \beta^Q), e^{p\xi(a,y)} - 1\rangle \leq f(a).
$$

In view of the first equation in (4), we get

$$
\Gamma(y, \delta, \omega, \beta^{\mathscr{L}}, \beta^Q) \leq f(a)
$$

and this ends the required proof. ■

Now we provide the strong duality theorem, which provides the condition, under which the Wolfe type dual problem is solvable and the global maximum can be attained.

THEOREM 3 *Let $b^* \in X$ be a local minimum of the MPVC, such that the VC-ACQ holds at b^*. Then, there exist Lagrange multipliers $\bar{\delta} \in \mathbb{R}^v, \bar{\omega} \in \mathbb{R}^Q, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^Q, \bar{\rho}, \bar{\kappa} \in \mathbb{R}^l$, such that $(b^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^Q, \bar{\rho}, \bar{\kappa})$ is a feasible point of the VC-WD(b^*) and*

$$\sum_{\alpha=1}^{v} \bar{\delta}_\alpha g_\alpha(b^*) + \sum_{\gamma=1}^{q} \bar{\omega}_\gamma h_\gamma(b^*) - \sum_{\alpha=1}^{l} \bar{\beta}_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(b^*) + \sum_{\alpha=1}^{l} \bar{\beta}_\alpha^Q Q_\alpha(b^*) = 0. \tag{14}$$

Moreover, if one of the following condition holds:

(i) $\Gamma(.,\delta,\omega,\beta^{\mathscr{L}},\beta^Q)$ is p-invex at $y \in X \cup prS_w(b^*)$ w.r.t. the kernel function ξ;

(ii) $f,\ g_\alpha(\alpha \in \tau_g^+(b^*)),\ h_\gamma(\gamma \in \tau_h^+(b^*)),\ -h_\gamma(\gamma \in \tau_h^-(b^*)),$
$-\mathscr{L}_\alpha(\alpha \in \tau_{+0}(b^*) \cup \tau_{+-}(b^*) \cup \tau_{00}(b^*) \cup \tau_{0-}(b^*) \cup \tau_{0+}^+(b^*)),$
$-\mathscr{L}_\alpha(\alpha \in \tau_{0+}^-(b^*)),\ -Q_\alpha(\alpha \in \tau_{0+}(b^*)),$
$Q_\alpha(\alpha \in \tau_{00}(b^*) \cup \tau_{+0}(b^*) \cup \tau_{0-}(b^*) \cup \tau_{+-}(b^*))$
are p-invex at $y \in X \cup prS_w(b^*)$ for the same real number $p \neq 0$ and w.r.t. the common kernel function ξ;

then, $(b^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^Q, \bar{\rho}, \bar{\kappa})$ is a global maximum of the VC-WD(b^*), i.e.,

$$\Gamma(b^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^Q) \geq \Gamma(y,\delta,\omega,\beta^{\mathscr{L}},\beta^Q), \forall (y,\delta,\omega,\beta^{\mathscr{L}},\beta^Q) \in S_w(b^*)$$

and

$$f(b^*) = \Gamma(b^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^Q).$$

PROOF Since b^* is a local minimum of the MPVC and the VC-ACQ condition is satisfied at b^*, by Theorem 1, one can see that there exist Lagrange multipliers $\bar{\delta} \in \mathbb{R}^v, \bar{\omega} \in \mathbb{R}^Q, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^Q, \bar{\rho}, \bar{\kappa} \in \mathbb{R}^l$, such that the conditions (2) and (3) [from Theorem 2] hold and hence $(b^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^Q, \bar{\rho}, \bar{\kappa})$ is a feasible point of the VC-WD(b^*). By Theorem 2, we get

$$f(b^*) \geq \Gamma(y,\delta,\omega,\beta^{\mathscr{L}},\beta^Q),\ \forall (y,\delta,\omega,\beta^{\mathscr{L}},\beta^Q,\rho,\kappa) \in S_w(b^*). \tag{15}$$

By combining (14) and (15), we get

$$\Gamma(b^*,\delta,\omega,\beta^{\mathscr{L}},\beta^Q) \geq \Gamma(y,\delta,\omega,\beta^{\mathscr{L}},\beta^Q),\ \forall (y,\delta,\omega,\beta^{\mathscr{L}},\beta^Q,\rho,\kappa) \in S_w(b^*),$$

that is, $(b^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^Q, \bar{\rho}, \bar{\kappa})$ is a global maximum of the VC-WD(b^*). ■

Next, we provide the converse duality theorem.

THEOREM 4 *Let $a \in X$ be any feasible solution of the MPVC and let*

$$(y^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^{Q}, \bar{\rho}, \bar{\kappa})$$

be a feasible point of the VC-WD such that

$$\begin{aligned} &\bar{\delta}_\alpha g_\alpha(y^*) \geq 0,\ \alpha = 1,2,...,v,\\ &\bar{\omega}_\gamma h_\gamma(y^*) = 0,\ \gamma = 1,2,...,q,\\ &-\bar{\beta}_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(y^*) \geq 0,\ \alpha = 1,2,...,l,\\ &\bar{\delta}_\alpha^{Q} Q_\alpha(y^*) \geq 0,\ \alpha = 1,2,...,l. \end{aligned}$$

In addition, if one of the following conditions holds:

(1) $\Gamma(., \delta, \omega, \beta^{\mathscr{L}}, \beta^{Q})$ is p-invex at $y^* \in X \cup prS_w$ w.r.t. the kernel function ξ;

(2) $f,\ g_\alpha(\alpha \in \tau_g^+(a)),\ h_\gamma(\gamma \in \tau_h^+(a)),\ -h_\gamma(\gamma \in \tau_h^-(a)),$
$-\mathscr{L}_\alpha(\alpha \in \tau_{+0}^+(a) \cup \tau_{+-}^+(a) \cup \tau_{00}^+(a) \cup \tau_{0-}^+(a) \cup \tau_{0+}^+(a)),$
$-\mathscr{L}_\alpha(\alpha \in \tau_{0+}^-(a)), Q_\alpha(\alpha \in \tau_{+0}^{++}(a) \cup \tau_{+-}^{++}(a))$
are p-invex at $y^* \in X \cup prS_w$for the same real number$p \neq 0$
and w.r.t. the common kernel function ξ;

Then, y^* will be a global minimum of the MPVC.

PROOF Assume that y^* is not a global minimum of the MPVC, i.e., there exists $\tilde{a} \in X$ such that

$$f(\tilde{a}) < f(y^*). \tag{16}$$

(1) Since $\tilde{a}$ and $(y^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^{Q}, \bar{\rho}, \bar{\kappa})$ constitute the feasible point for the MPVC and the VC-WD, respectively. Now, using the hypothesis of the theorem, we obtain

$$\begin{aligned} &\sum_{\alpha=1}^{v} \bar{\delta}_\alpha g_\alpha(\tilde{a}) + \sum_{\gamma=1}^{q} \bar{\omega}_\gamma h_\gamma(\tilde{a}) - \sum_{\alpha=1}^{l} \bar{\beta}_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(\tilde{a}) + \sum_{\alpha=1}^{l} \bar{\beta}_\alpha^{Q} Q_\alpha(\tilde{a}) \\ &\qquad \leq \sum_{\alpha=1}^{v} \bar{\delta}_\alpha g_\alpha(y^*) + \sum_{\gamma=1}^{q} \bar{\omega}_\gamma h_\gamma(y^*) - \sum_{\alpha=1}^{l} \bar{\beta}_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(y^*) + \sum_{\alpha=1}^{l} \bar{\beta}_\alpha^{Q} Q_\alpha(y^*). \end{aligned} \tag{17}$$

By combining (16) and (17), we get

$$\Gamma(\tilde{a}, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^{Q}) < \Gamma(y^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^{Q}).$$

Using p-invexity of $\Gamma(., \delta, \omega, \beta^{\mathscr{L}}, \beta^{Q})$ w.r.t. the common kernel function ξ, at $y^* \in X \cup prS_w$; we get

$$\frac{1}{p} \langle \nabla\, \Gamma(y^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^{Q}), e^{p\xi(\tilde{a}, y^*)} - 1 \rangle < 0,$$

and this contradicts the dual constraint of the VC-WD(a), which completes the proof.
(2) Since $\tilde{a}$ and $(y^*, \overline{\delta}, \overline{\omega}, \overline{\beta}^{\mathscr{L}}, \overline{\beta}^{Q}, \overline{\rho}, \overline{\kappa})$ constitute the feasible point for the MPVC and the VC-WD, respectively, then it follows that

$$\begin{aligned}
&g_\alpha(\tilde{a}) \leq g_\alpha(y^*),\ \alpha \in \tau_g^+(\tilde{a}),\\
&h_\gamma(\tilde{a}) = h_\gamma(y^*),\ \gamma \in \tau_h^+(\tilde{a}) \cup \tau_h^-(\tilde{a}),\\
&-\mathscr{L}_\alpha(\tilde{a}) \leq -\mathscr{L}_\alpha(y^*),\ \alpha \in \tau_{+0}^+(\tilde{a}) \cup \tau_{+-}^+(\tilde{a}) \cup \tau_{00}^+(\tilde{a}) \cup \tau_{0-}^+(\tilde{a}) \cup \tau_{0+}^+(\tilde{a}),\\
&-\mathscr{L}_\alpha(\tilde{a}) \geq -\mathscr{L}_\alpha(y^*),\ \alpha \in \tau_{0+}^-(\tilde{a}),\\
&Q_\alpha(\tilde{a}) \leq Q_\alpha(y^*),\ \alpha \in \tau_{+0}^{++}(\tilde{a}) \cup \tau_{+-}^{++}(\tilde{a}).
\end{aligned}$$

