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Dedicated to Professor J. Klamka
on the occasion of his 70th birthday

Design of the oscillatory systems with the extremal dynamic properties

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Abstract. In this article the problem of determination of such coefficients $a_1, a_2, ..., a_n$ and eigenvalues $s_1, s_2, ..., s_n$ of the characteristic equation which yield required extremal values of the solution $x(t)$ at the extremal value τ of time is solved. The extremal values of $x(\tau)$ and τ are treated as functions of the roots $s_1, s_2, ..., s_n$. The analytical formulae enable us to design the systems with prescribed dynamic properties. For solution of the problem the properties of symmetrical equations are used. The method is illustrated by an example of the equation of 4-th degree. The regions of the different kinds of the roots: real, with one pair of complex and two pairs of complex roots are illustrated. A practical problem is shown.

Key words: extremal dynamic properties, oscillatory systems, symmetrical equations, regions of the roots.

1. Introduction

The oscillations can be observed both in the mechanical and in the electrical systems. These oscillations are caused mainly by the exchange of the kinetic and potential energy in the system. Great oscillations of the suspension of the car can lead to its destruction.

In the article an analytic method is proposed, which enables the design of the system with prescribed values of the amplitude and period of the oscillations.

2. Statement of the problem

Calculation of conditions and extremum of the extreme value of the dynamic error [1].

Case 1

Let us consider the differential equation determining the dynamic error in a linear control system of n-th order with lumped and constant parameters:

$$\frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + ... + a_{n-1} \frac{dx}{dt} + a_n x = 0. \tag{1}$$

The initial conditions are determined by the force function and the system's parameters.

Let us assume in general, that

$$x^{(i)}(0) = c_{i+1} \neq 0 \qquad \text{for} \qquad i = 0, 1, ..., n-1.$$

We assume further that the characteristic equation of Eq. (1) has m different real roots and $2p$ different complex roots.

It is evident that

$$m + 2p = n.$$

We denote by s_k real roots and

$$\alpha_k + j\omega_k = r_k, \quad \alpha_k - j\omega_k = \widehat{r}_k, \quad (k = 1, 2, ..., p).$$

The solution of Eq. (1) takes the form

$$x(t) = \sum_{k=1}^{m} A_k e^{s_k t} + \sum_{k=1}^{p} \left[B_k \cos(\omega_k t) + C_k \sin(\omega_k t)\right] e^{\alpha_k t}, \tag{2}$$

where A_k, B_k, C_k, s_k, α_k, ω_k are real numbers.

The necessary conditions for the dynamic error $x(t)$ to attain an extreme value at $t = \tau$ is given by the relation:

$$\frac{dx}{dt} = \sum_{k=1}^{m} A_k s_k e^{s_k t} + \sum_{k=1}^{p} [(-B_k \sin \omega_k \tau + C_k \cos \omega_k \tau)\omega_k + (B_k \cos \omega_k \tau + C_k \sin \omega_k \tau)\alpha_k]\, e^{\alpha_k \tau} = 0. \tag{3}$$

The constants are determined from

$$x^{(i)}(0) = c_{i+1} = \sum_{k=1}^{m} A_k s_k^i + \sum_{k=1}^{p} \left[B_k Re(r_k^i) + C_k Im(r_k^i)\right], \tag{4}$$

$$(i = 0, 1, ..., n-1).$$

The extreme value of the dynamic error is

$$x(\tau) = \sum_{k=1}^{m} A_k e^{s_k \tau} + \sum_{k=1}^{p} \left[B_k \cos(\omega_k \tau) + C_k \sin(\omega_k \tau)\right] e^{\alpha_k \tau}. \tag{5}$$

The extremum of extreme value of the dynamic error given by Eq. (5), computed with regard to the parameters s_k, α_k, ω_k, is obtained by putting the respective partial derivatives of $x(\tau)$ equal to zero.

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Denoting by

$$\left(\frac{\partial x(\tau)}{\partial s_k}\right)^*, \qquad \left(\frac{\partial x(\tau)}{\partial \alpha_k}\right)^*, \qquad \left(\frac{\partial x(\tau)}{\partial \omega_k}\right)^*$$

the partial derivatives of expression (5) for the constant τ we may write

$$\left.\begin{aligned} \frac{\partial x(\tau)}{\partial s_k} &= \left(\frac{\partial x(\tau)}{\partial s_k}\right)^* + \frac{\partial x(\tau)}{\partial \tau}\frac{\partial \tau}{\partial s_k} \\ \frac{\partial x(\tau)}{\partial \alpha_k} &= \left(\frac{\partial x(\tau)}{\partial \alpha_k}\right)^* + \frac{\partial x(\tau)}{\partial \tau}\frac{\partial \tau}{\partial \alpha_k} \\ \frac{\partial x(\tau)}{\partial \omega_k} &= \left(\frac{\partial x(\tau)}{\partial \omega_k}\right)^* + \frac{\partial x(\tau)}{\partial \tau}\frac{\partial \tau}{\partial \omega_k} \end{aligned}\right\}. \tag{6}$$

However, we have from Eq. (3)

$$\frac{\partial x(\tau)}{\partial \tau} = 0$$

and therefore

$$\left.\begin{aligned} \frac{\partial x(\tau)}{\partial s_k} &= \left(\frac{\partial x(\tau)}{\partial s_k}\right)^* \\ \frac{\partial x(\tau)}{\partial \alpha_k} &= \left(\frac{\partial x(\tau)}{\partial \alpha_k}\right)^* \\ \frac{\partial x(\tau)}{\partial \omega_k} &= \left(\frac{\partial x(\tau)}{\partial \omega_k}\right)^* \end{aligned}\right\}. \tag{7}$$

