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DETERMINATION OF RANDOM FRICTION FORCES ON THE BIOLOGICAL SURFACES OF A HUMAN HIP JOINT WITH A PHOSPHOLIPID BILAYER

WYZNACZANIE LOSOWYCH SIŁ TARCIA NA BIOLOGICZNYCH POWIERZCHNIACH STAWU BIODRA Z DWUWARSTWĄ FOSFOLIPIDÓW

Key words: human hip joint, friction forces, hydrodynamic lubrication, phospholipids bilayer, analytical stochastic estimation of solutions, apparent viscosity depended on the gap height variations.

Abstract The paper presented concerns a new mathematical form of the stochastic theory of hydrodynamic friction forces occurring on real human hip joint surfaces with a phospholipids bilayer. This paper particularly presents a new review of stochastic analytical considerations realized by the authors for friction forces estimation during hydrodynamic lubrication of biological surfaces performed on the basis of the gap height measurements in the human hip joint. After numerous experimental measurements, it directly follows that the random unsymmetrical increments and decrements of the gap height of human joints have an important influence on the load carrying capacities and finally on the friction forces and wear of cooperating cartilage surfaces. The main topic demonstrates the impact of the variations of expectancy values and the standard deviation of the human joint gap height on the friction forces occurring in the human joint.

Moreover, an evident connection is observed between the apparent dynamic viscosity and the features of the cartilage surface coated by the phospholipid cells. Hence, after the abovementioned remarks, follows the corollary that the influence of the gap height stochastic variations and random surfaces coated by the PL cells tend indirectly from the apparent viscosity into the friction force variations. The synthetic, complex elaborations of the results obtained indicate the influence of the random roughness and stochastic growth of living biological cartilage surfaces on the friction forces distribution.

Słowa kluczowe: staw biodrowy człowieka, siły tarcia, hydrodynamiczne smarowanie, dwuwarstwa fosfolipidów, stochastyczne analityczne oszacowania rozwiązań, lepkość pozorna zależna od zmian wysokości szczeliny.

Streszczenie Celem badawczym podjętym w pracy jest wpływ zmian losowych wysokości szczeliny stawu na zmiany wartości sił nośnych oraz sił tarcia i współczynników tarcia. Do oddziaływań losowych zaliczamy między innymi: ciągłe zmiany architektury kształtów mikrochropowatości powierzchni chrząstki stawowej z udziałem komórek fosfolipidów, ciągły wzrost żywych komórek chrząstki stawowej, mikrometrowej wielkości odkształcenia hipersprężystej warstwy wierzchniej chrząstki stawowej. Głównym rezultatem pracy jest oszacowanie funkcji oczekiwanej sił tarcia i współczynników tarcia dla biołożysk w postaci ogólnych wzorów analitycznych w zależności od pomierzonych wartości funkcji gęstości prawdopodobieństwa. Według informacji autora przedstawiony w niniejszej pracy model stochastyczny układu równań ruchu i energii dla hydrodynamicznej teorii smarowania jest nowy, ponieważ ujmuje losowe zmiany wszystkich parametrów jednocześnie w odróżnieniu od dotychczasowych modeli probabilistycznych ograniczających się jedynie do zmian losowych wysokości szczeliny biołożyska. W pracy wprowadzono nowe pojęcie funkcji oczekiwanej w odróżnieniu od wartości oczekiwanej.

Przedstawiony cel badawczy jest w niniejszej pracy realizowany poprzez badanie rozwiązań analitycznych stochastycznego modelu hydrodynamicznego smarowania stawu w postaci układu równań ruchu oraz przez probabilistyczną analizę parametrów smarowania stawu w zakresie odchyleń standardowych w powiązaniu z wynikami doświadczalnie pomierzonych wysokości szczeliny stawu.

W pracy uwzględnia się wpływ wysokości i kształtu mikroszczeliny stawu na wartość lepkości pozornej cieczy synowialnej o własnościach nienewtonowskich w ruchu ustalonym pomiędzy dwoma współpracującymi powierzchniami chrząstek stawowych.

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INTRODUCTION

The superficial layer of articular hip joint cartilage is coated by a phospholipids bilayer (PL bilayer) of several nanometre thickness or multiple phospholipid layers **[L. 1]**. Regarding human joints, various substances accumulate around the layer, including, among others, lubricin (in approx. 4%), hyaluronic acid (HA) producing enzymes, and single phospholipids. The results of the latest numerous studies carried out by means of an atomic force microscope (AFM) have demonstrated that the phospholipid layer fairly significantly controls the course of hydrodynamic lubrication of surfaces by the biological fluids **[L. 1]**.

During the synovial fluid flow in the human hip joint gap, the cartilage superficial layer and movable phospholipid bilayer change the gap height of the spherical human hip joint **[L. 4–6]**. On the basis of the latest numerous measurements performed by the AFM, it follows that the total gap height ε_T has random increments or decrements in comparison to the height of the nominal mean value. The abovementioned changes in δ_1 can be caused by random micro vibration, by the non-continuous random loading of the joint, stochastic changes of roughness geometry, or random and body growth of the living cartilage cells **[L. 2–3**, **7]**. Random changes of the joint gap height have direct and indirect influences on the stochastic friction forces. The direct influences are visible in analytical solutions presenting the fiction forces dependent on the gap height. The indirect influences of the random gap height variations on the friction forces are provoked by the stochastic changes of synovial fluid dynamic viscosity. The random increments (decrements) of the gap height imply decreases (increases) of the average velocity of synovial fluid in the joint gap, i.e. the share rate decreases (increases), respectively. Hence, for a non-Newtonian fluid, it follows that the viscosity of synovial fluid increases (decreases). Finally, the friction forces depend strongly on synovial fluid viscosity.

The to-date numerous experimental studies on the effects exerted by the phospholipid membrane on the course of hydrodynamic surface lubrication process were, in general, biased by the chemical character, both in theoretical and experimental aspects **[L. 8–13]**. None of those studies have been the subject of any biotribological verification based on analytical-numerical hydrodynamics. This fact inspired the authors to undertake studies of the random friction forces on the biological surfaces with phospholipid bilayer determination in the scope of bio-tribology and biomechanics. The aims of this paper are the direct and indirect influences of random gap height variations on the stochastic value of friction forces on the biological surfaces with a phospholipid bilayer.