By the p-invexity of the functions, considered in the hypothesis of the theorem, w.r.t. the common kernel function ξ, we get

$$\begin{aligned}
&\frac{1}{p}\langle \nabla g_\alpha(y^*), e^{p\xi(\tilde{a},y^*)} - 1\rangle \leq 0,\ \overline{\delta}_\alpha > 0,\ \alpha \in \tau_g^+(\tilde{a}),\\
&\frac{1}{p}\langle \nabla h_\gamma(y^*), e^{p\xi(\tilde{a},y^*)} - 1\rangle \leq 0,\ \overline{\omega}_\gamma > 0,\ \gamma \in \tau_h^+(\tilde{a}),\\
&\frac{1}{p}\langle \nabla h_\gamma(y^*), e^{p\xi(\tilde{a},y^*)} - 1\rangle \geq 0,\ \overline{\omega}_\gamma < 0,\ \gamma \in \tau_h^-(\tilde{a}),\\
&-\frac{1}{p}\langle \nabla \mathscr{L}_\alpha(y^*), e^{p\xi(\tilde{a},y^*)} - 1\rangle \leq 0,\ \overline{\beta}_\alpha^{\mathscr{L}} \geq 0,\\
&\quad \alpha \in \tau_{+0}^+(\tilde{a}) \cup \tau_{+-}^+(\tilde{a}) \cup \tau_{00}^+(\tilde{a}) \cup \tau_{0-}^+(\tilde{a}) \cup \tau_{0+}^+(\tilde{a}),\\
&-\frac{1}{p}\langle \nabla \mathscr{L}_\alpha(y^*), e^{p\xi(\tilde{a},y^*)} - 1\rangle \geq 0,\ \overline{\beta}_\alpha^{\mathscr{L}}, \leq 0\ \alpha \in \tau_{+0}^+(\tilde{a}),\\
&\frac{1}{p}\langle \nabla Q_\alpha(y^*), e^{p\xi(\tilde{a},y^*)} - 1\rangle \leq 0,\ \overline{\beta}_\alpha^{Q} \geq 0,\ \alpha \in \tau_{+0}^{++}(\tilde{a}) \cup \tau_{+-}^{++}(\tilde{a}),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{1}{p}\left\langle \sum_{\alpha=1}^{v} \overline{\delta}_\alpha \nabla g_\alpha(y^*) + \sum_{\gamma=1}^{q} \overline{\omega}_\gamma \nabla h_\gamma(y^*) - \sum_{\alpha=1}^{l} \overline{\beta}_\alpha^{\mathscr{L}} \nabla \mathscr{L}_\alpha(y^*) + \right.\\
&\qquad \left. \sum_{\alpha=1}^{l} \overline{\beta}_\alpha^{Q} \nabla Q_\alpha(y^*), e^{p\xi(\tilde{a},y^*)} - 1 \right\rangle \leq 0.
\end{aligned}$$

By connecting the above inequality and (4), we get

$$\frac{1}{p}\langle \nabla f(y^*), e^{p\xi(\tilde{a},y^*)} - 1\rangle \geq 0.$$

By the p-invexity of f, w.r.t. the kernel function ξ, the above implies that

$$f(\tilde{a}) \geq f(y^*),$$

which contradicts our hypothesis and hence the required proof is complete. ■

We now give the restricted converse duality theorem.

THEOREM 5 *Let $b^* \in X$ be any feasible solution of the MPVC and let $(\underline{y}^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^{Q}, \bar{\rho}, \bar{\kappa})$ be a feasible point of the VC-WD, such that $f(b^*) = \Gamma(y^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^{Q})$. Moreover, if one of the following conditions holds:*

(1) $\Gamma(., \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^{Q})$ is p-invex at $y^* \in X \cup prS_w$ w.r.t. the kernel function ξ,

(2) $f,\ g_\alpha(\alpha \in \tau_g^+(b^*)),\ h_\gamma(\gamma \in \tau_h^+(b^*)), -h_\gamma(\gamma \in \tau_h^-(b^*)),$
$-\mathscr{L}_\alpha(\alpha \in \tau_{+0}^+(b^*) \cup \tau_{+-}^+(b^*) \cup \tau_{00}^+(b^*) \cup \tau_{0-}^+(b^*) \cup \tau_{0+}^+(b^*)),$
$-\mathscr{L}_\alpha(\alpha \in \tau_{0+}^-(b^*)),\ Q_\alpha(\alpha \in \cup\tau_{+0}^{++}(b^*) \cup \tau_{+-}^{++}(b^*))$
are p-invex at $y^* \in X \cup prS_w$ for the same real number $p \neq 0$
and w.r.t. the common kernel function ξ;

Then, b^* is a global minimum of the MPVC.

PROOF Assume that $b^* \in X$ is not a global minimum of the MPVC, then there exists $\tilde{a} \in X$ such that

$$f(\tilde{a}) < f(b^*).$$

Now, using the assumptions of the theorem, we deduce that $f(b^*) < \Gamma(y^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^{Q})$. This is a contradiction to Theorem 2. This completes the proof. ■

Now, we provide the strict converse duality theorem.

THEOREM 6 *Let $b^* \in X$ be a local minimum for the MPVC, such that the VC-ACQ holds at b^*. Assume that the conditions of Theorem 3 hold and $(y^*, \tilde{\delta}, \tilde{\omega}, \tilde{\beta}^{\mathscr{L}}, \tilde{\beta}^{Q}, \tilde{\rho}, \tilde{\kappa})$ is a global maximum of the VC-WD(b^*). If one of the following conditions holds:*

(1) $\Gamma(., \tilde{\delta}, \tilde{\omega}, \tilde{\beta}^{\mathscr{L}}, \tilde{\beta}^{Q})$ is strictly p-invex at $y^* \in X \cup prS_w$ w.r.t. the kernel function ξ;

(2) f is strictly p-invex and $g_\alpha(\alpha \in \tau_g^+(b^*)),\ h_\gamma(\gamma \in \tau_h^+(b^*)),\ -h_\gamma(\gamma \in \tau_h^-(b^*)),$
$-\mathscr{L}_\alpha(\alpha \in \tau_{+0}(b^*) \cup \tau_{+-}(b^*) \cup \tau_{00}(b^*) \cup \tau_{0-}(b^*) \cup \tau_{0+}^+(b^*)), -\mathscr{L}_\alpha(\alpha \in \tau_{0+}^-(b^*)),$
$-Q_\alpha(\alpha \in \tau_{0+}(b^*)),\ Q_\alpha(\alpha \in \tau_{00}(b^*) \cup \tau_{+0}(b^*) \cup \tau_{0-}(b^*) \cup \tau_{+-}(b^*))$
are p-invex at $y^* \in X \cup prS_w$ for the same real number $p \neq 0$
and w.r.t. the common kernel function ξ; then $b^* = y^*$.

PROOF (1) Assume that $b^* \neq y^*$. By Theorem 3, there exist Lagrange multipliers $\bar{\delta} \in \mathbb{R}^v, \bar{\omega} \in \mathbb{R}^Q, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^{Q}, \bar{\rho}, \bar{\kappa} \in \mathbb{R}^l$, such that $(y^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^{Q}, \bar{\rho}, \bar{\kappa})$ is a global maximum of the $VC-WD(b^*)$. Hence,

$$f(b^*) = \Gamma(b^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^{Q}) = \Gamma(y^*, \tilde{\delta}, \tilde{\omega}, \tilde{\beta}^{\mathscr{L}}, \tilde{\beta}^{Q}). \tag{18}$$

Using the feasibility of b^* for MPVC and the feasibility of $(y^*, \tilde{\delta}, \tilde{\omega}, \tilde{\beta}^{\mathscr{L}}, \tilde{\beta}^{Q}, \tilde{\rho}, \tilde{\kappa})$ for the VC-WD(b^*), we obtain

$$
\begin{aligned}
&g_\alpha(b^*) < 0, \tilde{\delta}_\alpha \geq 0,\ \alpha \notin \tau_g(b^*),\\
&g_\alpha(b^*) = 0, \tilde{\delta}_\alpha \geq 0,\ \alpha \in \tau_g(b^*),\\
&h_\gamma(b^*) = 0, \tilde{\omega}_\gamma \in \mathbb{R},\ \gamma \in \tau_h(b^*),\\
&-\mathscr{L}_\alpha(b^*) < 0, \tilde{\beta}^{\mathscr{L}}_\alpha \geq 0, \alpha \in \tau_+(b^*),\\
&-\mathscr{L}_\alpha(b^*) = 0, \tilde{\beta}^{\mathscr{L}}_\alpha \in \mathbb{R},\ \alpha \in \tau_0(b^*),\\
&Q_\alpha(b^*) > 0, \tilde{\beta}^{Q}_\alpha = 0,\ \alpha \in \tau_{0+}(b^*),\\
&Q_\alpha(b^*) = 0, \tilde{\beta}^{Q}_\alpha \geq 0,\ \alpha \in \tau_{00}(b^*) \cup \tau_{+0}(b^*),\\
&Q_\alpha(b^*) < 0, \tilde{\beta}^{Q}_\alpha \geq 0,\ \alpha \in \tau_{0-}(b^*) \cup \tau_{+-}(b^*),
\end{aligned}
$$

that is,

$$
\sum_{\alpha=1}^{v} \tilde{\delta}_\alpha g_\alpha(b^*) + \sum_{\gamma=1}^{q} \tilde{\omega}_\gamma h_\gamma(b^*) - \sum_{\alpha=1}^{l} \tilde{\beta}^{\mathscr{L}}_\alpha \mathscr{L}_\alpha(b^*) + \sum_{\alpha=1}^{l} \tilde{\beta}^{Q}_\alpha Q_\alpha(b^*) \leq 0. \tag{19}
$$