We obtain the following conditions:

$$\left.\begin{aligned} &\sum_{k=1}^{m} \frac{\partial A_k}{\partial s_j} e^{s_k \tau} + A_j \tau e^{s_j \tau} \\ &\qquad + \sum_{k=1}^{p} \left(\frac{\partial B_k}{\partial s_j}\cos\omega_k\tau + \frac{\partial C_k}{\partial s_j}\sin\omega_k\tau\right) e^{\alpha_k\tau} = 0 \\ &\qquad j = 1, 2, \ldots\ldots, m \\ &\sum_{k=1}^{m} \frac{\partial A_k}{\partial \alpha_j} e^{s_k \tau} + \sum_{k=1}^{p} \left(\frac{\partial B_k}{\partial \alpha_j}\cos\omega_k\tau + \frac{\partial C_k}{\partial \alpha_j}\sin\omega_k\tau\right) e^{\alpha_k\tau} \\ &\qquad + \left(B_j\cos\omega_j\tau + C_j\sin\omega_j\tau\right) e^{\alpha_j\tau}\tau = 0 \\ &\sum_{k=1}^{m} \frac{\partial A_k}{\partial \omega_j} e^{s_k \tau} + \sum_{k=1}^{p} \left(\frac{\partial B_k}{\partial \omega_j}\cos\omega_k\tau + \frac{\partial C_k}{\partial \omega_j}\sin\omega_k\tau\right) e^{\alpha_k\tau} \\ &\qquad + \left(C_j\cos\omega_j\tau - B_j\sin\omega_j\tau\right) e^{\alpha_j\tau}\tau = 0 \\ &\qquad j = 1, 2, \ldots\ldots, p \end{aligned}\right\} \tag{8}$$

In this way we have a system of n linear and homogenous equations with n unknowns

$$e^{s_k\tau}, \quad e^{\alpha_k\tau}\sin\omega_k\tau, \quad e^{\alpha_k\tau}\cos\omega_k\tau.$$

The determinant of system (8) must vanish if there are not to be all zero solutions. The same determinant (after being reflected about one of the main diagonals) is:

$$|D + A\tau|, \tag{9}$$

where D and A are matrices determined by the following equations:

$$\left.\begin{aligned} D &= \sum_{j=1}^{m}\sum_{k=1}^{m} \frac{\partial A_j}{\partial s_k} E_{jk} + \sum_{j=1}^{p}\sum_{k=1}^{m} \\ &\cdot \left(\frac{\partial B_j}{\partial s_k} E_{m+2j-1,k} + \frac{\partial C_j}{\partial s_k} E_{m+2j,k}\right) + \sum_{j=1}^{m}\sum_{k=1}^{p} \\ &\cdot \left(\frac{\partial A_j}{\partial \alpha_k} E_{j,m+2k-1} + \frac{\partial A_j}{\partial \omega_k} E_{j,m+2k}\right) + \sum_{j=1}^{p}\sum_{k=1}^{p} \\ &\cdot \left[\left(\frac{\partial B_j}{\partial \alpha_k} E_{m+2j-1,m+2k-1} + \frac{\partial B_j}{\partial \omega_k} E_{m+2j-1,m+2k}\right)\right. \\ &\left. + \left(\frac{\partial C_j}{\partial \alpha_k} E_{m+2j,m+2k-1} + \frac{\partial C_j}{\partial \omega_k} E_{m+2j,m+2k}\right)\right], \\ A &= \sum_{j=1}^{m} A_j E_{jj} + \sum_{j=1}^{p} \\ &\cdot \left[B_j\left(E_{m+2j-1,m+2j-1} - E_{m+2j,m+2j}\right)\right. \\ &\left. + C_j\left(E_{m+2j-1,m+2j} + E_{m+2j,m+2j-1}\right)\right], \end{aligned}\right\} \tag{10}$$

$$\begin{aligned} E_{jk} &= \left(e_{\mu,\nu}^{(jk)}\right)_{\mu,\nu=1,\ldots,n} \\ e_{\mu,p}^{(jk)} &= \delta_{\mu j}\delta_{\nu k} = \begin{cases} 0 & \text{for} \quad \mu = j, \nu = k \\ 1 & \text{for all other cases} \end{cases} \end{aligned} \tag{11}$$

Finally, we have

$$|D + A\tau| = 0 \tag{12}$$

and system (8) yields for unknown τ (after some algebraic manipulations) the following equation:

$$(-1)^n \tau^n \prod_{k=1}^{m} A_k \prod_{k=1}^{p} \left(B_k^2 + C_k^2\right) = 0 \tag{13}$$

Case 2

It might be asked whether the time τ, corresponding to the extreme value of the dynamic error, attains an extreme value with respect to the parameters s_k, α_k, ω_k. To investigate this we assume that

$$\left.\begin{aligned} &\frac{\partial \tau}{\partial s_k} = 0 \qquad (k = 1, \ldots., m) \\ &\frac{\partial \tau}{\partial \alpha_k} = \frac{\partial \tau}{\partial \omega_k} = 0 \qquad (k = 1, \ldots., p) \end{aligned}\right\}. \tag{14}$$

We compute the partial derivatives of Eq. (8), taking into account Eq. (14).

$$\begin{aligned} &\sum_{k=1}^{m} \frac{\partial A_k}{\partial s_j} s_k e^{s_k\tau} + (1 + s_j\tau) A_j e^{s_j\tau} \\ &+ \sum_{k=1}^{p} \left[\left(\frac{\partial B_k}{\partial s_j}\cos\omega_k\tau + \frac{\partial C_k}{\partial s_j}\sin\omega_k\tau\right)\alpha_k\right. \\ &\left. + \left(\frac{\partial C_k}{\partial s_j}\cos\omega_k\tau - \frac{\partial B_k}{\partial s_j}\sin\omega_k\tau\right)\right] e^{s_k\tau} = 0, \\ &(j = 1, \ldots.., m), \end{aligned} \tag{15}$$

$$\sum_{k=1}^{m}\frac{\partial A_k}{\partial \alpha_j}s_k e^{s_k\tau}+\sum_{k=1}^{p}\left(\frac{\partial B_k}{\partial \alpha_j}\cos\omega_k\tau+\frac{\partial C_k}{\partial \alpha_j}\sin\omega_k\tau\right)\alpha_k$$
$$+\left[\left(B_j\cos\omega_j\tau+C_j\sin\omega_j\tau\right)\left(1+\alpha_j\tau\right)\right.$$
$$\left.+\left(C_j\cos\omega_j\tau-B_j\sin\omega_j\tau\right)\omega_j\tau\right]e^{\alpha_j\tau}=0,$$
$$(j=1,......,p), \tag{16}$$