RANDOM GAP HEIGHT WITH PROBABILITY DENSITY FUNCTION AND STANDARD DEVIATION

The calculations describing the bonehead of the hip joint are performed in spherical coordinates (φ, r, ϑ), where φ is in the circumferential direction, r is in the radial direction, and ϑ is in the meridian direction of the sphere. The geometry of the human spherical hip joint is presented in **Fig. 1.**

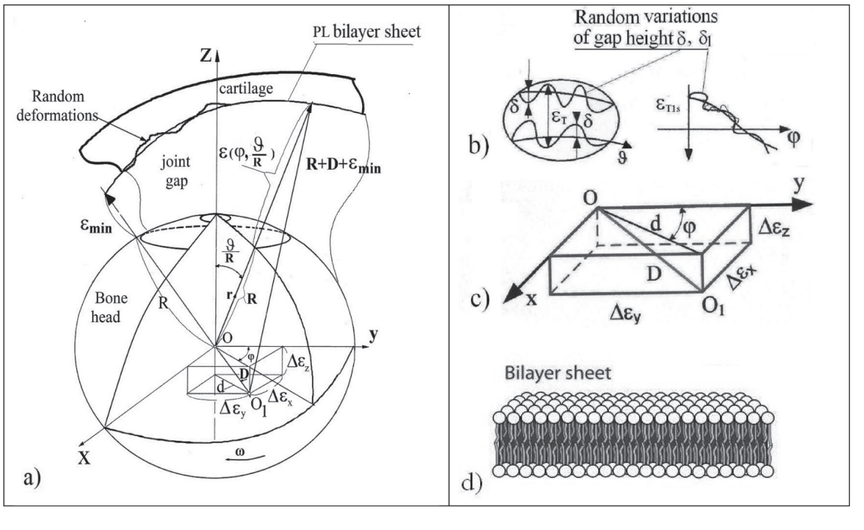

Fig. 1. Human gap height for random deformable cartilage surface: a) joint gap between hip bone head and sleeve (acetabulum), b) random variations δ (random correction parameter) of the gap height ε_T, c) eccentricity D, i.e. the distance between the centre O of the bonehead and the centre O_1 of the sleeve, d) phospholipid (PL) bilayer sheet on the internal sleeve surface; whereas, R is the radius of the bone head, and ε_{min} is the least height of the gap height

Rys. 1. Wysokość szczeliny dla losowo odkształcalnej powierzchni chrząstki stawowej: a) szczelina stawu pomiędzy głową stawu biodra i panewką, b) losowa zmiany δ wysokości ε_T szczeliny, c) mimośrodowość D, czyli odległość środka O sferycznej głowy i środka O_1 panewki, d) warstwa fosfolipidów na wewnętrznej powierzchni panewki, gdzie na rysunkach oznaczono: R – promień głowy kostnej, ε_{min} – najmniejsza wysokość szczeliny

Lubrication region Ω is defined by the following inequalities: $0<\varphi<\pi$; $\pi R/8< \vartheta<\pi R/2$, and $\vartheta = \vartheta_1 R$. The gap height in the hip joint between two nominal smooth spherical cartilage surfaces and deformed by the stochastic variations can be presented in the following dimensional form **[L. 6]**:

$$\varepsilon_T(\varphi,\vartheta_1,\delta_1)=\varepsilon_0\varepsilon_{T1}(\varphi,\vartheta_1,\delta_1)\equiv \varepsilon_0\varepsilon_{T1s}(\varphi,\vartheta_1)[1+\delta_1(\varphi,\vartheta_1)] \tag{1a}$$

$$\varepsilon_T(\delta_1=0)=\varepsilon_0\varepsilon_{T1s}(\varphi,\vartheta_1)=\Delta\varepsilon_x\cos\sin\vartheta_1+\Delta\varepsilon_y\sin\varphi\sin\vartheta_1-\Delta\varepsilon_z\cos\vartheta_1-R+[\Delta\varepsilon_x\cos\varphi\sin_1+\Delta\varepsilon_y\sin\varphi\sin\vartheta_1-\Delta\varepsilon_z\cos_1)^2+(R+\varepsilon_{min})(R+2D+\varepsilon_{min})]^{0.5} \tag{1b}$$

The expected value of the arbitrary random variable correction δ_1 and the expected function of the total gap height is defined by the following expressions **[L. 14]**:

$$EX(\delta_1)=\int_{-\infty}^{+\infty}(\delta_1)\times f(\delta_1)d\delta_1 \tag{2a}$$

$$EX(\varepsilon_{T1})=EX[\varepsilon_{T1s}(1+\delta_1)]=\varepsilon_{T1s}\left[1+EX(\delta_1)\right]=\varepsilon_{T1s}\left[1+\int_{-\infty}^{+\infty}(\delta_1)\times f(\delta_1)d\delta_1\right] \tag{2b}$$

We denote the following: EX – expected operator, δ_1 – considered dimensionless random variable correction, f – probability density function obtained from the measurements. Function f assigns probability P for variable random correction δ_1. Standard deviation σ for random variable correction has the following form **[L. 14]**:

$$\sigma\equiv\sqrt{EX(\delta_1)^2-EX^2(\delta_1)} \tag{3}$$

In the measurements performed **[L. 4]**, the values of the probability density function f are the probabilities assigned to the positive or negative gap height correction values δ_1 due to random roughness and asperities of cartilage in individual points on the joint surfaces. The measurements of the gap height corrections δ_1 due to the random roughness are performed on the cartilage sample (2 mm×2 mm) of the bonehead for a normal hip joint using mechanical sensor **[L. 4]**. The results obtained are presented in **Figs. 2a** and **b.**

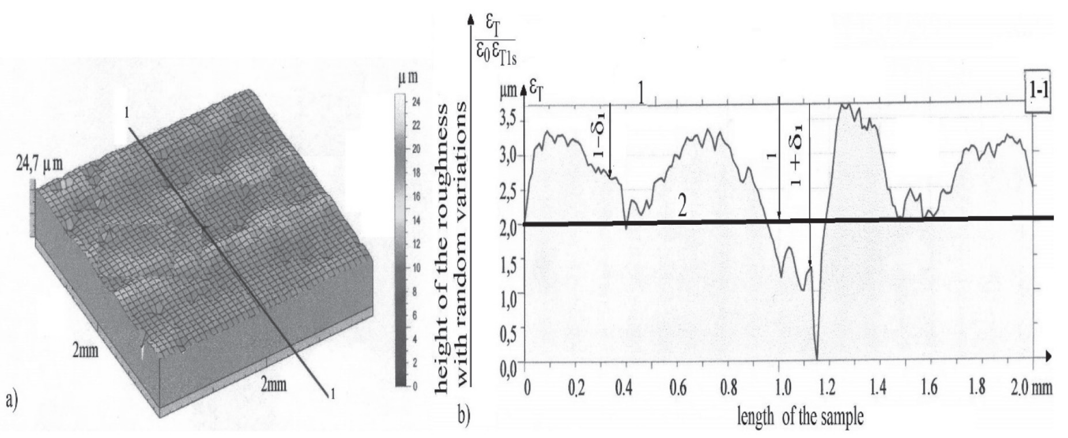

Fig. 2. Random changes of joint gap height ε_T due to random roughness: a) measurement of rough values on the normal cartilage sample (2 mm×2 mm) taken from bone head of human hip joint, b) asperities of normal cartilage surface along the cross-section 1-1 of normal cartilage sample, 1 – middle axis of the gap height between two smooth cooperating surfaces, 2 – throw of the smooth surface restricted the gap without random roughness variations