By combining (18) and (19), we get

$$
\Gamma(b^*, \tilde{\delta}, \tilde{\omega}, \tilde{\beta}^{\mathscr{L}}, \tilde{\beta}^{Q}) \leq\ \Gamma(y^*, \tilde{\delta}, \tilde{\omega}, \tilde{\beta}^{\mathscr{L}}, \tilde{\beta}^{Q}). \tag{20}
$$

By the strict p-invexity of $\Gamma(., \tilde{\delta}, \tilde{\omega}, \tilde{\beta}^{\mathscr{L}}, \tilde{\beta}^{Q})$, w.r.t. the common kernel function ξ it follows that

$$
\frac{1}{p}\langle \nabla\, \Gamma(y^*, \tilde{\delta}, \tilde{\omega}, \tilde{\beta}^{\mathscr{L}}, \tilde{\beta}^{Q}), e^{p\xi(b^*,y^*)} - 1\rangle < 0.
$$

This contradicts the first equation in (4) and this ends the required proof.

(2) Using the strict p-invexity of f at y^*, w.r.t. the kernel function ξ, we get

$$
f(b^*) - f(y^*) > \frac{1}{p}\langle \nabla f(y^*), e^{p\xi(b^*,y^*)} - 1\rangle. \tag{21}
$$

By the p-invexity of $g_\alpha(\alpha \in \tau_g^+(b^*)),\ h_\gamma(\gamma \in \tau_h^+(b^*)), h_\gamma(\gamma \in \tau_h^-(b^*))$, $-\mathscr{L}_\alpha(\alpha \in \tau_{+0}(b^*) \cup \tau_{+-}(b^*) \cup \tau_{00}(b^*) \cup \tau_{0-}(b^*) \cup \tau_{0+}^+(b^*))$, $-\mathscr{L}_\alpha(\alpha \in \tau_{0+}^-(b^*)),\ -Q_\alpha(\alpha \in \tau_{0+}(b^*))$, $Q_\alpha(\alpha \in \tau_{00}(b^*) \cup \tau_{+0}(b^*) \cup \tau_{0-}(b^*) \cup \tau_{+-}(b^*))$, at $y^* \in X \cup prS_w(b^*), b^* \in X$, w.r.t. the common kernel function ξ and

$(y^*, \tilde{\delta}, \tilde{\omega}, \tilde{\beta}^{\mathscr{L}}, \tilde{\beta}^{Q}, \tilde{\rho}, \tilde{\kappa}) \in S_w(b^*)$, we get

$$\begin{aligned}
&g_\alpha(y^*) + \frac{1}{p}\langle \nabla g_\alpha(y^*),\ e^{p\xi(b^*,y^*)} - 1\rangle \leq g_\alpha(b^*) \leq 0,\ \tilde{\delta}_\alpha > 0,\ \alpha \in \tau_g^+(b^*),\\
&h_\gamma(y^*) + \frac{1}{p}\langle \nabla h_\gamma(y^*),\ e^{p\xi(b^*,y^*)} - 1\rangle \leq h_\gamma(b^*) = 0,\ \tilde{\omega}_\gamma > 0,\ \gamma \in \tau_h^+(b^*),\\
&h_\gamma(y^*) + \frac{1}{p}\langle \nabla h_\gamma(y^*),\ e^{p\xi(b^*,y^*)} - 1\rangle \geq h_\gamma(b^*) = 0,\ \tilde{\omega}_\gamma < 0,\ \gamma \in \tau_h^-(b^*),\\
&-\mathscr{L}_\alpha(y^*) - \frac{1}{p}\langle \nabla \mathscr{L}_\alpha(y^*),\ e^{p\xi(b^*,y^*)} - 1\rangle \leq -\mathscr{L}_\alpha(b^*) \leq 0,\ \tilde{\beta}_\alpha^{\mathscr{L}} \geq 0,\\
&\qquad\qquad \alpha \in \tau_{+0}(b^*) \cup \tau_{+-}(b^*) \cup \tau_{00}(b^*) \cup \tau_{0-}(b^*) \cup \tau_{0+}^+(b^*),\\
&-\mathscr{L}_\alpha(y^*) - \frac{1}{p}\langle \nabla \mathscr{L}_\alpha(y^*),\ e^{p\xi(b^*,y^*)} - 1\rangle \leq -\mathscr{L}_\alpha(b^*) = 0,\\
&\qquad\qquad \tilde{\beta}_\alpha^{\mathscr{L}} < 0,\ \alpha \in \tau_{0+}^-(b^*),\\
&Q_\alpha(y^*) + \frac{1}{p}\langle \nabla Q_\alpha(y^*),\ e^{p\xi(b^*,y^*)} - 1\rangle \geq Q_\alpha(b^*) > 0,\ \tilde{\beta}_\alpha^{Q} = 0,\ \alpha \in \tau_{0+}(b^*),\\
&Q_\alpha(y^*) + \frac{1}{p}\langle \nabla Q_\alpha(y^*),\ e^{p\xi(b^*,y^*)} - 1\rangle \leq Q_\alpha(b^*) = 0,\\
&\qquad\qquad \tilde{\beta}_\alpha^{Q} \geq 0,\ \alpha \in \tau_{+0}(b^*) \cup \tau_{00}(b^*),\\
&Q_\alpha(y^*) + \frac{1}{p}\langle \nabla Q_\alpha(y^*),\ e^{p\xi(b^*,y^*)} - 1\rangle \leq Q_\alpha(b^*) < 0,\\
&\qquad\qquad \tilde{\beta}_\alpha^{Q} \geq 0,\ \alpha \in \tau_{0-}(b^*) \cup \tau_{+-}(b^*),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\sum_{\alpha=1}^{v} \tilde{\delta}_\alpha g_\alpha(y^*) + \sum_{\gamma=1}^{q} \tilde{\omega}_\gamma h_\gamma(y^*) - \sum_{\alpha=1}^{l} \tilde{\beta}_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(y^*) + \sum_{\alpha=1}^{l} \tilde{\beta}_\alpha^{Q} Q_\alpha(y^*)\\
&+\frac{1}{p}\Bigg\langle \sum_{\alpha=1}^{v} \tilde{\delta}_\alpha \nabla g_\alpha(y^*) + \sum_{\gamma=1}^{q} \tilde{\omega}_\gamma \nabla h_\gamma(y^*) - \sum_{\alpha=1}^{l} \tilde{\beta}_\alpha^{\mathscr{L}} \nabla \mathscr{L}_\alpha(y^*)\\
&+ \sum_{\alpha=1}^{l} \tilde{\beta}_\alpha^{Q} \nabla Q_\alpha(y^*), e^{p\xi(b^*,y^*)} - 1 \Bigg\rangle \leq 0.
\end{aligned} \tag{22}$$

By combining (21) and (3.1), we get

$$\Gamma(y^*, \tilde{\delta}, \tilde{\omega}, \tilde{\beta}^{\mathscr{L}}, \tilde{\beta}^{Q}) < f(b^*).$$

This contradicts (18) and this ends the required proof. ■

Now, we provide an example in order to validate the theorems.

EXAMPLE 1 *Consider the optimization problem*

$$\begin{aligned} \min\ & f(a) = a_1^2 + 3a_2^2 \\ & s.t. \mathscr{L}_1(a) = \frac{a_2}{3} \geq 0, \\ & Q_1(a)\mathscr{L}_1(a) = \frac{a_1 a_2}{3} \leq 0, \end{aligned} \tag{23}$$

with $n = 2,\ m = q = 0,\ l = 1$. *The feasible set* X *is given by*

$$X := \left\{ (a_1, a_2 \in \mathbb{R}^2) : \frac{a_2}{3} \geq 0, \frac{a_1 a_2}{3} \leq 0 \right\}.$$

For any feasible point $a \in X$, *the Wolfe dual model VC-WD*(a) *to the MPVC* (23) *is given by*

$$\begin{aligned} \max\ & \Gamma(y, \beta_1^{\mathscr{L}}, \beta_1^Q) = y_1^2 + 3y_2^2 - \beta_1^{\mathscr{L}} \frac{y_2}{3} + \beta_1^Q y_1 \\ & s.t.\ \nabla\, \Gamma(y, \beta_1^{\mathscr{L}}, \beta_1^Q) = \left(2y_1 + \beta_1^Q, 6y_2 - \frac{\beta_1^{\mathscr{L}}}{3} \right) = (0,0), \\ & \beta_1^Q = \kappa_1 a_1, \kappa_1 \geq 0, \\ & \beta_1^{\mathscr{L}} = \rho_1 - \kappa_1 \frac{a_2}{3}, \rho_1 \geq 0. \end{aligned} \tag{24}$$

(1) To show that any feasible point $b^* \in X$ *is a global minimum of the MPVC using Theorem 5, we have to show that* $f(b^*) = \Gamma(y^*, \bar{\beta}_1^{\mathscr{L}}, \bar{\beta}_1^Q)$ *for some* $(y^*, \bar{\beta}_1^{\mathscr{L}}, \bar{\beta}_1^Q) \in S_w$ *such that the hypothesis of Theorem 5 holds at* y^* *on* $X \cup pr_{R^2} S_w$.