$$\sum_{k=1}^{m}\frac{\partial A_k}{\partial \omega_j}s_k e^{s_k\tau}+\sum_{k=1}^{p}\left[\left(\frac{\partial B_k}{\partial \omega_j}\cos\omega_k\tau+\frac{\partial C_k}{\partial \omega_j}\sin\omega_k\tau\right)\alpha_k\right.$$
$$\left.+\left(\frac{\partial C_k}{\partial \omega_j}\cos\omega_k\tau-\frac{\partial B_k}{\partial \omega_j}\sin\omega_k\tau\right)\omega_k\right]e^{\alpha_k\tau}$$
$$+\left[\left(C_j\cos\omega_j\tau-B_j\sin\omega_j\tau\right)\left(1+\alpha_j\tau\right)\right.$$
$$\left.-\left(B_j\cos\omega_j\tau+C_j\sin\omega_j\tau\right)\omega_j\tau\right]e^{\alpha_j\tau}=0,$$
$$(j=1,...,p). \tag{17}$$

Let

$$F=\sum_{\mu=1}^{m}s_\mu E_{\mu,\mu}+\sum_{\mu=1}^{p}$$
$$\cdot\left[\alpha_\mu\left(E_{m+2\mu-1,m+2\mu-1}+E_{m+2\mu,m+2\mu-1}\right)\right.$$
$$\left.+\omega_\mu\left(E_{m+2\mu-1,m+2\mu}-E_{m+2\mu,m+2\mu-1}\right)\right]. \tag{18}$$

Equations (15)–(17) yield, after by equating the determinant to zero

$$|FD+A+FA\tau|=0, \tag{19}$$

$$(-1)^p\prod_{k=1}^{m}A_k\prod_{k=1}^{p}\left(B_k^2+C_k^2\right)\prod_{k=1}^{m}s_k\prod_{k=1}^{p}\left(\alpha_k^2+\omega_k^2\right)\tau^{n-1}$$
$$\cdot\left[\tau+\sum_{k=1}^{m}\frac{1}{s_k}+\sum_{k=1}^{p}\left(\frac{1}{r_k}+\frac{1}{\hat{r}_k}\right)\right]=0. \tag{20}$$

From (20) it results that

$$\tau=0 \tag{21}$$

or, using Vieta's formulae

$$\tau=-\left[\sum_{k=1}^{m}\frac{1}{s_k}+\sum_{k=1}^{p}\left(\frac{1}{r_k}+\frac{1}{\hat{r}_k}\right)\right]=\frac{a_{n-1}}{a_n}. \tag{22}$$

The set of Eqs. (17) gives also another necessary condition, which was presented in [2].

In the paper [2] another necessary condition was found, i.e.:

$$D_n(\tau)=\begin{vmatrix} c_1 & c_2 & c_3 & c_4 & ... & c_{n-1} & c_n \\ -\frac{a_{n-2}}{a_n} & \tau & -1 & 0 & ... & 0 & 0 \\ -\frac{a_{n-3}}{a_n} & 0 & \tau & -2 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ -\frac{a_1}{a_n} & 0 & 0 & 0 & ... & \tau & 2-n \\ -\frac{1}{a_n} & 0 & 0 & 0 & ... & 0 & \tau \end{vmatrix}=0. \tag{23}$$

After substituting $\tau=\dfrac{a_{n-1}}{a_n}$ into (23) we obtain the relation between initial conditions $c_{i+1},\ i=0,1,...,n-1$ and coefficients $a_j,\ j=1,2,...,n$.

$$D_n=\begin{vmatrix} c_1 & c_2 & c_3 & c_4 & ... & c_{n-1} & c_n \\ a_{n-2} & -a_{n-1} & a_n & 0 & ... & 0 & 0 \\ a_{n-3} & 0 & -a_{n-1} & 2a_n & ... & 0 & 0 \\ a_{n-4} & 0 & 0 & -a_{n-1} & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ a_1 & 0 & 0 & 0 & ... & -a_{n-1} & (n-2)a_n \\ 1 & 0 & 0 & 0 & ... & 0 & -a_{n-1} \end{vmatrix}=0. \tag{24}$$

3. Solution of the problem

It is a very difficult problem to determine the roots $s_1, s_2, ..., s_n$ which fulfill the necessary conditions $\tau = \frac{a_{n-1}}{a_n}$ and $D_n = 0$. The solution of algebraic equation with n higher than $n = 4$ is possible only using an additional assumption [3]. For that reason we use the properties of symmetrical algebraic equations. From the theoretical point of view such equations can be solved up to 9-th degree, which is satisfactory for practical applications.

3.1. Symmetrical algebraic equations. In what follows we use the equations

$$a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + ... + a_2 z^2 + a_1 z + a_0 = 0, \quad a_0 \neq 0 \tag{25}$$

in which the coefficients of all terms in equal distance from the beginning and from the end are equal. It is easy to see that if the equation has a root z_k, then it also has a root $\frac{1}{z_k}$.

Let the degree be even, $n = 2k$.

Divide Eq. (25) by z^k and arrange the appropriate terms to obtain

$$\begin{aligned} &a_0\left(z^k + \frac{1}{z^k}\right) + a_1\left(z^{k-1} + \frac{1}{z^{k-1}}\right) \\ &+ ... + a_{k-1}\left(z + \frac{1}{z}\right) + a_k = 0. \end{aligned} \tag{26}$$

Putting

$$y = z + \frac{1}{z} \tag{27}$$

we obtain that

$$\left.\begin{aligned} &y^2 = z^2 + \frac{1}{z^2} + 2 \\ &... \\ &y^k = \left(z^k + \frac{1}{z^k}\right) + k\left(z^{k-2} + \frac{1}{z^{k-2}}\right) \\ &\qquad + \frac{k}{2}\left(z^{k-4} + \frac{1}{z^{k-4}}\right) + ... \end{aligned}\right\}. \tag{28}$$

Hence we have

$$\left.\begin{aligned} z + \frac{1}{z} &= y \\ z^2 + \frac{1}{z^2} &= y^2 - 2 \\ z^3 + \frac{1}{z^3} &= y^3 - 3y \\ &... \end{aligned}\right\}. \tag{29}$$