Rys. 2. Zmiany losowe wysokości ε_T szczeliny stawu powodowane losową chropowatością powierzchni: a) pomiar chropowatości na próbce zdrowej chrząstki (2 mm×2 mm) głowy kostnej stawu biodrowego człowieka, b) nierówności powierzchni zdrowej chrząstki wzdłuż przekroju poprzecznego 1-1 prezentowanej próbki, 1 – środkowa oś wysokości szczeliny pomiędzy dwoma współpracującymi gładkimi powierzchniami, 2 – rzut gładkiej powierzchni ograniczającej szczelinę bez losowych zmian chropowatości

GOVERNING EQUATIONS IN SPHERICAL COORDINATES

The lubrication problem is described by the following equations: the equilibrium of momentum equations, the continuity equation, the energy equation, and the Young-Kelvin Laplace equation. We take into account the following random expected functions: hydrodynamic pressure $EXp(\varphi,\vartheta)$, temperature $EXT(\varphi,r,\vartheta)$, synovial fluid velocity components $EXv_i(\varphi,r,\vartheta)$ for $i=\varphi,r,\vartheta$; random dynamic viscosity of synovial fluid $EX\eta_T(\varphi,r,\vartheta)$, and random joint gap height $EX\varepsilon_T(\varphi,\vartheta)$. Here, synovial fluid density variations are neglected, i.e. incompressible liquid is considered. In the equations, the known dependence has been used between interfacial energy γ and the exponent of hydrogen ion concentration p_H and cartilage surface wettability We **[L. 15–18]**. The influences of the electrostatic field on the viscosity of synovial fluid are taken into account **[L. 19–22]**. We apply the boundary layer simplifications in the abovementioned system of hydrodynamic equations, i.e. we neglect the terms of the order of radial clearance equal to value 10^{-4}. After calculations and term ordering, we obtain the following system of hydrodynamic lubrication equations for the phospholipid bilayer in a stochastic form **[L. 19–22]**:

$$0=-\frac{1}{R\sin\vartheta_1}\frac{\partial EX(p)}{\partial\phi}+\frac{\partial}{\partial r}\left[EX\eta_T(\phi,r,\vartheta)\frac{\partial EXv_\phi}{\partial r}\right]+\rho_e E_\phi \tag{4}$$

$$0=\frac{\partial EX(p)}{\partial r} \tag{5}$$

$$-\frac{\rho EXv_\phi^2}{R}ctg\vartheta_1=-\frac{1}{R}\frac{\partial EX(p)}{\partial\vartheta_1}+\frac{\partial}{\partial r}\left[EX\eta_T(\phi,r,\vartheta_1)\frac{\partial EXv_\vartheta}{\partial r}\right]+\rho_e E_\vartheta \tag{6}$$

$$\frac{\partial EXv_\phi}{\partial\phi}+R\sin\vartheta_1\frac{\partial EXv_r}{\partial r}+\frac{\partial}{\partial\vartheta_1}\left(\sin\vartheta_1 EXv_\vartheta\right)=0 \tag{7}$$

$$\frac{\partial}{\partial r}\left(\kappa\frac{\partial EX(T)}{\partial r}\right)+EX\eta_T(\phi,r,\vartheta)\left[\left(\frac{\partial EXv_\phi}{\partial r}\right)^2+\left(\frac{\partial EXv_\vartheta}{\partial r}\right)^2\right]=J^2/\sigma_e \tag{8}$$

where the expected function of apparent viscosity η_T in [Pa s] has the following form:

$$EX\eta_T(\phi,r,\vartheta)=EX\eta_T(n,p_H,We,T,\gamma,E)\equiv$$
$$=\frac{\gamma_{\max}(\mathrm{p}_H,\mathrm{We})+kEX\left(A^{-1}\right)\cdot EX(T)\ln L}{\delta_v\cdot EX(v_0)}\left[1+\delta_E(p_H,E)E^2\right]\left(\sqrt{\left(\frac{\partial EXv_{\phi1}}{\partial r_1}\right)^2+\left(\frac{\partial EXv_{\vartheta1}}{\partial r_1}\right)^2}\right)^{n-1} \tag{9}$$

$$0<L\equiv\frac{\left(\sqrt{L_k}+1\right)^2}{(L_a+1)(L_b+1)}<1,\ L_a\equiv\frac{K_a}{a_H^+},\ L_b\equiv\frac{a_H^+}{K_b},\ L_k\equiv L_aL_b,\quad (L_a+1)(L_b+1)>\left(\sqrt{L_k}+1\right)^2 \tag{10}$$

We denote the following: $0\le r\le EX\varepsilon_T(\varphi,\vartheta)$, $0<\varphi<\pi$; $\pi R/8<<\pi R/2$, $\vartheta=\vartheta_1 R$, $\boldsymbol{J}^2/\sigma_e\kappa$ [K/m²] and $EX(p)=p(1+EX\delta_p)$, $EX(T)=T(1+EX\delta_T)$, $EXv_i=v_i(1+EX\delta_{vi})$, $i=1, 2, 3$. Unknown, variable random corrections of the pressure, temperature, and lubricant velocity components are denoted by δ_p, δ_T and δ_{vi}.

We denote the following: $\boldsymbol{E}$ in [V/m] – electric intensity vector with components E_φ, E_r, and E_ϑ; $\boldsymbol{J}$ in [A/m²] – electric current density vector, EX(T) in [K] – expected function of random temperature variations in synovial fluid, ρ_e in [C/m³=As/m³] – electric space charge in synovial fluid, κ in [W/mK] – thermal conductivity coefficient for synovial fluid, γ in [J/m² = N/m] – interfacial energy, γ_{max} – is the maximum interfacial energy of the lipid membrane, K_a[J] – acid equilibrium constant (denotes how much energy is needed to stretch the bi-layer), K_b *in* [J] – base equilibrium constant (denotes how much energy is needed to bend or flex the bi-layer), a_H in [J] – protons energy activity, A in [m²] – the region of cartilage surface coated by the PL molecules, ρ in [kg/m^3] − synovial fluid density, σ_e in [S/m = sA^2/Nm^2] − electrical conductivity of PL bi-layer, k = 1.38054·10^{-23} [J/K] − Boltzmann constant, δ_v – dimensionless collagen fibres concentration in bio-fluid, (2<δ_v<6), δ_E in [m^2/V^2] − experimental coefficient describing the influence of the electric field intensity $\boldsymbol{E}$ and ion hydrogen concentration p_H in bio-fluid on the dynamic viscosity, and n-dimensionless flow index (0.8<n<1.2).