The feasible set S_W *of the VC-WD is given by*

$$\begin{aligned} S_w := \{ (y_1, y_2, \beta_1^Q, \beta_1^{\mathscr{L}}, \rho_1, \kappa_1) :& 2y_1 + \beta_1^Q = 0, \\ & 6y_2 - \frac{\beta_1^{\mathscr{L}}}{3} = 0, \\ & \beta_1^Q = \kappa_1 \mathscr{L}_1(a) : \kappa_1 \geq 0, \\ & \beta_1^{\mathscr{L}} = \rho_1 - \kappa_1 Q_1(a) : \rho_1 \geq 0 \}. \end{aligned}$$

Also

$$f(a_1, a_2) = \Gamma(y_1, y_2, \beta_1^{\mathscr{L}}, \beta_1^Q) = -\frac{(\beta_1^{\mathscr{L}})^2}{108} - \frac{(\beta_1^Q)^2}{4} \leq 0.$$

This is only possible if $\beta_1^{\mathscr{L}} = 0, \beta_1^Q = 0$, *and* $b^* = (0,0)$. *That is*

$$b^* = (0,0)^T \in X, (y, \beta_1^Q, \beta_1^{\mathscr{L}}, \rho_1, \kappa_1) = (0,0,0,0,0) \in S_w(b^*),$$

and so, we get

$$f(b^*) = 0 = \Gamma(0,0,0).$$

It can be verified that the hypothesis of Theorem 5 holds. That is, b^* is a global minimum of (23). So, Theorem 5 is verified.

(2) We can get $y_1 = -\frac{\beta_1^Q}{2}$, $y_2 = \frac{\beta_1^{\mathscr{L}}}{18}$ by (24) and

$$\Gamma(y, \beta_1^{\mathscr{L}}, \beta_1^Q) = -\frac{(\beta_1^{\mathscr{L}})^2}{108} - \frac{(\beta_1^Q)^2}{4} \leq 0.$$

Since $f(a) = a_1^2 + 3a_2^2 \geq 0$, we can get $f(a) \geq \Gamma(y, \beta_1^{\mathscr{L}}, \beta_1^Q)$. Hence, Theorem 2 is verified.

(3) Since $\nabla \mathscr{L}_1 = (0,1/3)^T, \nabla Q_1 = (1,0)^T$. So, we obtain that (23) satisfies VC-ACQ. By Theorem 1, there exist Lagrange multipliers $\beta_1^{\mathscr{L}}, \beta_1^Q, \rho_1, \nu_1 \in \mathbb{R}$ such that $(0, \beta_1^{\mathscr{L}}, \beta_1^Q, \rho_1, \nu_1)$ is a feasible point of the VC-WD(0), and

$$-\beta_1^{\mathscr{L}} \mathscr{L}_1(0) + \beta_1^Q Q_1(0) = 0.$$

Hence, $(0, \beta_1^{\mathscr{L}}, \beta_1^Q, \rho_1, \nu_1)$ is a global maximum of the VC-WD(0) and $f(0) = 0 = \Gamma(0, \beta_1^{\mathscr{L}}, \beta_1^Q)$. Theorem 3 is verified.

3.2. Mond–Weir type dual model

Now, we shall be discussing the Mond–Weir type dual for MPVC. For $a \in X$, the Mond–Weir type dual of the MPVC, VC-MWD(a) for short, is as follows:

$$\begin{aligned}
&\max\ f(y) \\
&\quad s.t. \nabla \Gamma(y, \delta, \omega, \beta^{\mathscr{L}}, \beta^Q) = 0, \\
&\quad \delta_\alpha \geq 0, \delta_\alpha g_\alpha(y) \geq 0,\ \alpha = 1,2,...,v, \\
&\quad \kappa_\gamma h_\gamma(y) = 0,\ \gamma = 1,2,.,q, \\
&\quad \beta_\alpha^Q Q_\alpha(y) \geq 0,\ \alpha = 1,2,.,l, \\
&\quad \beta_\alpha^Q = \kappa_\alpha \mathscr{L}_\alpha(a), \kappa_\alpha \geq 0,\ \alpha = 1,2,.,l, \\
&\quad -\beta_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(y) \geq 0,\ \alpha = 1,2,.,l, \\
&\quad \beta_\alpha^{\mathscr{L}} = \rho_\alpha - \kappa_\alpha Q_\alpha(a), \rho_\alpha \geq 0,\ \alpha = 1,2,.,l.
\end{aligned} \tag{25}$$

Let $S_{MW}(a) \subseteq \mathbb{R}^n \times \mathbb{R}^v \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$ denote the feasible set, i.e.

$$\begin{aligned} S_{MW}(a) = \{(y,\delta,\omega,\beta^{\mathscr{L}},\beta^{Q},\rho,\kappa) : \nabla\,\Gamma(y,\delta,\omega,\beta^{\mathscr{L}},\ \beta^{Q}) = 0,\\ \delta_\alpha \geq 0, \delta_\alpha g_\alpha(y) \geq 0, \alpha = 1,2,...,v,\\ \kappa_\gamma h_\gamma(y) = 0,\ \gamma = 1,2,.,q,\\ \beta^{Q}_\alpha Q_\alpha(y) \geq 0,\ \alpha = 1,2,.,l,\\ \beta^{Q}_\alpha = \kappa_\alpha \mathscr{L}_\alpha(a),\ \kappa_\alpha \geq 0,\ \alpha = 1,2,.,l,\\ -\beta^{\mathscr{L}}_\alpha \mathscr{L}_\alpha(y) \geq 0,\ \alpha = 1,2,.,l,\\ \beta^{\mathscr{L}}_\alpha = \rho_\alpha - \kappa_\alpha Q_\alpha(a),\ \rho_\alpha \geq 0,\ \alpha = 1,2,.,l\}. \end{aligned} \tag{26}$$

We denote

$$prS_{MW}(a) = \{y \in \mathbb{R}^n : (y,\delta,\omega,\beta^{\mathscr{L}},\beta^{Q},\rho,\kappa) \in S_{MW}(a)\}$$

as the projection of the set $S_{MW}(a)$ on $\mathbb{R}^n$.

Similar to the Wolfe dual, there is another dual problem that we consider, which is denoted by VC-MWD, and is defined as follows:

$$\begin{aligned} &\max f(y)\\ &\quad s.t. (y,\delta,\omega,\beta^{\mathscr{L}},\beta^{Q},\rho,\kappa) \in \cap_{a\in X} S_{MW}(a). \end{aligned}$$

The set of all feasible points of the VC-MWD is denoted by $S_{MW} = \cap_{a\in X} S_{MW}(a)$ and the projection of the set S_{MW} on $\mathbb{R}^n$ is denoted by prS_{MW}.

THEOREM 7 *(Weak duality) Let $a \in X$ and $(y,\delta,\omega,\beta^{\mathscr{L}},\beta^{Q},\rho,\kappa) \in S_{MW}$ be feasible points for the MPVC and the VC-MWD, respectively. Moreover, if one of the following conditions holds:*

(1) $f(.)$ is pseudo-p-invex and

$$\sum_{\alpha=1}^{v} \delta_\alpha g_\alpha(.) + \sum_{\gamma=1}^{q} \omega_\gamma h_\gamma(.) - \sum_{\alpha=1}^{l} \beta^{\mathscr{L}}_\alpha \mathscr{L}_\alpha(.) + \sum_{\alpha=1}^{l} \beta^{Q}_\alpha Q_\alpha(.)$$

is quasi-p-invex at $y \in X \cup prS_{MW}$, w.r.t. the common kernel function ξ;

(2) $f(.)$ is pseudo-p-invex and $g_\alpha(\alpha \in \tau^+_g(a)),\ h_\gamma(\gamma \in \tau^+_h(a)),\ -h_\gamma(\gamma \in \tau^-_h(a))$, $-\mathscr{L}_\alpha(\alpha \in \tau^+_{+0}(a) \cup \tau^+_{+-}(a) \cup \tau^+_{00}(a) \cup \tau^+_{0-}(a) \cup \tau^+_{0+}(a))$, $-\mathscr{L}_\alpha(\alpha \in \tau^-_{0+}(a)), Q_\alpha(\alpha \in \tau^{++}_{+0}(a) \cup \tau^{++}_{+-}(a))$ are quasi-p-invex at $y \in X \cup prS_{MW}$ for the same real number $p \neq 0$ and w.r.t. the common kernel function ξ;

then, $f(a) \geq f(y)$.