In particular, for the equation of the 4-th degree we must solve three equations of the second degree: one for the determination of the values of y and two for determination the values of unknown z. The equation of the odd degree $n = 2k + 1$ has always the root equal to $z = -1$. After dividing the equation

$$\begin{aligned} &a_0 z^{2k+1} + a_1 z^{2k} + a_2 z^{2k-1} + ... + a_k z^{k+1} \\ &+ ... + a_2 z^2 + a_1 z + a_0 = 0 \end{aligned} \tag{30}$$

by $(z + 1)$ we obtain the equation of the even degree

$$b_0 z^{2k} + b_1 z^{2k-1} + ... + b_1 z^k + b_0 = 0 \tag{31}$$

where

$$\left.\begin{aligned} &b_0 = a_0 \\ &b_1 = -a_0 + a_1 \\ &b_2 = a_0 - a_1 + a_2 \\ &... \\ &b_k = (-1)^k a_0 + (-1)^{k-1} a_1 + (-1)^{k-2} a_2 + ... + a_k \end{aligned}\right\}. \tag{32}$$

3.2. Determination of the curves bounding the regions with different kinds of roots. We apply the well-known relation for the discriminant of Eq. (25):

$$\Delta_n = V_n^2 = \prod_{\substack{k,l=1 \\ k>l}}^{n} (z_k - z_l)^2, \tag{33}$$

where V_n is the Vandermonde determinant.

In the paper [4] there is presented an example of the 3-rd degree.

We illustrate the method by an example of equation of the 4-th degree.

4. Particular example, $n = 4$

Let us consider the differential equation

$$\frac{d^4 x(t)}{dt^4} + a_1 \frac{d^3 x(t)}{dt^3} + a_2 \frac{d^2 x(t)}{dt^2} + a_3 \frac{dx(t)}{dt} + a_4 x(t) = 0 \tag{34}$$

with initial conditions

$$x(0) = c_1, \quad x^{(1)}(0) = c_2, \quad x^{(2)}(0) = c_3, \quad x^{(3)}(0) = c_4.$$

The characteristic equation of (34) is

$$s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0. \tag{35}$$

We assume that the roots s_1, s_2, s_3, s_4 have negative real parts.

We want to obtain simple analytic formulae for s_i by using symmetrization of Eq. (34).

We put

$$s = \sqrt[4]{a_4} z, \qquad a_4 > 0. \tag{36}$$

Then we obtain the equation

$$z^4 + \frac{a_1}{\sqrt[4]{a_4}} z^3 + \frac{a_2}{\sqrt[4]{a_4^2}} z^2 + \frac{a_3}{\sqrt[4]{a_4^3}} z + 1 = 0. \tag{37}$$

We denote

$$b_1 = \frac{a_1}{\sqrt[4]{a_4}}, \quad b_2 = \frac{a_2}{\sqrt[4]{a_4^2}}, \quad b_3 = \frac{a_3}{\sqrt[4]{a_4^3}} \tag{38}$$

and assume that

$$\left.\begin{aligned} &b_1 = b_3 \\ &\text{or} \\ &a_1 = \frac{a_3}{\sqrt{a_4}} \end{aligned}\right\}. \tag{39}$$

Then Eq. (37) takes a form

$$z^4 + b_1 z^3 + b_2 z^2 + b_1 z + 1 = 0 \tag{40}$$

which is symmetric.

We observe that the extremal time for Eq. (35) is

$$\tau_1 = \frac{a_3}{a_4} \tag{41}$$

with the necessary condition

$$D_4 = \begin{vmatrix} c_1 & c_2 & c_3 & c_4 \\ a_2 & -a_3 & a_4 & 0 \\ a_1 & 0 & -a_3 & 2a_4 \\ 1 & 0 & 0 & -a_3 \end{vmatrix} = 0. \tag{42}$$

After symmetrization we have for Eq. (40) that

$$\tau_1 = b_1. \tag{43}$$

The condition (42) has the form

$$D_4 = \begin{vmatrix} c_1 & c_2 & c_3 & c_4 \\ b_2 & -b_1 & 1 & 0 \\ b_1 & 0 & -b_1 & 2 \\ 1 & 0 & 0 & -b_1 \end{vmatrix} = 0 \tag{44}$$

or

$$c_1 b_1^3 + \left(b_1^2 + b_2 b_1^2 + 2\right) c_2 + \left(2b_1 + b_1^3\right) c_3 + b_1^2 c_4 = 0. \tag{45}$$

Using Eq. (28) for solution Eq. (40) we put

$$\left.\begin{aligned} y &= z + \frac{1}{z} \\ y^2 &= z^2 + \frac{1}{z^2} + 2 \end{aligned}\right\} \tag{46}$$

and obtain

$$y^2 + b_1 y + b_2 - 2 = 0. \tag{47}$$

The roots of (47) are

$$\left.\begin{aligned} y_1 &= -\frac{1}{2}b_1 + \frac{1}{2}\sqrt{b_1^2 - 4b_2 + 8} \\ y_2 &= -\frac{1}{2}b_1 - \frac{1}{2}\sqrt{b_1^2 - 4b_2 + 8} \end{aligned}\right\}. \tag{48}$$

From Eq. (46) we have

$$z^2 - zy + 1 = 0. \tag{49}$$

Substitution of (48) to Eq. (49) gives the roots of Eq. (40)

$$\left.\begin{aligned} z_1 &= -\frac{1}{4}b_1 + \frac{1}{4}\sqrt{b_1^2 - 4b_2 + 8} \\ &\quad + \frac{1}{4}\sqrt{2b_1^2 - 2b_1\sqrt{b_1^2 - 4b_2 + 8} - 4b_2 - 8} \\ z_2 &= -\frac{1}{4}b_1 - \frac{1}{4}\sqrt{b_1^2 - 4b_2 + 8} \\ &\quad + \frac{1}{4}\sqrt{2b_1^2 + 2b_1\sqrt{b_1^2 - 4b_2 + 8} - 4b_2 - 8} \\ z_3 &= -\frac{1}{4}b_1 + \frac{1}{4}\sqrt{b_1^2 - 4b_2 + 8} \\ &\quad - \frac{1}{4}\sqrt{2b_1^2 - 2b_1\sqrt{b_1^2 - 4b_2 + 8} - 4b_2 - 8} \\ z_4 &= -\frac{1}{4}b_1 - \frac{1}{4}\sqrt{b_1^2 - 4b_2 + 8} \\ &\quad - \frac{1}{4}\sqrt{2b_1^2 + 2b_1\sqrt{b_1^2 - 4b_2 + 8} - 4b_2 - 8} \end{aligned}\right\}. \tag{50}$$