The system of partial differential Equations (4)–(8) determines following expected values of unknown random functions, namely: three components of synovial fluid velocity $EX[v_i(\varphi,r,\vartheta)]$ in [m/s] for $i = \varphi,r,\vartheta;$ hydrodynamic pressure $EX\ [p(\varphi,\vartheta)]$ in [N],temperature $EX\ [T(\varphi,r,\vartheta)]$ in [K]. Bio-fluid dynamic viscosity decreases if index δ_v increases from 2 to 6.Coefficient δ_v describes the concentration c_c of collagen fibres inside synovial fluid. For δ_v = 2, we have c_c=100 000 mol/mm^3 and for δ_v = 6, concentration has value c_c=100 mol/mm^3.

Apparent viscosity Expression (9) had been derived from known surface energy $\gamma(p_H,We)$. This function describes dependences between surface energy and exponent of hydrogen ion concentration p_H and wettability We of the cooperating cartilage surfaces.

The surface energy γ had been transformed in an analytical form on the apparent dynamic viscosity η_T for synovial fluid. The abovementioned transformation was graphically illustrated in paper **[L. 19–22]** for two phospholipids kinds PC and PS presenting dimensionless values indicated for acid pKa and base pKb equilibrium constant. Increments of pKa parameter implies viscosity increments in interval 2<p_H<4 and increases decrements of dynamic viscosity in interval 4<p_H<10.

The dynamic viscosity of synovial fluid decreases with decrements of wettability We for the determined values of δ_v, T, and v_0. Decrements of wettability from 70° to 50° denote transition from hydrophobic to hydrophilic properties of cooperating cartilage surfaces swimming round by the synovial fluid.

The value of coefficient δ_E in [m^2/V^2] for synovial fluid has not been exactly determined experimentally. According to the author's knowledge, for E = 10 V/m and p_H = 8, we have δ_E = 0.0003 m^2/V^2 **[L. 23]**. Hence, dimensionless increments of synovial fluid dynamic viscosity with PL caused by the electrostatic field have value of $1+\delta_E E^2$ = 1.03, i.e. barely 3%. The dimensionless values of parameters L_a L_b were explained in Expression (10). Because 0<L<1, $ln(L)$ is a negative value; whereas, the inequality $\gamma_{max}>-kA^{-1}T\ lnL$ is always valid. Symbol v_0 in [m/s] denotes the characteristic dimensional value of synovial fluid. The symbols $v_{\varphi 1}$, $v_{\vartheta 1}$ occurring in Expression (10) denote the dimensionless forms of dimensional synovial fluid velocity components v_φ [m/s] and v_ϑ [m/s] in directions φ, and ϑ; whereas, $v_\varphi = v_0 v_{\varphi 1}$, $v_\vartheta = v_0 v_{\vartheta 1}$. The left side of Equation (6) describes the centrifugal forces occurring during the rotational motion of one of two spherical surfaces.

INTEGRALS FOR RANDOM HYDRODYNAMIC LUBRICATION

Integration of the system (4)–(8) of partial differential equations presenting the lubrication of two cooperating spherical cartilage surfaces will be performed in spherical coordinates.

The random variable gap height ε_T filled up with synovial fluid (without cavitation) is restricted by the motionless cartilage surface on the sleeve (acetabulum) and by the movable bio-surface on the bone head with angular velocity ω. Therefore, we impose the following boundary conditions on the expectancy values of synovial fluid velocity components:

$$EXv_\varphi = \omega R\sin\vartheta_1 \text{ for } r = 0,\ EXv_\varphi = 0 \text{ for } r = EX(\varepsilon_T) \quad (11)$$

$$EXv_r = 0 \text{ for } r = 0, \text{ and } EXv_r = 0 \text{ for } r = EX(\varepsilon_T) \quad (12)$$

$$EXv_\vartheta = 0 \text{ for } \vartheta = 0,\ EXv_\vartheta = 0 \text{ for } \vartheta = EX(\varepsilon_T) \quad (13)$$

Imposing the condition (11) on the general solution of Equation (4), we obtain the following expected value of synovial fluid velocity component in circumferential φ direction **[L. 22]**.

$$EXv_\phi(\phi,r,\vartheta) = \left(\frac{1}{R\sin\vartheta_1}\frac{\partial EX(p)}{\partial\phi} - M_\phi\right)A_\eta + (1-A_s)\omega R\sin\vartheta_1 \quad (14a)$$

Whereas, the functions A_s**[L. 1]**, A_η[m^4/Ns] are as follows:

$$A_s(\phi,r,\vartheta) \equiv \frac{\int_0^r \frac{dr}{EX(\eta_T)}}{\int_0^{EX(\varepsilon_T)} \frac{dr}{EX(\eta_T)}}, \qquad A_\eta(\phi,r,\vartheta) \equiv \int_0^r \frac{rdr}{EX(\eta_T)} - \frac{\left(\int_0^r \frac{dr}{EX(\eta_T)}\right)\left(\int_0^{EX(\varepsilon_T)} \frac{rdr}{EX(\eta_T)}\right)}{\int_0^{EX(\varepsilon_T)} \frac{dr}{EX(\eta_T)}} \quad (14b)$$

where $0 \leq r \leq EX\varepsilon_T(\varphi, \vartheta)$, $0<\varphi<\pi$; $\pi R/8<\vartheta<\pi R/2$, $\vartheta = \vartheta_1 R$; and the expected value of apparent viscosity $EX(\eta_T)$ is defined by (9). Putting solution (14a) into Equation (6) and imposing boundary condition (13), we obtain the following expected value of synovial fluid velocity component in meridian ϑ direction of the spherical surface:

$$EXv_{\vartheta}(\phi,r,\vartheta)=\left(\frac{1}{R}\frac{\partial EX(p)}{\partial \vartheta_1}-M_{\vartheta}\right)A_{\eta}-\frac{\rho}{R}A_p ctg\,\vartheta_1 \tag{15a}$$