PROOF Since $a \in X$ and $(y, \delta, \omega, \beta^{\mathscr{L}}, \beta^{Q}, \rho, \kappa) \in S_{MW}$ it follows that

$$\begin{aligned}
&g_\alpha(a) \leq 0,\ \delta_\alpha \geq 0,\ \alpha = 1, 2, ..., v,\\
&h_\gamma(a) = 0,\ \omega_\gamma \in \mathbb{R},\ \gamma \in \tau_h,\\
&-\mathscr{L}_\alpha(a) < 0,\ \beta_\alpha^{\mathscr{L}} \geq 0,\ \alpha \in \tau_+(a),\\
&-\mathscr{L}_\alpha(a) = 0,\ \beta_\alpha^{\mathscr{L}} \in \mathbb{R},\ \alpha \in \tau_0(a),\\
&Q_\alpha(a) > 0,\ \beta_\alpha^{Q} = 0,\ \alpha \in \tau_{0+}(a),\\
&Q_\alpha(a) = 0,\ \beta_\alpha^{Q} \geq 0,\ \alpha \in \tau_{00}(a) \cup \tau_{+0}(a),\\
&Q_\alpha(a) < 0,\ \beta_\alpha^{Q} \geq 0,\ \alpha \in \tau_{0-}(a) \cup \tau_{+-}(a).
\end{aligned}$$

By (25), this implies that

$$\begin{aligned}
&\sum_{\alpha=1}^{v} \delta_\alpha g_\alpha(a) + \sum_{\gamma=1}^{q} \omega_\gamma h_\gamma(a) - \sum_{\alpha=1}^{l} \beta_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(a) + \sum_{\alpha=1}^{l} \beta_\alpha^{Q} Q_\alpha(a)\\
&\qquad \leq \sum_{\alpha=1}^{v} \delta_\alpha g_\alpha(y) + \sum_{\gamma=1}^{q} \omega_\gamma h_\gamma(y) - \sum_{\alpha=1}^{l} \beta_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(y) + \sum_{\alpha=1}^{l} \beta_\alpha^{Q} Q_\alpha(y).
\end{aligned}$$

Combining the quasi-p-invexity of

$$\sum_{\alpha=1}^{v} \delta_\alpha g_\alpha(.) + \sum_{\gamma=1}^{q} \omega_\gamma h_\gamma(.) - \sum_{\alpha=1}^{l} \beta_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(.) + \sum_{\alpha=1}^{l} \beta_\alpha^{Q} Q_\alpha(.)$$

w.r.t. the common kernel function ξ, we get

$$\begin{aligned}
\frac{1}{p}\Bigg\langle &\sum_{\alpha=1}^{v} \delta_\alpha \nabla g_\alpha(y) + \sum_{\gamma=1}^{q} \omega_\gamma \nabla h_\gamma(y) - \sum_{\alpha=1}^{l} \beta_\alpha^{\mathscr{L}} \nabla \mathscr{L}_\alpha(y)\\
&+ \sum_{\alpha=1}^{l} \beta_\alpha^{Q} \nabla Q_\alpha(y), e^{p\xi(a,y)} - 1 \Bigg\rangle \leq 0.
\end{aligned}$$

By connecting the above inequality and the first equation of (25), we get

$$\frac{1}{p}\langle \nabla f(y), e^{p\xi(a,y)} - 1 \rangle \geq 0.$$

Application of pseudo-p-invexity of f w.r.t. the kernel function ξ leads us to

$$f(a) \geq f(y)$$

and this is the required proof.

(2) By $a \in X$, and $(y, \delta, \omega, \beta^{\mathscr{L}}, \beta^{Q}, \rho, \kappa) \in S_{MW}$, it follows that

$$\begin{aligned}
&g_\alpha(a) \le g_\alpha(y),\ \alpha \in \tau_g^+(a),\\
&h_\gamma(a) = h_\gamma(y),\ \gamma \in \tau_h^+(a) \cup \tau_h^-(a),\\
&-\mathscr{L}_\alpha(a) \le -\mathscr{L}_\alpha(y),\ \alpha \in \tau_{+0}^+(a) \cup \tau_{+-}^+(a) \cup \tau_{00}^+(a) \cup \tau_{0-}^+(a) \cup \tau_{0+}^+(a),\\
&-\mathscr{L}_\alpha(a) \ge -\mathscr{L}_\alpha(y),\ \alpha \in \tau_{0+}^-(a),\\
&Q_\alpha(a) \le Q_\alpha(y),\ \alpha \in \tau_{+0}^{++}(a) \cup \tau_{+-}^{++}(a).
\end{aligned}$$

By the quasi-p-invexity of $g_\alpha(\alpha \in \tau_g^+(a)),\ h_\gamma(\gamma \in \tau_h^+(a)), h_\gamma(\gamma \in \tau_h^-(a))$, $-\mathscr{L}_\alpha(\alpha \in \tau_{+0}^+(a) \cup \tau_{+-}^+(a) \cup \tau_{00}^+(a) \cup \tau_{0-}^+(a) \cup \tau_{0+}^+(a)),\ \mathscr{L}_\alpha(\alpha \in \tau_{0+}^-(a))$, $Q_\alpha(\alpha \in \tau_{+0}^{++}(a) \cup \tau_{+-}^{++}(a))$ for the same real number $p \neq 0$ and w.r.t. the common kernel function ξ, we get

$$\begin{aligned}
&\frac{1}{p}\langle \nabla g_\alpha(y),\ e^{p\xi(a,y)} - 1\rangle \le 0, \bar{\delta}_\alpha > 0,\ \alpha \in \tau_g^+(a),\\
&\frac{1}{p}\langle \nabla h_\gamma(y),\ e^{p\xi(a,y)} - 1\rangle \le 0, \bar{\omega}_\gamma > 0,\ \gamma \in \tau_h^+(a),\\
&\frac{1}{p}\langle \nabla h_\gamma(y),\ e^{p\xi(a,y)} - 1\rangle \ge 0, \bar{\omega}_\gamma < 0,\ \gamma \in \tau_h^-(a),\\
&-\frac{1}{p}\langle \nabla \mathscr{L}_\alpha(y),\ e^{p\xi(a,y)} - 1\rangle \le 0,\ \bar{\beta}_\alpha^{\mathscr{L}} \ge 0,\ \alpha \in \tau_{+0}^+(a) \cup \tau_{+-}^+(a) \cup \tau_{00}^+(a) \cup \tau_{0-}^+(a) \cup \tau_{0+}^+(a),\\
&-\frac{1}{p}\langle \nabla \mathscr{L}_\alpha(y),\ e^{p\xi(a,y)} - 1\rangle \ge 0, \bar{\beta}_\alpha^{\mathscr{L}} \le 0,\ \alpha \in \tau_{+0}^-(a),\\
&\frac{1}{p}\langle \nabla Q_\alpha(y),\ e^{p\xi(a,y)} - 1\rangle \le 0,\ \bar{\beta}_\alpha^{Q} \ge 0,\ \alpha \in \tau_{+0}^{++}(a) \cup \tau_{+-}^{++}(a).
\end{aligned}$$

Using the above inequalities and (2), we get

$$\frac{1}{p}\left\langle \sum_{\alpha=1}^{v} \delta_\alpha \nabla g_\alpha(y) + \sum_{\gamma=1}^{q} \omega_\gamma \nabla h_\gamma(y) - \sum_{\alpha=1}^{l} \beta_\alpha^{\mathscr{L}} \nabla \mathscr{L}_\alpha(y) + \sum_{\alpha=1}^{l} \beta_\alpha^{Q} \nabla Q_\alpha(y), e^{p\xi(a,y)} - 1 \right\rangle \le 0.$$

By combining the above inequality and (25), we obtain

$$\frac{1}{p}\langle \nabla f(y), e^{p\xi(a,y)} - 1\rangle \ge 0.$$

By the pseudo-p-invexity of f, w.r.t. the kernel function ξ, we get

$$f(a) \ge f(y)$$

and this ends the required proof. ■

We now provide the strong duality theorem.