Knowing the roots of Eq. (40) we can calculate the discriminant (33)

$$\begin{aligned} \Delta_4 &= \prod_{\substack{k,l=1 \\ k>l}}^{4} (z_k - z_l)^2 \\ &= -\left(2b_1 + b_2 + 2\right)\left(2b_1 - b_2 - 2\right)\left(b_1^2 - 4b_2 + 8\right)^2. \end{aligned} \tag{51}$$

From Eq. (51) we obtain that

$$\Delta_4 = 0 \tag{52}$$

if

$$b_2 = 2b_1 - 2 \tag{53}$$

or

$$b_2 = -2b_1 - 2 \tag{54}$$

or

$$b_2 = \frac{1}{4}b_1^2 + 2. \tag{55}$$

For the stability of the system, the Hurwitz determinant H must be positive

$$H_4 = \begin{vmatrix} b_1 & 1 & 0 & 0 \\ b_1 & b_2 & b_1 & 1 \\ 0 & 1 & b_1 & b_2 \\ 0 & 0 & 0 & 1 \end{vmatrix} > 0. \tag{56}$$

From (56) we obtain the following stability conditions

$$\left.\begin{aligned} b_1 &> 0 \\ b_2 &> 0 \\ b_2 &> 2 \end{aligned}\right\}. \tag{57}$$

The limit of stability is for $b_2 = 2$.

From (57) it follows that the case (54) is not allowed.

In Fig. 1 we illustrate the regions for different kinds of roots according to the values of the discriminant (51), that means $\Delta_4 < 0$, $\Delta_4 > 0$ and $p < 0$ or $p > 0$, $q < 0$, $q > 0$.

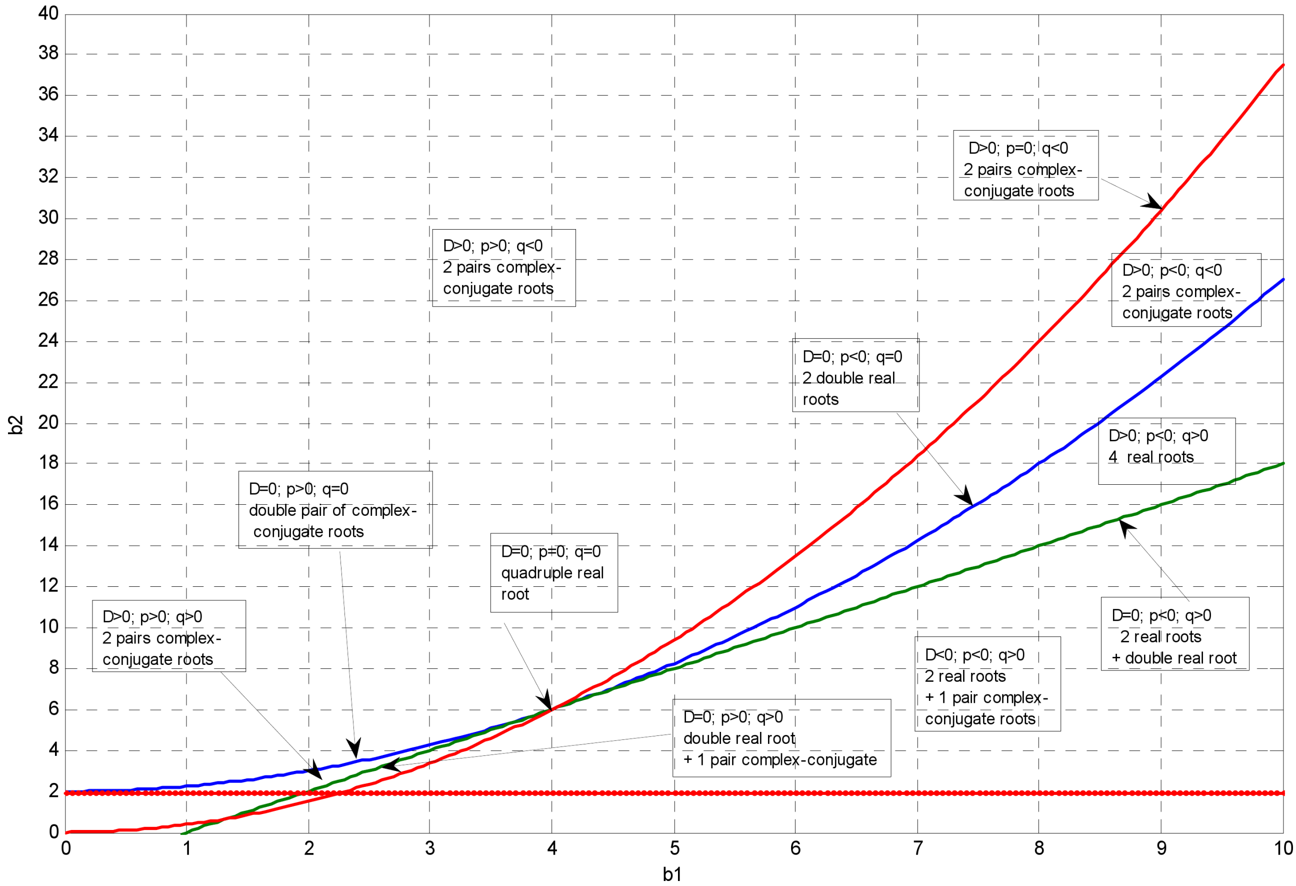

Fig. 1. Regions of roots for different values of b_1 and b_2, according to the values of Δ, p and q