The last term on the right hand of Equation (15a), including term A_p, describes the influences of the centrifugal forces on the synovial fluid velocity component in the meridian direction of the spherical bonehead during the motion. The term A_p [m^6/Ns^3] has the following form:

$$\begin{aligned}A_p(\phi,r,\vartheta)\equiv&\left(\frac{1}{R\sin\vartheta_1}\frac{\partial EX(p)}{\partial\phi}-M_{\phi}\right)^2 A_{\rho 1}(\phi,r,\vartheta)+\\ &-2\omega\left(\frac{\partial EX(p)}{\partial\phi}-RM_{\phi}\sin\vartheta_1\right)A_{\rho 2}(\phi,r,\vartheta)+\left(\omega R\sin\vartheta_1\right)^2 A_{\rho 3}(\phi,r,\vartheta)\end{aligned} \tag{15b}$$

Whereas, the help-functions $A_{\rho 1}$ in [m^{12}/N^{3}s^3], $A_{\rho 2}$ in [m^8/N^{2}s^2], $A_{\rho 3}$ in [m^4/Ns] are as follows:

$$A_{\rho 1}(\phi,r,\vartheta)\equiv\int_0^r\left(\frac{1}{EX(\eta_T)}\int_0^r A_{\eta}^2 dr\right)dr-A_s\int_0^{EX(\varepsilon_T)}\left(\frac{1}{EX(\eta_T)}\int_0^r A_{\eta}^2 dr\right)dr \tag{15c}$$

$$A_{\rho 2}(\phi,r,\vartheta)\equiv\int_0^r\left(\frac{1}{EX(\eta_T)}\int_0^r(1-A_s)A_{\eta}dr\right)dr-A_s\int_0^{EX(\varepsilon_T)}\left(\frac{1}{EX(\eta_T)}\int_0^r(1-A_s)A_{\eta}dr\right)dr \tag{15d}$$

$$A_{\rho 3}(\phi,r,\vartheta)\equiv\int_0^r\left(\frac{1}{EX(\eta_T)}\int_0^r(1-A_s)^2 dr\right)dr-A_s\int_0^{EX(\varepsilon_T)}\left(\frac{1}{EX(\eta_T)}\int_0^r(1-A_s)^2 dr\right)dr \tag{15e}$$

Magnetic influences on the velocity component v_{φ}, v_{ϑ} are described by the term $M_i=\rho_e E_i$ in [N/m^3] for $i = \varphi, \vartheta$. By imposing one of the boundary conditions (12), i.e. $v_r = 0$ for $r = 0$, on the continuity Equation (7), then after integration with respect to the variable r and after term ordering, we obtain the following form of the expectancy value of the synovial fluid velocity component in radial r direction:

$$EXv_r(\phi,r,\vartheta)=-\frac{1}{R\sin\vartheta_1}\int_0^r\frac{\partial EXv_{\phi}}{\partial\phi}dr-\frac{1}{R\sin\vartheta_1}\int_0^r\frac{\partial\left(\sin\vartheta_1 EXv_{\vartheta}\right)}{\partial\vartheta_1}dr \tag{16}$$

Now, we put solutions (14a), (15a) into Expression (16). We impose the second boundary condition (12), i.e. $EXv_r = 0$ for $r = EX(\varepsilon_T)$, on the velocity component (16). Hence, after calculations and term ordering ,we obtain the following equation determining the expected function of the hydrodynamic pressure $EXp(\varphi,\vartheta)$:

$$\begin{aligned}&\frac{1}{\sin\vartheta_1}\frac{\partial}{\partial\phi}\left[\left(\frac{\partial EX(p)}{\partial\phi}-RM_{\phi}\sin\vartheta_1\right)\left(\int_0^{EX(\varepsilon_T)}A_{\eta}dr\right)\right]+\frac{\partial}{\partial\vartheta_1}\left[\sin\vartheta_1\left(\frac{\partial EX(p)}{\partial\vartheta_1}-RM_{\vartheta_1}\right)\left(\int_0^{EX(\varepsilon_T)}A_{\eta}dr\right)\right]+\\ &-\rho\frac{\partial}{\partial\vartheta_1}\left(\cos\vartheta_1\int_0^{EX(\varepsilon_T)}A_p dr\right)=\omega R^2\sin\vartheta_1\frac{\partial}{\partial\phi}\left[\int_0^{EX(\varepsilon_T)}A_s dr-EX(\varepsilon_T)\right]\end{aligned} \tag{17}$$

where $0 \leq r \leq \varepsilon_T(\varphi, \vartheta)$, $0<\varphi<\pi$; $\pi R/8<\vartheta<\pi R/2$, $\vartheta = \vartheta_1 R$.

Now, we put the expected functions of synovial fluid velocity components (14a), (15a) into energy Equation (8) with a constant value of the thermal conductivity coefficient κ of synovial fluid. After transformations and term ordering, we obtain the following partial differential equation determining the expected function of temperature in synovial fluid:

$$\frac{\partial^2 EX(T)}{\partial r_2^2}+\frac{EX(\eta_T)}{\kappa}\left\{\left[\left(\frac{1}{R\sin\vartheta_1}\frac{\partial EX(p)}{\partial\phi}-M_\phi\right)\frac{\partial A_\eta}{\partial r}-\omega R\sin\vartheta_1\frac{\partial A_s}{\partial r}\right]^2+\left[\left(\frac{1}{R}\frac{\partial EX(p)}{\partial\vartheta_1}-M_{\vartheta_1}\right)\frac{\partial A_\eta}{\partial r}\right]^2\right\}+$$
$$+\frac{EX(\eta_T)}{\kappa}\frac{\rho}{R}ctg\vartheta_1\frac{\partial A_p}{\partial r}\left[\frac{\rho}{R}ctg\vartheta_1\frac{\partial A_p}{\partial r}-2\left(\frac{1}{R}\frac{\partial EX(p)}{\partial\vartheta_1}-M_{\vartheta_1}\right)\frac{\partial A_\eta}{\partial r}\right]=\frac{M_T}{\kappa} \tag{18}$$

for: $0\le r\le EX\varepsilon_T(\varphi,\ \vartheta)$, $0<\varphi<\pi$; $\pi R/8<\vartheta<\pi R/2$, $\vartheta=\vartheta_1 R$ and $M_T/\kappa=\boldsymbol{J}^2/\sigma\kappa$ in [K/m²].

The derivative of the function (15b) occurring in temperature Equation (18) has the following form:

$$\frac{\partial A_p}{\partial r}\equiv\left(\frac{1}{R\sin\vartheta_1}\frac{\partial EX(p)}{\partial\phi}-M_\phi\right)^2\frac{\partial}{\partial r}\left[A_{\rho1}(\phi,r,\vartheta_1)\right]-2\omega\left(\frac{\partial EX(p)}{\partial\phi}-RM_\phi\sin\vartheta_1\right)\frac{\partial}{\partial r}\left[A_{\rho2}(\phi,r,\vartheta_1)\right]+$$
$$+\left(\omega R\sin\vartheta_1\right)^2\frac{\partial}{\partial r}\left[A_{\rho3}(\phi,r,\vartheta_1)\right] \tag{19}$$

Differentiating the formulae (15c, d, e) with respect to the variable r, we obtain derivatives occurring in the right hand of expression (19).