THEOREM 8 *Let $b^* \in X$ be a local minimum of the MPVC, such that the VC-ACQ holds at b^*. Then, there exist Lagrange multipliers $\bar{\delta} \in \mathbb{R}^v, \bar{\omega} \in \mathbb{R}^p, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^Q, \bar{\rho}, \bar{\kappa} \in \mathbb{R}^l$, such that $(b^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^Q, \bar{\rho}, \bar{\kappa})$ is a feasible point of the VC-MWD(b^*), that is, $(b^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^Q, \bar{\rho}, \bar{\kappa}) \in S_{MW}(b^*)$. Moreover, Theorem 7 holds, and then $(b^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^Q)$ is a global maximum of the VC-MWD(b^*).*

PROOF Since $b^* \in X$ is a local minimum of the MPVC and the VC-ACQ condition is satisfied at b^*, by Theorem 1, it follows that there exist Lagrange multipliers $\bar{\delta} \in \mathbb{R}^v, \bar{\omega} \in \mathbb{R}^p, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^Q, \bar{\rho}, \bar{\kappa} \in \mathbb{R}^l$, such that the conditions (2) and (3) hold and hence $(b^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^Q, \bar{\rho}, \bar{\kappa})$ is a feasible point of VC-MWD(b^*). By Theorem 7, it follows that

$$f(b^*) \geq f(y), \forall (y, \delta, \omega, \beta^{\mathscr{L}}, \beta^Q, \rho, \kappa) \in S_{MW}(b^*)$$

and hence $(b^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^Q, \bar{\rho}, \bar{\kappa}) \in S_{MW}(b^*)$ is a global maximum of the VC-MWD. ■

THEOREM 9 *(Converse duality) Let $a \in X$ and $(y^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^Q, \bar{\rho}, \bar{\kappa}) \in S_{MW}$ be feasible points for the MPVC and the VC-MWD, respectively. In addition, if one of the following conditions holds:*

(1) $f(.)$ is pseudo-p-invex and

$$\sum_{\alpha=1}^{v} \bar{\delta}_\alpha g_\alpha(.) + \sum_{\gamma=1}^{q} \bar{\omega}_\gamma h_\gamma(.) - \sum_{\alpha=1}^{l} \bar{\beta}_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(.) + \sum_{\alpha=1}^{l} \bar{\beta}_\alpha^Q Q_\alpha(.)$$

is quasi-p-invex at $y \in X \cup prS_{MW}$, w.r.t. the common kernel function ξ ;

(2) f is pseudo-p-invex and $g_\alpha(\alpha \in \tau_g^+(a))$, $h_\gamma(\gamma \in \tau_h^+(a))$, $-h_\gamma(\gamma \in \tau_h^-(a))$, $-\mathscr{L}_\alpha(\alpha \in \tau_{+0}^+(a) \cup \tau_{+-}^+(a) \cup \tau_{00}^+(a) \cup \tau_{0-}^+(a) \cup \tau_{0+}^+(a))$, $-\mathscr{L}_\alpha(\alpha \in \tau_{0+}^-(a)), Q_\alpha(\alpha \in \tau_{+0}^{++}(a) \cup \tau_{+-}^{++}(a))$ are quasi-p-invex at $y \in X \cup prS_{MW}$ for the same real number $p \neq 0$ and w.r.t. the common kernel function ξ;

then y^* is a global minimum of the MPVC.

PROOF Assume that y^* is not a global minimum of the MPVC, that is, there exists $\tilde{a} \in X$, such that $f(\tilde{a}) < f(y^*)$.

(1) By the pseudo-p-invexity of $f(.)$ w.r.t. the kernel function ξ, we get

$$\frac{1}{p} \langle \nabla f(y^*), e^{p\xi(\tilde{a}, y^*)} - 1 \rangle < 0. \tag{27}$$

Since $\tilde{a} \in X$ and $(y^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^{Q}, \bar{\rho}, \bar{\kappa}) \in S_{MW}$, we get

$$\begin{aligned}
&\bar{\delta}_\alpha g_\alpha(\tilde{a}) \le \bar{\delta}_\alpha g_\alpha(y^*),\ \alpha = 1,2,...,v,\\
&\bar{\omega}_\gamma h_\gamma(\tilde{a}) = \bar{\omega}_\gamma h_\gamma(y^*),\ \gamma = 1,2,...,q,\\
&-\bar{\beta}_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(\tilde{a}) \le -\bar{\beta}_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(y^*),\ \alpha = 1,2,...,l,\\
&\bar{\beta}_\alpha^{Q} Q_\alpha(\tilde{a}) \le \bar{\beta}_\alpha^{Q} Q_\alpha(y^*),\ \alpha = 1,2,...,l,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\sum_{\alpha=1}^{v} \bar{\delta}_\alpha g_\alpha(\tilde{a}) + \sum_{\gamma=1}^{q} \bar{\omega}_\gamma h_\gamma(\tilde{a}) - \sum_{\alpha=1}^{l} \bar{\beta}_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(\tilde{a}) + \sum_{\alpha=1}^{l} \bar{\beta}_\alpha^{Q} Q_\alpha(\tilde{a})\\
&\qquad \le \sum_{\alpha=1}^{v} \bar{\delta}_\alpha g_\alpha(y^*) + \sum_{\gamma=1}^{q} \bar{\omega}_\gamma h_\gamma(y^*) - \sum_{\alpha=1}^{l} \bar{\beta}_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(y^*) + \sum_{\alpha=1}^{l} \bar{\beta}_\alpha^{Q} Q_\alpha(y^*).
\end{aligned}$$

By the quasi-p-invexity of

$$\sum_{\alpha=1}^{v} \bar{\delta}_\alpha g_\alpha(.) + \sum_{\gamma=1}^{q} \bar{\omega}_\gamma h_\gamma(.) - \sum_{\alpha=1}^{l} \bar{\beta}_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(.) + \sum_{\alpha=1}^{l} \bar{\beta}_\alpha^{Q} Q_\alpha(.),$$

w.r.t. the common kernel function ξ, we get

$$\begin{aligned}
&\frac{1}{p}\Bigg\langle \sum_{\alpha=1}^{v} \bar{\delta}_\alpha \nabla g_\alpha(y^*) + \sum_{\gamma=1}^{q} \bar{\omega}_\gamma \nabla h_\gamma(y^*) - \sum_{\alpha=1}^{l} \bar{\beta}_\alpha^{\mathscr{L}} \nabla \mathscr{L}_\alpha(y^*)\\
&+ \sum_{\alpha=1}^{l} \bar{\beta}_\alpha^{Q} \nabla Q_\alpha(y^*), e^{p\xi(\tilde{a},y^*)} - 1 \Bigg\rangle \le 0. \qquad (28)
\end{aligned}$$

By combining the inequalities (27) and (3.2), we get

$$\frac{1}{p}\langle \nabla\, \Gamma(y^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^{Q}), e^{p\xi(\tilde{a},y^*)} - 1\rangle < 0,$$

which contradicts (25) and this is the required proof.
(2) Since $\tilde{a} \in X$ and $(y^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^{Q}, \bar{\rho}, \bar{\kappa}) \in S_{MW}$, we get

$$\begin{aligned}
&\bar{\delta}_\alpha g_\alpha(\tilde{a}) \le \bar{\delta}_\alpha g_\alpha(y^*),\ \alpha = 1,2,...,v,\\
&\bar{\omega}_\gamma h_\gamma(\tilde{a}) = \bar{\omega}_\gamma h_\gamma(y^*),\ \gamma = 1,2,...,q,\\
&-\bar{\beta}_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(\tilde{a}) \le -\bar{\beta}_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(y^*),\ \alpha = 1,2,...,l,\\
&\bar{\beta}_\alpha^{Q} Q_\alpha(\tilde{a}) \le \bar{\beta}_\alpha^{Q} Q_\alpha(y^*),\ \alpha = 1,2,...,l.
\end{aligned}$$

Using the above inequalities and (2), we conclude that

$$\begin{aligned}
&g_\alpha(\tilde{a}) \le g_\alpha(y^*),\ \alpha \in \tau_g^+(\tilde{a}),\\
&h_\gamma(\tilde{a}) = h_\gamma(y^*),\ \gamma \in \tau_h^+(\tilde{a}) \cup \tau_h^-(\tilde{a}),\\
&-\mathscr{L}_\alpha(\tilde{a}) \le -\mathscr{L}_\alpha(y^*),\ \alpha \in \tau_{+0}^+(\tilde{a}) \cup \tau_{+-}^+(\tilde{a}) \cup \tau_{00}^+(\tilde{a}) \cup \tau_{0-}^+(\tilde{a}) \cup \tau_{0+}^+(\tilde{a}),\\
&-\mathscr{L}_\alpha(\tilde{a}) \ge -\mathscr{L}_\alpha(y^*),\ \alpha \in \tau_{0+}^-(\tilde{a}),\\
&Q_\alpha(\tilde{a}) \le Q_\alpha(y^*),\ \alpha \in \tau_{+0}^{++}(\tilde{a}) \cup \tau_{+-}^{++}(\tilde{a}),
\end{aligned}$$

and by the quasi-p-invexity of $g_\alpha(\alpha \in \tau_g^+(a)),\ h_\gamma(\gamma \in \tau_h^+(a)), h_\gamma(\gamma \in \tau_h^-(a))$, $-\mathscr{L}_\alpha(\alpha \in \tau_{+0}^+(a) \cup \tau_{+-}^+(a) \cup \tau_{00}^+(a) \cup \tau_{0-}^+(a) \cup \tau_{0+}^+(a)), \mathscr{L}_\alpha(\alpha \in \tau_{0+}^-(a))$, $Q_\alpha(\alpha \in \tau_{+0}^{++}(a) \cup \tau_{+-}^{++}(a))$, w.r.t. the common kernel function ξ, this implies that