Different regions of the roots

The equation

$$s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0. \tag{58}$$

Substituting

$$s = y - \frac{a_1}{4} \tag{59}$$

to Eq. (58) gives

$$y^4 + py^2 + qy + r = 0, \tag{60}$$

where

$$p = a_2 - \frac{3}{8}a_1^2, \tag{61}$$

$$q = -\frac{1}{2}a_1 a_2 + \frac{1}{8}a_1^3 + a_3, \tag{62}$$

$$r = -\frac{1}{4}a_1 a_3 + \frac{1}{16}a_1^2 a_2 - \frac{3}{256}a_1^4 + a_4 \tag{63}$$

and the discriminant

$$\Delta_4 = 16rp^4 - 4p^3q^2 - 128p^2r^2 + 144pq^2r - 27q^4 + 256r^3. \tag{64}$$

After symmetrization we obtain

$$p = b_2 - \frac{3}{8}b_1^2, \tag{65}$$

$$q = \frac{1}{8}b_1^3 - \frac{1}{2}b_1 b_2 + b_1, \tag{66}$$

$$r = -\frac{1}{4}b_1^2 + \frac{1}{16}b_1^2 b_2 - \frac{3}{256}b_1^4 + 1, \tag{67}$$

$$\Delta_4 = -\left(2b_1 + b_2 + 2\right)\left(2b_1 - b_2 - 2\right)\left(b_1^2 - 4b_2 + 8\right)^2, \tag{68}$$

$$p = 0 \quad \text{for} \quad b_2 = \frac{3}{8}b_1^2, \tag{69}$$

$$q = 0 \quad \text{for} \quad b_2 = \frac{1}{4}b_1^2 + 2, \tag{70}$$

$$\Delta_4 = 0 \quad \text{for} \quad b_2 = 2b_1 - 2 \tag{71}$$

or

$$b_2 = \frac{1}{4}b_1^2 + 2. \tag{72}$$

The limit of stability is for

$$\left.\begin{matrix} b_1 > 0 \\ b_2 = 2 \end{matrix}\right\}. \tag{73}$$

In particular, for $b_2 = 2$, $b_1 = \dfrac{4}{\sqrt{3}}$ we have $r = 0$.

Using the curves which are determined by relations (69), (70), (71) and (72) we can establish Fig. 1 and Table 1, illustrating the different regions of the roots.

Table 1
Different regions of roots

	Two pairs complex-conjugate			Two real + two complex	Four real	Contradictionary inequalities		
D	>0	>0	>0	<0	>0	<0	<0	<0
p	>0	>0	<0	<0	<0	<0	>0	>0
q	>0	<0	<0	>0	>0	<0	<0	>0

In particular we have that:
for $b_1 = 4$ and $b_2 = 6$: one quadruple real root
for $b_1 > 4$ and $b_2 = \frac{1}{4}b_1^2 + 2$: two double real roots
for $0 < b_1 < 4$ and $b_2 = \frac{1}{4}b_1^2 + 2$: double pair of complex-conjugate roots
for $b_1 > 4$ and $b_2 = 2b_1 - 2$: two different real roots and one double real root
for $0 < b_1 < 4$ and $b_2 = 2b_1 - 2$: one double real root and one pair of complex-conjugate root.

In conclusion we see that there are three different regions which include two pairs of complex-conjugate roots, one region with one pair of complex roots and two real roots and finally one region with four real roots.

The determination of the coefficients b_1 and b_2 from the necessary conditions (45) and (3) is very difficult. For that reason we calculate from these equations the initial conditions $\frac{c_2}{c_1}$, $\frac{c_3}{c_1}$, $\frac{c_4}{c_1}$, $c_1 \neq 0$.

Equation (1) in this case is as follows:

$$\frac{d^4z}{dt^4} + b_1\frac{d^3z}{dt^3} + b_2\frac{d^2z}{dt^2} + b_1\frac{dz}{dt} + 1 = 0. \tag{74}$$

The solution of Eq. (74) takes a form

$$\begin{aligned} z(t) = &-\frac{(z_2z_3z_4c_1 - z_2z_4c_2 - z_2z_3c_2 + z_2c_3 + z_4c_3 + z_3c_3 - c_4 - z_3z_4c_2)\,e^{z_1t}}{(z_4 - z_1)(z_3 - z_1)(z_1 - z_2)} \\ &+\frac{(z_1z_3z_4c_1 - z_1z_3c_2 - z_1z_4c_2 + z_1c_3 + z_4c_3 + z_3c_3 - c_4 - z_3z_4c_2)\,e^{z_2t}}{(z_1 - z_2)(z_4 - z_2)(z_3 - z_2)} \\ &-\frac{(z_1c_3 + z_1z_2z_4c_1 - c_4 - z_1z_2c_2 - z_2z_4c_2 + z_2c_3 + z_4c_3 - z_1z_4c_2)\,e^{z_3t}}{(z_3 - z_1)(z_3 - z_2)(z_3 - z_4)} \\ &+\frac{(z_1z_2z_3c_1 - z_1z_2c_2 + z_2c_3 - z_2z_3c_2 + z_1c_3 - c_4 - z_1z_3c_2 + z_3c_3)\,e^{z_4t}}{(z_4 - z_2)(z_3 - z_4)(z_4 - z_1)}. \end{aligned} \tag{75}$$

The derivative

$$\begin{aligned} \frac{dz(t)}{dt} = &-\frac{(z_2z_3z_4c_1 - z_2z_4c_2 - z_2z_3c_2 + z_2c_3 + z_4c_3 + z_3c_3 - c_4 - z_3z_4c_2)\,z_1e^{z_1t}}{(z_4 - z_1)(z_3 - z_1)(z_1 - z_2)} \\ &+\frac{(z_1z_3z_4c_1 - z_1z_3c_2 - z_1z_4c_2 + z_1c_3 + z_4c_3 + z_3c_3 - c_4 - z_3z_4c_2)\,z_2e^{z_2t}}{(z_1 - z_2)(z_4 - z_2)(z_3 - z_2)} \\ &-\frac{(z_1c_3 + z_1z_2z_4c_1 - c_4 - z_1z_2c_2 - z_2z_4c_2 + z_2c_3 + z_4c_3 - z_1z_4c_2)\,z_3e^{z_3t}}{(z_3 - z_1)(z_3 - z_2)(z_3 - z_4)} \\ &+\frac{(z_1z_2z_3c_1 - z_1z_2c_2 + z_2c_3 - z_2z_3c_2 + z_1c_3 - c_4 - z_1z_3c_2 + z_3c_3)\,z_4e^{z_4t}}{(z_4 - z_2)(z_3 - z_4)(z_4 - z_1)}. \end{aligned} \tag{76}$$

The necessary condition for extremal τ is $\left.\frac{dz}{dt}\right|_{\tau} = 0$.