Determination of the expected function of temperature $EX[T(\alpha_1,\alpha_2,\alpha_3)]$ from the second order partial differential Equation (18) requires two boundary conditions. Variations of the expected temperature values below and above the characteristic environmental temperature T_0 lead finally to the constant value of temperature f_c on the movable bonehead surface and tend to the unknown temperature $f_p(\varphi,\vartheta)$ on the motionless acetabulum surface. The aforementioned conditions take the following form:

$$EX[T(\varphi,r,\vartheta)] = T_0+f_c \text{ for } r=0 \tag{20a}$$

$$EX[T(\varphi,r,\vartheta)] = T_0+f_p(\varphi,\vartheta) \text{ for } r=EX(\varepsilon_T) \tag{20b}$$

To determine the unknown function of the temperature $f_p(\varphi,\vartheta)$ in [K] on the acetabulum surface, we take into account the condition of heat flux flow density q_c in [W/m²] from the bone head surface through the synovial fluid onto the acetabulum surface. Such condition takes the following form:

$$\kappa\frac{\partial EX(T)}{\partial r}=-q_c \quad \text{for} \quad r=0 \tag{21}$$

RANDOM FRICTION FORCES, FRICTION COEFFICIENT, AND LOAD CARRYING CAPACITY

The time components of the expected random values of friction forces $F_{R\varphi}$ in [N], $F_{R\vartheta}$ in [N] in spherical directions φ and ϑ, occurring in human joint gaps with PL bilayer, have the following forms:

$$EX\left(F_{R\phi}\right)=\iint_\Omega\left(EX\left(\eta_T\right)\frac{\partial EX\left(v_\phi\right)}{\partial r}\right)_{r=EX(\varepsilon_T)}R^2 d\phi d\vartheta_1 \tag{22}$$

$$EX(F_{R\vartheta})=\iint_\Omega\left(EX\left(\eta_T\right)\frac{\partial EX\left(v_\vartheta\right)}{\partial r}\right)_{r=EX(\varepsilon_T)}R^2 d\phi d\vartheta_1 \tag{23}$$

where $0\le r\le\varepsilon_T(\varphi,\ \vartheta)$, $0<\varphi<\pi$; $\pi R/8<\vartheta<\pi R/2$, $\vartheta=\vartheta_1 R$, $\eta_T(\varphi,\ r,\ \vartheta)$, synovial fluid viscosity, $EX(\varepsilon_T)=EX[\varepsilon_T(\varphi,\ \vartheta)]$ – the expected value of random gap height, $\Omega(\varphi,\ \vartheta_1)$ – lubrication surface.

Putting the derivatives of the expected functions of random synovial fluid velocity components (14a), (15a) into (23) and taking into account derivatives (14b) and (20), we finally obtain the expected functions of random friction force components $F_{R\varphi}$ in [N], and $F_{R\vartheta}$ in [N] in the following form:

$$EX\left(F_{R\phi}\right)=\iint_{\Omega}\left(\frac{\partial EX(p)}{\partial\phi}-RM_{\phi}\sin\vartheta_1\right)\left[EX\left[\varepsilon_T(\phi,\vartheta_1)\right]-\frac{\int_0^{EX[\varepsilon_T(\phi,\vartheta_1)]}\frac{rdr}{EX\left[\eta_T(\phi,r,\vartheta)\right]}}{\int_0^{EX[\varepsilon_T(\phi,\vartheta_1)]}\frac{dr}{EX\left[\eta_T(\phi,r,\vartheta)\right]}}\right]d\phi Rd\vartheta_1+ \tag{24}$$

$$-\iint_{\Omega}\left[\frac{\omega R^3\sin^2\vartheta_1}{\int_0^{EX[\varepsilon_T(\phi,\vartheta_1)]}\frac{dr}{EX\left[\eta_T(\phi,r,\vartheta)\right]}}\right]d\phi d\vartheta_1$$

$$EX\left(F_{R\vartheta}\right)=\iint_{\Omega}\left(\frac{\partial EX(p)}{\partial\vartheta_1}-RM_{\vartheta 1}\right)\left[EX\left[\varepsilon_T(\phi,\vartheta_1)\right]-\frac{\int_0^{EX[\varepsilon_T(\phi,\vartheta_1)]}\frac{rdr}{EX\left[\eta_T(\phi,r,\vartheta)\right]}}{\int_0^{EX[\varepsilon_T(\phi,\vartheta_1)]}\frac{dr}{EX\left[\eta_T(\phi,r,\vartheta)\right]}}\right]R\sin\vartheta_1 d\phi d\vartheta_1+$$
$$-\rho\iint_{\Omega}\cos\vartheta_1\left\{\left(\frac{1}{R\sin\vartheta_1}\frac{\partial EX(p)}{\partial\phi}-M_{\phi}\right)^2\cdot EX\left(F_{\rho 1}\right)-2\omega\left(\frac{\partial EX(p)}{\partial\phi}-RM_{\phi}\sin\vartheta_1\right)\cdot EX\left(F_{\rho 2}\right)\right.$$
$$\left.+\left(\omega R\sin\vartheta_1\right)^2\cdot EX\left(F_{\rho 3}\right)\right\}d\phi Rd\vartheta_1 \tag{25}$$

The expected functions of parameters presenting the centrifugal effects of friction forces $EXF_{\rho 1}$ in [m^9/N^{2}s^2], $EXF_{\rho 2}$ in [m^5/Ns], $EXF_{\rho 3}$ in [m] occurring in the three last terms on the right hand of Equation (25), by virtue of (23), (15a, b, c), and (19), have the following form:

$$EX\left(F_{\rho 1}\right)\equiv\left\{EX\left(\eta_T\right)\frac{\partial}{\partial r}\left[A_{\rho 1}(\phi,r,\vartheta)\right]\right\}_{r=EX(\varepsilon_T)},EX\left(F_{\rho 2}\right)\equiv\left\{EX\left(\eta_T\right)\frac{\partial}{\partial r}\left[A_{\rho 2}(\phi,r,\vartheta)\right]\right\}_{r=EX(\varepsilon_T)}$$
$$EX\left(F_{\rho 3}\right)\equiv\left\{EX\left(\eta_T\right)\frac{\partial}{\partial r}\left[A_{\rho 3}(\phi,r,\vartheta)\right]\right\}_{r=EX(\varepsilon_T)} \tag{26}$$

where $0\le r\le\varepsilon_T(\varphi,\vartheta)$, $0<\varphi<\pi$; $\pi R/8<\vartheta<\pi R/2$, $\vartheta = \vartheta_1 R, \eta_T(\varphi, r, \vartheta)$.