$$\begin{aligned}
&\frac{1}{p}\langle \nabla g_\alpha(y^*),\ e^{p\xi(\tilde{a},y^*)} - 1\rangle \le 0, \bar{\delta}_\alpha > 0,\ \alpha \in \tau_g^+(\tilde{a}),\\
&\frac{1}{p}\langle \nabla h_\gamma(y^*),\ e^{p\xi(\tilde{a},y^*)} - 1\rangle \le 0, \bar{\omega}_\gamma > 0,\ \gamma \in \tau_h^+(\tilde{a}),\\
&\frac{1}{p}\langle \nabla h_\gamma(y^*),\ e^{p\xi(\tilde{a},y^*)} - 1\rangle \ge 0, \bar{\omega}_\gamma < 0,\ \gamma \in \tau_h^-(\tilde{a}),\\
&-\frac{1}{p}\langle \nabla \mathscr{L}_\alpha(y^*),\ e^{p\xi(\tilde{a},y^*)} - 1\rangle \le 0,\\
&\bar{\beta}_\alpha^{\mathscr{L}}, \ge 0,\ \alpha \in \tau_{+0}^+(\tilde{a}) \cup \tau_{+-}^+(\tilde{a}) \cup \tau_{00}^+(\tilde{a}) \cup \tau_{0-}^+(\tilde{a}) \cup \tau_{0+}^+(\tilde{a}),\\
&-\frac{1}{p}\langle \nabla \mathscr{L}_\alpha(y^*),\ e^{p\xi(\tilde{a},y^*)} - 1\rangle \ge 0, \bar{\beta}_\alpha^{\mathscr{L}}, \le 0,\ \alpha \in \tau_{+0}^-(\tilde{a}),\\
&\frac{1}{p}\langle \nabla Q_\alpha(y^*),\ e^{p\xi(\tilde{a},y^*)} - 1\rangle \le 0, \bar{\beta}_\alpha^{Q} \ge 0,\ \alpha \in \tau_{+0}^{++}(\tilde{a}) \cup \tau_{+-}^{++}(\tilde{a}).
\end{aligned}$$

From the above inequalities and (2), we get

$$\begin{aligned}
&\frac{1}{p}\left\langle \sum_{\alpha=1}^{v} \delta_\alpha \nabla g_\alpha(y^*) + \sum_{\gamma=1}^{q} \omega_\gamma \nabla h_\gamma(y^*) - \sum_{\alpha=1}^{l} \beta_\alpha^{\mathscr{L}} \nabla \mathscr{L}_\alpha(y^*)\right.\\
&\left. + \sum_{\alpha=1}^{l} \beta_\alpha^{Q} \nabla Q_\alpha(y^*), e^{p\xi(\tilde{a},y^*)} - 1 \right\rangle \le 0.
\end{aligned}$$

By combining the above inequality and (25), we get

$$\frac{1}{p}\langle \nabla f(y^*), e^{p\xi(\tilde{a},y^*)} - 1\rangle \ge 0.$$

By the pseudo-p-invexity of f, w.r.t. the kernel function ξ, this implies that

$$f(\tilde{a}) \ge f(y^*)$$

and this ends the required proof. ■

Now, we provide a restricted converse duality theorem.

THEOREM 10 *Let $b^* \in X$ and $(y^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^{Q}, \bar{\rho}, \bar{\kappa}) \in S_{MW}$ be feasible points for the MPVC and the VC-MWD, respectively, such that $f(b^*) = f(y^*)$. If the hypothesis of Theorem 7 holds at $y^* \in X \cup prS_{MW}$, then b^* is a global minimum of the MPVC.*

PROOF Assume that $b^* \in X$ is not a global minimum of the MPVC, then there exists $\tilde{a} \in X$ such that

$$f(\tilde{a}) \leq f(b^*).$$

Using the assumptions of the theorem, we get

$$f(\tilde{a}) \leq f(y^*),$$

this contradicts Theorem 7 and hence this constitutes the required proof. ■

THEOREM 11 *(Strict converse duality) Let $b^* \in X$ be a local minimum for the MPVC such that the VC-ACQ holds at b^*. Assume that the conditions of Theorem 8 hold and $(y^*, \tilde{\delta}, \tilde{\omega}, \tilde{\beta}^{\mathscr{L}}, \tilde{\beta}^{Q}, \tilde{\rho}, \tilde{\kappa})$ is a global maximum of the VC-WD(b^*). If one of the following conditions holds:*

(i) $f(.)$ *is* strictly pseudo-p-invex and

$$\sum_{\alpha=1}^{v} \delta_\alpha g_\alpha(.) + \sum_{\gamma=1}^{q} \omega_\gamma h_\gamma(.) - \sum_{\alpha=1}^{l} \beta_\alpha^{\mathscr{L}} \mathscr{L}_\alpha(.) + \sum_{\alpha=1}^{l} \beta_\alpha^{Q} Q_\alpha(.)$$

is quasi-p-invex at $y \in X \cup prS_{MW}$, w.r.t. the common kernel function ξ;

(ii) $f(.)$ is strictly pseudo-p-invex and

$g_\alpha(\alpha \in \tau_g^+(a)),\ h_\gamma(\gamma \in \tau_h^+(a)),\ -h_\gamma(\gamma \in \tau_h^-(a)),$

$-\mathscr{L}_\alpha(\alpha \in \tau_{+0}^+(a) \cup \tau_{+-}^+(a) \cup \tau_{00}^+(a) \cup \tau_{0-}^+(a) \cup \tau_{0+}^+(a)),\ -\mathscr{L}_\alpha(\alpha \in \tau_{0+}^-(a)),$

$Q_\alpha(\alpha \in \tau_{+0}^{++}(a) \cup \tau_{+-}^{++}(a))$ are strictly quasi-p-invex at$y \in X \cup prS_{MW}$

for the same real number $p \neq 0$ and w.r.t. the common kernel function ξ;

then $b^* = y^*$.

PROOF (1) Assume that $b^* \neq y^*$. By Theorem 8, there exist Lagrange multipliers $\bar{\delta} \in \mathbb{R}^v, \bar{\omega} \in \mathbb{R}^p, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^{Q}, \bar{\rho}, \bar{\kappa} \in \mathbb{R}^l$, such that $(y^*, \bar{\delta}, \bar{\omega}, \bar{\beta}^{\mathscr{L}}, \bar{\beta}^{Q}, \bar{\rho}, \bar{\kappa})$ is the global maximum of the VC-MWD(b^*). Hence,

$$f(b^*) = f(y^*). \tag{29}$$

Since $b^* \in X$ and $(y^*, \tilde{\delta}, \tilde{\omega}, \tilde{\beta}^{\mathscr{L}}, \tilde{\beta}^{Q}, \tilde{\rho}, \tilde{\kappa}) \in S_{MW}$, it follows that

$$
\begin{aligned}
&g_\alpha(b^*) \le 0,\ \tilde{\delta}_\alpha \ge 0,\ \alpha = 1,2,...,v,\\
&h_\gamma(b^*) = 0,\ \tilde{\omega}_\gamma \in \mathbb{R},\ \gamma = 1,2,...,q,\\
&-\mathscr{L}_\alpha(b^*) < 0,\ \tilde{\beta}^{\mathscr{L}}_\alpha \ge 0,\ \alpha \in \tau_+(b^*),\\
&-\mathscr{L}_\alpha(b^*) = 0,\ \tilde{\beta}^{\mathscr{L}}_\alpha \in \mathbb{R},\ \alpha \in \tau_0(b^*),\\
&Q_\alpha(b^*) > 0,\ \tilde{\beta}^{Q}_\alpha = 0,\ \alpha \in \tau_{0+}(b^*),\\
&Q_\alpha(b^*) = 0,\ \tilde{\beta}^{Q}_\alpha \ge 0,\ \alpha \in \tau_{00}(b^*) \cup \tau_{+0}(b^*),\\
&Q_\alpha(b^*) < 0,\ \tilde{\beta}^{Q}_\alpha \ge 0,\ \alpha \in \tau_{0-}(b^*) \cup \tau_{+-}(b^*).
\end{aligned}
$$

By (25), this implies that

$$
\begin{aligned}
&\sum_{\alpha=1}^{v} \tilde{\delta}_\alpha g_\alpha(b^*) + \sum_{\gamma=1}^{q} \tilde{\omega}_\gamma h_\gamma(b^*) - \sum_{\alpha=1}^{l} \tilde{\beta}^{\mathscr{L}}_\alpha \mathscr{L}_\alpha(b^*) + \sum_{\alpha=1}^{l} \tilde{\beta}^{Q}_\alpha Q_\alpha(b^*)\\
&\le \sum_{\alpha=1}^{v} \tilde{\delta}_\alpha g_\alpha(y^*) + \sum_{\gamma=1}^{q} \tilde{\omega}_\gamma h_\gamma(y^*) - \sum_{\alpha=1}^{l} \tilde{\beta}^{\mathscr{L}}_\alpha \mathscr{L}_\alpha(y^*) + \sum_{\alpha=1}^{l} \tilde{\beta}^{Q}_\alpha Q_\alpha(y^*).
\end{aligned}
$$

Using the quasi-p-invexity of

$$
\sum_{\alpha=1}^{v} \tilde{\delta}_\alpha g_\alpha(.) + \sum_{\gamma=1}^{q} \tilde{\omega}_\gamma h_\gamma(.) - \sum_{\alpha=1}^{l} \tilde{\beta}^{\mathscr{L}}_\alpha \mathscr{L}_\alpha(.) + \sum_{\alpha=1}^{l} \tilde{\beta}^{Q}_\alpha Q_\alpha(.)
$$

w.r.t. the common kernel function ξ, we get

$$
\begin{aligned}
&\frac{1}{p}\Bigg\langle \sum_{\alpha=1}^{v} \tilde{\delta}_\alpha \nabla g_\alpha(y^*) + \sum_{\gamma=1}^{q} \tilde{\omega}_\gamma \nabla h_\gamma(y^*) - \sum_{\alpha=1}^{l} \tilde{\beta}^{\mathscr{L}}_\alpha \nabla \mathscr{L}_\alpha(y^*)\\
&\qquad + \sum_{\alpha=1}^{l} \tilde{\beta}^{Q}_\alpha \nabla Q_\alpha(y^*), e^{p\xi(b^*,y^*)} - 1 \Bigg\rangle \le 0.
\end{aligned}
$$

Using the above inequality and the first equation in (25), we get

$$
\frac{1}{p}\langle \nabla f(y^*), e^{p\xi(b^*,y^*)} - 1 \rangle \ge 0.
$$

By the strict pseudo-p-invexity of f w.r.t. the kernel function ξ, we conclude that

$$
f(b^*) > f(y^*).
$$

This is a contradiction to (29) and so we get the required proof.