From the technological point of view we require the values of τ and $x(\tau)$.

According to (22) we know that for Eq. (74) $\tau = b_1$, and we assume the value of $\frac{z(\tau)}{c_1}$, $c_1 \neq 0$.

Assuming the values of b_2 we can calculate the three initial conditions $\frac{c_2}{c_1}$, $\frac{c_3}{c_1}$, $\frac{c_4}{c_1}$, $c_1 \neq 0$ from Eqs. (45), (75) and (76).

In the special, very interesting case when $c_2 = 0$, which gives the minimum of $z(\tau)$, we need only two equations, namely (45) and (76). There are linear equations for $\frac{c_3}{c_1}$ and $\frac{c_4}{c_1}$ with the variable coefficients b_1 and b_2.

In the Table 2 there are the calculated values of $\frac{c_3}{c_1}$, $\frac{c_4}{c_1}$ and extremal value $\frac{z_e}{c_1}$ as functions of parameters b_1 and b_2 for the region of the real roots. These relations are illustrated in Fig. 2. One representative example is shown in Fig. 3.

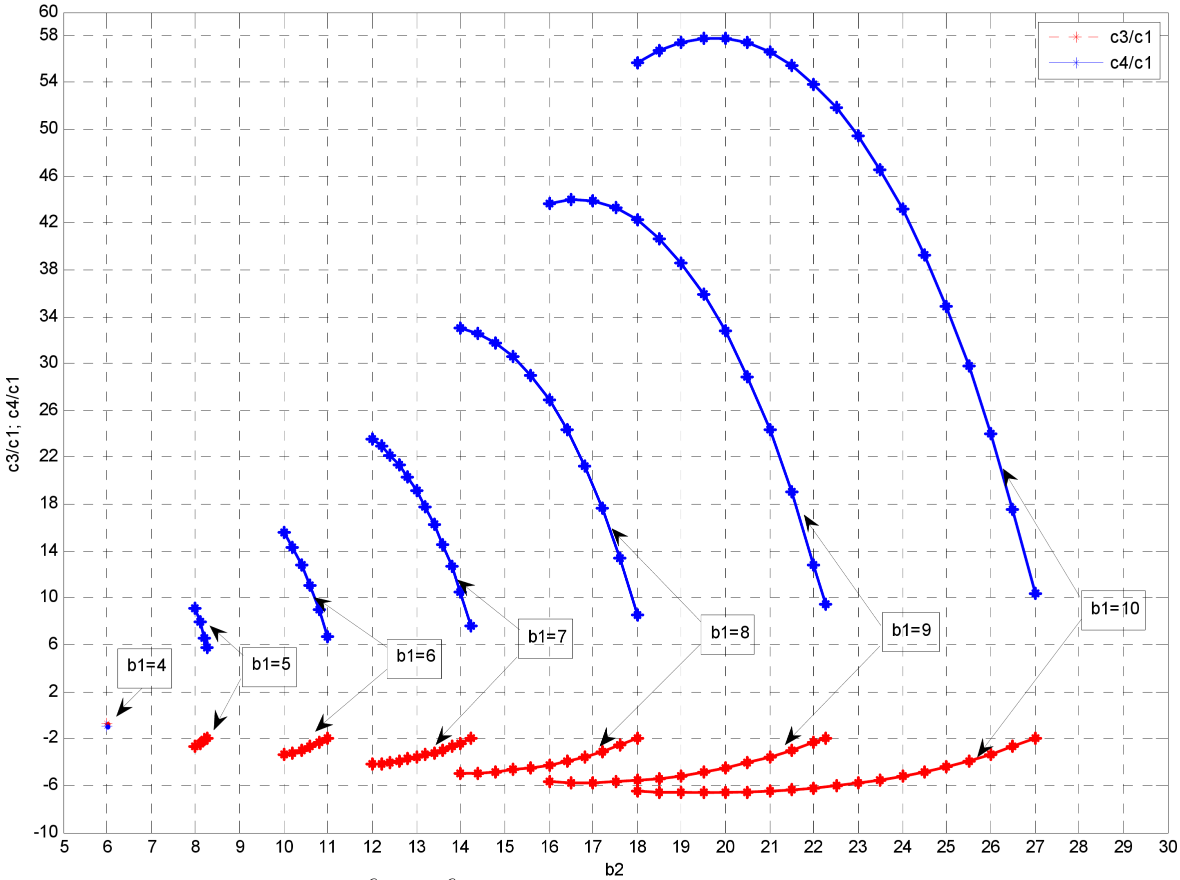

Fig. 2. Calculated values of $\frac{c_3}{c_1}$ and $\frac{c_4}{c_1}$ as a function of b_2 for desired $b_1 = \tau$ (the region of real roots)

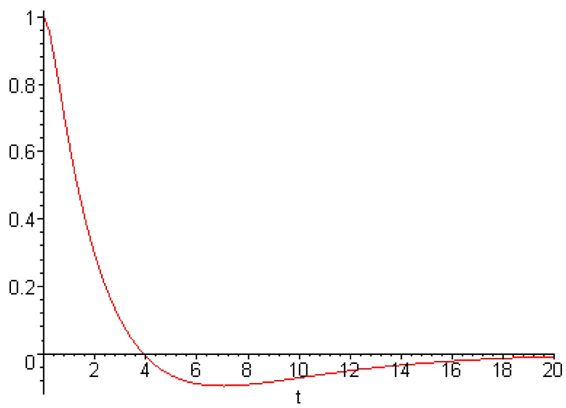

Fig. 3. The response of the system for $b_1 = \tau = 7$, $b_2 = 14$, $c_2 = 0$, $c_1 = 1$ and calculated c_3 and c_4 (the region of real roots)