The expected value of the load carrying capacity ***C*** in [N] situated in the opposite side to the load *W* direction has the following form:

$$EXC_{tot}^{(sph)}=\sqrt{\left[\int_{R\pi/8}^{R\pi/2}\left(\int_0^{\phi_k}EXp(\phi,\vartheta)R(\sin\phi)(\sin_R^{\vartheta})d\phi\right)d\vartheta\right]^2+\left[\int_{R\pi/8}^{R\pi/2}\left(\int_0^{\phi_k}EXp(\phi,\vartheta)R(\cos\phi)(\sin_R^{\vartheta})d\phi\right)d\vartheta\right]^2} \tag{27}$$

where symbol φ_k denotes the end coordinate of the film in circumferential direction and $0 \le \varphi < 2\pi\theta_1$, $0 \le \theta_1 < 1$, $R\pi/8 \le \vartheta \le R\pi/2$, $\vartheta = R\vartheta_1$, symbol $EX(p)$in [Pa] denotes the expected function of the hydrodynamic pressure. Taking into account the Coulomb Law for friction forces, and adhesion expected value $EX(A_D)$ in [N], the expected friction coefficient $EX\mu$ in **[L. 1]** has the following form:

$$EX\mu = \frac{\left| \boldsymbol{e}_1 EX\left(F_{R\phi}\right) + \boldsymbol{e}_3 EX\left(F_{R\vartheta}\right) \right| - EX\left(A_D\right)}{EX(C)} \tag{28}$$

where $\boldsymbol{e}_1, \boldsymbol{e}_3$ are the unit vectors in circumferential φ and meridian ϑ directions.

ESTIMATION OF RANDOM VALUES OF TRIBOLOGY PARAMETERS

In this section, the estimations are an elaboration of random lubrication parameters, such as load carrying capacity and friction forces by virtue of the analytical solutions obtained without any numerical calculations on the grounds of probabilistic transformations and inequalities.

First, we find the probabilistic density function f of the random variable correction function δ_1 defined in Section 2 in Equation (2). Equation (2a) gives the expected value of correction function, i.e. $EX(\delta_1) = m$, and from Equation (3), we calculate standard deviation σ. Hence, from Expressions (2b) and (1a and b), we obtain an estimation of the expected value of the gap height in the form of the following inequality:

$$(1+m-\sigma)\,\varepsilon_T(\delta_1 = 0) \le EX(\varepsilon_T) \le (1+m+\sigma)\varepsilon_T(\delta_1 = 0) \tag{29}$$

where $\varepsilon_T(\delta_1 = 0)$ denotes the gap height without random corrections.

The random increments (decrements) of the joint gap height imply decreases (increases) in the random average velocities of synovial fluid and, simultaneously, the decreases (increases) in the shear rates during the synovial fluid flow, respectively. Hence, for non-Newtonian synovial fluid flow, we obtain increases (decreases) in the dynamic viscosity of synovial fluid. Thus, it follows that the random increments (decrements) of the joint gap height imply the increases (decreases) in the synovial dynamic viscosity. Therefore, the following inequality is valid:

$$(1+m-\sigma)\cdot\eta_T\left(\delta_1 = 0\right) \le EX(\eta_T) \le (1+m+\sigma)\cdot\eta_T\left(\delta_1 = 0\right) \tag{30}$$

Symbol $\eta_T(\delta_1 = 0) = \eta_T(We,pH,T,\Theta)$ denotes the viscosity obtained from Equation (9) for gap height without random corrections (1b).

The expected value of load carrying capacity for the hip joint was obtained from (27) for $C = pS$, where $S = R^2 d\varphi d\vartheta_1$ denotes the lubricated surface. By virtue of EXp obtained from (17) and after inequalities (29) and (30), we obtain the following estimation of the expected value of load carrying capacity:

$$\frac{R^2 S\omega\eta_T\left(\delta_1 = 0\right)}{\varepsilon_T^2\left(\delta_1 = 0\right)}\frac{(1+m-\sigma)}{(1+m+\sigma)^2} \le EX(C) \le \frac{R^2 S\omega\eta_T\left(\delta_1 = 0\right)}{\varepsilon_T^2\left(\delta_1 = 0\right)}\frac{1+m+\sigma}{(1+m-\sigma)^2} \tag{31}$$

Taking into account the analytical solutions (24–26) and inequalities (29–30), we obtain the following estimation of the expected function of friction forces:

$$\frac{SR\omega\eta_T\left(\delta_1 = 0\right)}{\varepsilon_T\left(\delta_1 = 0\right)}\frac{(1+m-\sigma)}{(1+m+\sigma)} \le EX(F_R) \le \frac{SR\omega\eta_T\left(\delta_1 = 0\right)}{\varepsilon_T\left(\delta_1 = 0\right)}\frac{1+m+\sigma}{(1+m-\sigma)} \tag{32}$$

Taking into account the analytical solutions (28) and inequalities (29–30) with dependence $\mu = F_R/C$, we obtain the following estimation of the expected values of friction coefficient:

$$(1+m-\sigma)\frac{\varepsilon_T\left(\delta_1 = 0\right)}{R} \le EX(\mu = F_R/C) \le (1+m+\sigma)\frac{\varepsilon_T\left(\delta_1 = 0\right)}{R} \tag{33}$$

EXAMPLE OF THE PROBABILITY DENSITY FUNCTION

The probability density function f dependent on the dimensionless random variable correction δ_1 is always determined in an experimental way. Based on many measurements of joint cartilage surface topography performed by the J. Cwanek **[L. 4]** and V.C. Mow **[L. 5]**, and papers and investigations **[L. 24–25]**, it follows that the gap height variations caused by the roughness and its asperities (see **Fig. 2**) create asymmetrical and symmetrical probability density functions. In the present considerations, only the case is taken into account where the probability average values of the sum of numerous asymmetrical variations tend to the symmetrical function. In the symmetrical density function f, the probability values of gap height increments assigned to the random variable correction δ_1 are symmetrical in relation to the decrements of the correction parameter. For example, by virtue of the experiments **[L. 4]**, we show the symmetrical probability density function described by the following equations:

$$f_S(\delta_1) \equiv \begin{cases} 1+\delta_1 & \text{for} \quad -1 \le \delta_1 \le 0, \\ 1-\delta_1 & \text{for} \quad 0 \le \delta_1 \le 1, \\ 0 & \text{for} \quad |\delta_1| > 1. \end{cases} \tag{34}$$