(2) Using $b^* \in X$ and $(y^*, \tilde{\delta}, \tilde{\omega}, \tilde{\beta}^{\mathscr{L}}, \tilde{\beta}^{Q}, \tilde{\rho}, \tilde{\kappa}) \in S_{MW}(b^*)$, we arrive at

$$
\begin{aligned}
&g_\alpha(b^*) \le g_\alpha(y^*),\ \alpha \in \tau_g^+(b^*),\\
&h_\gamma(b^*) = h_\gamma(y^*),\ \gamma \in \tau_h^+(b^*) \cup \tau_h^-(b^*),\\
&-\mathscr{L}_\alpha(b^*) \le -\mathscr{L}_\alpha(y^*),\ \alpha \in \tau_{+0}^+(b^*) \cup \tau_{+-}^+(b^*) \cup \tau_{00}^+(b^*) \cup \tau_{0-}^+(b^*) \cup \tau_{0+}^+(b^*),\\
&-\mathscr{L}_\alpha(b^*) \ge -\mathscr{L}_\alpha(y^*),\ \alpha \in \tau_{0+}^-(b^*),\\
&Q_\alpha(b^*) \le Q_\alpha(y^*),\ \alpha \in \tau_{+0}^{++}(b^*) \cup \tau_{+-}^{++}(b^*).
\end{aligned}
$$

By the quasi-p-invexity of $g_\alpha(\alpha \in \tau_g^+(a)),\ h_\gamma(\gamma \in \tau_h^+(a)),\ -h_\gamma(\gamma \in \tau_h^-(a)),$ $-\mathscr{L}_\alpha(\alpha \in \tau_{+0}^+(a) \cup \tau_{+-}^+(a) \cup \tau_{00}^+(a) \cup \tau_{0-}^+(a) \cup \tau_{0+}^+(a)), \mathscr{L}_\alpha(\alpha \in \tau_{0+}^-(a)),$ $Q_\alpha(\alpha \in \tau_{+0}^{++}(a) \cup \tau_{+-}^{++}(a))$ w.r.t. the common kernel function ξ, we get

$$
\begin{aligned}
&\frac{1}{p}\langle \nabla g_\alpha(y^*), e^{p\xi(b^*,y^*)} - 1\rangle \le 0,\ \bar{\delta}_\alpha > 0,\ \alpha \in \tau_g^+(b^*),\\
&\frac{1}{p}\langle \nabla h_\gamma(y^*), e^{p\xi(b^*,y^*)} - 1\rangle \le 0,\ \bar{\omega}_\gamma > 0,\ \gamma \in \tau_h^+(b^*),\\
&\frac{1}{p}\langle \nabla h_\gamma(y^*), e^{p\xi(b^*,y^*)} - 1\rangle \ge 0,\ \bar{\omega}_\gamma < 0,\ \gamma \in \tau_h^-(b^*),\\
&-\frac{1}{p}\langle \nabla \mathscr{L}_\alpha(y^*), e^{p\xi(b^*,y^*)} - 1\rangle \le 0,\\
&\qquad\qquad \bar{\beta}_\alpha^{\mathscr{L}}, \ge 0 \alpha \in \tau_{+0}^+(b^*) \cup \tau_{+-}^+(b^*) \cup \tau_{00}^+(b^*) \cup \tau_{0-}^+(b^*) \cup \tau_{0+}^+(b^*),\\
&-\frac{1}{p}\langle \nabla \mathscr{L}_\alpha(y^*), e^{p\xi(b^*,y^*)} - 1\rangle \ge 0,\ \bar{\beta}_\alpha^{\mathscr{L}}, \le 0\ \alpha \in \tau_{+0}^+(b^*),\\
&\frac{1}{p}\langle \nabla Q_\alpha(y^*), e^{p\xi(b^*,y^*)} - 1\rangle \le 0,\ \bar{\beta}_\alpha^{Q} \ge 0,\ \alpha \in \tau_{+0}^{++}(b^*) \cup \tau_{+-}^{++}(b^*).
\end{aligned}
$$

From the above inequalities and (2), we obtain

$$
\begin{aligned}
\frac{1}{p}\Bigg\langle &\sum_{\alpha=1}^{v} \tilde{\delta}_\alpha \nabla f(y^*) + \sum_{\gamma=1}^{q} \tilde{\omega}_\gamma \nabla h_\gamma(y^*) - \sum_{\alpha=1}^{l} \tilde{\beta}_\alpha^{\mathscr{L}} \nabla \mathscr{L}_\alpha(y^*)\\
&+ \sum_{\alpha=1}^{l} \tilde{\beta}_\alpha^{Q} \nabla Q_\alpha(y^*), e^{p\xi(b^*,y^*)} - 1 \Bigg\rangle \le 0.
\end{aligned}
$$

By combining the above inequality and (25), we get

$$\frac{1}{p}\langle \nabla f(y^*), e^{p\xi(b^*,y^*)} - 1\rangle \ge 0.$$

By the pseudo-p-invexity of f, w.r.t. the kernel function ξ, we obtain

$$f(b^*) \ge f(y^*).$$

This is a contradiction to (29) and this ends the required proof. ■

EXAMPLE 2 *Consider the MPVC of Example 1. For any feasible point $a \in X$,the VC-MWD (a) to the MPVC is given by*

$$\begin{aligned}
\max\ & f(y) = y_1^2 + 3y_2^2 \\
& s.t. \nabla\, \Gamma(y, \beta_1^Q, \beta_1^{\mathscr{L}}) = \left(2y_1 + \beta_1^Q, 6y_2 - \frac{\beta_1^{\mathscr{L}}}{3}\right) = (0,0), \\
& \beta_1^Q Q_1(y) = \beta_1^Q y_1 \geq 0, \\
& \beta_1^Q = \kappa_1 \frac{a_2}{3},\ \kappa_1 \geq 0, \\
& -\beta_1^{\mathscr{L}} \mathscr{L}_1(y) = -\beta_1^{\mathscr{L}} \frac{y_2}{3} \geq 0, \\
& \beta_1^{\mathscr{L}} = \rho_1 - \kappa_1 a_1,\ \rho_1 \geq 0.
\end{aligned} \tag{30}$$

(1) It is easy to verify that $(y^*, \beta_1^Q, \beta_1^{\mathscr{L}}, \rho_1, \kappa_1) = (0,0,0,0,0) \in S_{MW}(b^*)$, and hence $y^* := (0,0) \in pr_{\mathbb{R}^2} S_{MW}$. Also, we get

$$f(b^*) = 0 = f(y^*),$$

that is, the hypothesis of Theorem 9 holds. Hence, b^* is a global minimum of (30). So, Theorem 9 is verified.

(2) We can get $y_1 = -\frac{\beta_1^Q}{2}, y_2 = \frac{\beta_1^{\mathscr{L}}}{18}$ by (30). One also has

$$\Gamma(y, \beta_1^{\mathscr{L}}, \beta_1^Q) = -\frac{(\beta_1^{\mathscr{L}})^2}{108} - \frac{(\beta_1^Q)^2}{4} \leq 0.$$

Using (30), we get $f(y) \leq 0$. Since $f(a) = a_1^2 + 3a_2^2 \geq 0$, we can get $f(a) \geq f(y)$. Theorem 7 is verified.

(3) Since $\nabla \mathscr{L}_1 = (0, 1/3)^T, \nabla Q_1 = (1,0)^T$. So we obtain that (30) satisfies VC-ACQ. By Theorem 1, there exist Lagrange multipliers $\beta_1^{\mathscr{L}}, \beta_1^Q, \rho_1, \kappa_1 \in \mathbb{R}$ such that $(0, \beta_1^{\mathscr{L}}, \beta_1^Q,$ $\rho_1, \kappa_1)$ is a feasible point of the VC-WD(0). Taking into account the fact that $f(y) \geq$ $0, (0, \beta_1^{\mathscr{L}}, \beta_1^Q, \rho_1, \kappa_1)$ is a global maximum of the VC-MWD(0). Hence, Theorem 8 is verified.

4. Conclusions

In the present article, we have established the weak, strong, converse and restricted converse duality results under the assumptions of p-invexity, strict p-invexity, pseudo-p-invexity, strict pseudo-p-invexity and quasi-p-invexity. Also, the validity of the results is verified by an example.

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