Figure 3 illustrates in a graphical form the function (34), where the vertical axis describes the probability values. In the upper horizontal axis, the dimensionless random variable correction δ_1 is depicted. The lower parallel horizontal axis marks the dimensionless values $1+\delta_1$. Point *1* on the lower axis coincides with the point *0* in the upper axis. By putting (34) into Formula (2), we obtain the expected value of the joint gap height $m_S = 0$. By substituting (34) into (3), we obtain the value of standard deviation: $\sigma_S = 0.4082$. The interval of standard deviation in the form $(m_S-\sigma_S,\ m_S+\sigma_S)$, i.e. (–0.4082, +0.4082) is depicted in **Fig. 3** on the axis δ_1. The corresponding values marked on the second horizontal axis $1+\delta_1$ indicate the expected value of gap height variations from $0.5918 \times \varepsilon_0\varepsilon_{T1s}(\varphi, \vartheta_1)$ to $1.4082 \times \varepsilon_0\varepsilon_{T1s}(\varphi, \vartheta_1)$ where the function $\varepsilon_0\varepsilon_{T1s}(\varphi, \vartheta_1) = \varepsilon_T(\delta_1 = 0)$ determines the formula (1b) and denotes the gap height without random corrections. It is visible on the vertical axis in **Fig. 3** that the abovementioned gap height variations correspond to the probability changes from $P_s = 0.5918$ to $P_s = 1.0000$.

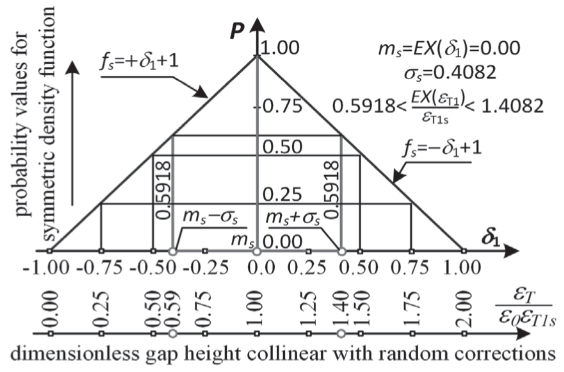

Fig. 3. The symmetrical probability density function f_S after (34) versus dimensionless random variable corrections of increments (positive corrections $\delta_1>0$) and decrements (negative corrections $\delta_1<0$) on the first axis and versus dimensionless gap height plus corrections i.e. $1 + \delta_1$ on the second coaxial parallel axis

Rys. 3. Symetryczna funkcja gęstości prawdopodobieństwa f_S według wzoru (34), zależna od bezwymiarowej losowej korekty przyrostów $\delta_1>0$ i ubytków $\delta_1<0$ wysokość szczeliny stawu (według pierwszej osi poziomej) oraz w uzależnieniu od bezwymiarowych losowych zmian wysokości całej szczeliny $1 + \delta_1$ (według drugiej współliniowej osi poziomej)

For the abovementioned measurements of gap height variations, we obtain $1+m-\sigma = 0.5919$ and $1+m+\sigma = 1.4082$. Thus, by virtue of inequalities (30–33), the following estimations for the expected values of the synovial fluid viscosity, load carrying capacity, friction forces, and friction coefficient take the following form:

$$0.5919\eta_T(\delta_1=0) \le EX(\eta_T) \le 1.4082\eta_T(\delta_1=0),\ 0.2984\frac{R^2S\omega\eta_T(\delta_1=0)}{\varepsilon_T^2(\delta_1=0)} \le EX(C) \le 4.0194\frac{R^2S\omega\eta_T(\delta_1=0)}{\varepsilon_T^2(\delta_1=0)},$$

$$0.4203\frac{SR\omega\eta_T(\delta_1=0)}{\varepsilon_T(\delta_1=0)} \le EX(F_R) \le 2.3790\frac{SR\omega\eta_T(\delta_1=0)}{\varepsilon_T(\delta_1=0)}, \tag{35}$$

$$0.5919\frac{\varepsilon_T(\delta_1=0)}{R} \le EX\left(\mu=\frac{F_R}{C}\right) \le 1.4082\frac{\varepsilon_T(\delta_1=0)}{R}$$

CONCLUSIONS

On the basis of the performed random analytical solutions and stochastic estimations with the elaborated example of symmetrical density function, the following is evident:

- The height of the human hip joint gap between two isotonic osmotic cartilage surfaces with the PL bilayer attains decreases and increases by about 40.8% in comparison to the gap height without any random corrections.
- The expected value of random friction forces occurring in the human hip joint gap between two isotonic osmotic cartilage surfaces with PL bilayer attains decreases from 0.4203 $F_R(\delta_1 = 0)$ and increases to 2,3790 $F_R(\delta_1 = 0)$ in comparison with the friction force $F_R(\delta_1 = 0)$ without any random corrections of the gap height.
- The expected value of the random friction coefficient occurring in the human hip joint gap between two isotonic osmotic cartilage surfaces with the PL bilayer and without adhesion forces attains decreases from 0.5919 μ ($\delta_1 = 0$) and increases to 1.4082 $\mu(\delta_1 = 0)$ in comparison to the friction coefficient μ ($\delta_1 = 0$) without any random corrections of the gap height.

DISCUSSION

The paper presented shows the non-Newtonian synovial fluid flow in a probabilistic sense for the human hip joint between two cooperating isotonic spherical deformable cartilage surfaces. The aforementioned deformability is caused, among others, by the random variations of the roughness or variations of topography of cartilage surface architecture caused by the continuous growth of the living tissue of cartilage.

The normal cartilage cooperating surfaces and superficial layer with PL bilayer have isotonic, iso-osmotic, and impermeable features. Bio-hydrodynamic lubrication of mentioned features has not been considered in literature until now.

Moreover, in contemporary scientific literature, the upper and lower estimation of expected values and simultaneously its probability values of load carrying capacity, friction forces, friction coefficients occurring during the bio-hydrodynamic lubrication of cooperating isotonic surfaces have not been derived. Mentioned estimations are obtained in this paper at first in a general form using standard deviations and then in particular numerical form for experimental measured data. Such possibility exclude the necessity of the comparison of obtained intervals of the estimation presented in this paper with foreseen and earlier results derived in literature.

The new results contained in this paper are obtained taking into account 3D variations of the dynamic viscosity of synovial fluid, particularly variations crosswise to the film thickness and non-Newtonian synovial fluid properties.

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