Table 2
Calculated values of $\frac{c_3}{c_1}$, $\frac{c_4}{c_1}$ and extremal value $\frac{z_e}{c_1}$ as a function of b_2, for desired $b_1 = \tau$ (the region of real roots)

	b_2	c_3/c_1	c_4/c_1	z_e/c_1
$b_1 = 4$, $c_2 = 0$	6	−0.679245283	−0.9433962264	−0.2484706482
	b_2	c_3/c_1	c_4/c_1	z_e/c_1
$b_1 = 5$, $c_2 = 0$	8	−2.617624782	9.135173888	−0.03364848487
	8.1	−2.389863918	7.905265195	−0.05388612399
	8.2	−2.136979825	6.539691016	−0.07584527425
	8.25	−1.999999956	5.799999839	−0.08755429817
	b_2	c_3/c_1	c_4/c_1	z_e/c_1
$b_1 = 6$, $c_2 = 0$	10	−3.402404662	15.54856285	−0.01697600359
	10.2	−3.202966053	14.28545166	−0.03059548085
	10.4	−2.967764948	12.79584467	−0.04579562809
	10.6	−2.692469396	11.05230621	−0.06279719315
	10.8	−2.371936048	9.022261736	−0.08185799389
	11	−2.000000044	6.666666995	−0.1032832451
	b_2	c_3/c_1	c_4/c_1	z_e/c_1
$b_1 = 7$, $c_2 = 0$	12	−4.196680909	23.57581808	−0.008374700879
	12.2	−4.112881804	22.96528172	−0.01406499475
	12.4	−4.010758671	22.22124176	−0.02031697029
	12.6	−3.889626399	21.33797805	−0.02716253005
	12.8	−3.74826571	20.30879306	−0.03463748129
	13	−3.585910017	19.12591582	−0.04278209987
	13.2	−3.401229646	17.78038747	−0.05164173296
	13.4	−3.192812588	16.26192029	−0.06126758383
	13.6	−2.959042807	14.55874049	−0.07171744313
	13.8	−2.698071966	12.65738158	−0.08305677028
	14	−2.407788289	10.54245749	−0.0953597931
	14.25	−2.000000014	7.57142857	−0.112223357
	b_2	c_3/c_1	c_4/c_1	z_e/c_1
$b_1 = 8$, $c_2 = 0$	14	−4.968478984	32.98995162	−0.004098064005
	14.4	−4.918750972	32.57969552	−0.009603254458
	14.8	−4.82170042	31.77902846	−0.01618774958
	15.2	−4.676045858	30.57737832	−0.02389987752
	15.6	−4.479653474	28.95714115	−0.0328066236
	16	−4.229539518	26.893701	−0.04299445956
	16.4	−3.921827338	24.3550756	−0.05457104061
	16.8	−3.5516614	21.3012065	−0.06766787904
	17.2	−3.113073026	17.68285252	−0.08244401846
	17.6	−2.598793318	13.44004491	−0.0990911605
	18	−1.999999995	8.49999981	−0.117840256
	b_2	c_3/c_1	c_4/c_1	z_e/c_1
$b_1 = 9$, $c_2 = 0$	16	−5.713414685	43.69037987	−0.001978385241
	16.5	−5.750949645	44.03653558	−0.005380995691
	17	−5.737717664	43.91450736	−0.009677120366
	17.5	−5.673621184	43.32339540	−0.01488141079
	18	−5.557659340	42.25396948	−0.02102410706
	18.5	−5.388022188	40.68953792	−0.02814984718
	19	−5.162122091	38.606237	−0.03631751054
	19.5	−4.87658356	35.97293733	−0.04560073231
	20	−4.527191754	32.75076841	−0.05608919669
	20.5	−4.108804888	28.89231179	−0.06789049313
	21	−3.615228095	24.34043678	−0.081328436
	21.5	−3.039042137	19.02672209	−0.0959685279
	22	−2.371376361	12.86936036	−0.1125786369
	22.25	−1.999999955	9.44443995	−0.1216150122
	b_2	c_3/c_1	c_4/c_1	z_e/c_1
$b_1 = 10$, $c_2 = 0$	18	−6.434525218	55.63215725	−0.00093787534
	18.5	−6.539897052	56.70694988	−0.00260483763
	19	−6.609580769	57.41772380	−0.00477255357
	19.5	−6.643686286	57.76560014	−0.00744384137
	20	−6.641995790	57.74835703	−0.01062633021
	20.5	−6.603987445	57.36067192	−0.01433211262
	21	−6.52884935	56.59426338	−0.01857762093
	21.5	−6.415485545	55.43795259	−0.02338353566
	22	−6.262511818	53.87762056	−0.02877490912
	22.5	−6.068245005	51.89609906	−0.03478136144
	23	−5.830684720	49.47298414	−0.04143740995
	23.5	−5.547491238	46.58441064	−0.04878285211
	24	−5.215950789	43.20269810	−0.05686333165
	24.5	−4.832936369	39.29595106	−0.06573101696
	25	−4.394857127	34.82754296	−0.07544539723
	25.5	−3.897596923	29.75548874	−0.0860743367
	26	−3.336439698	24.03168512	−0.0976952159
	26.5	−2.705977542	17.60097119	−0.1103964981
	27	−2.0	10.40000016	−0.1242794964

Similarly in the Table 3 the relations for the region of the two real roots and one pair of complex-conjugate roots is presented. This is illustrated in Fig. 4 and the representative example is shown in Fig. 5.

Table 3
Calculated values of $\frac{c_3}{c_1}$, $\frac{c_4}{c_1}$ and extremal value $\frac{z_e}{c_1}$ as a function of b_2, for desired $b_1 = \tau$ (the region of two real roots and one pair of complex-conjugate roots)

	b_2	c_3/c_1	c_4/c_1	z_e/c_1
$b_1 = 7$, $c_2 = 0$	2.1	6.53609094	−54.6200911	1.434661654
	3	6.400283046	−53.63063362	0.893003676
	4	5.358903921	−46.04344286	0.4607888096
	5	3.493249467	−32.45081754	0.1998854097
	6	1.281385317	−16.33580732	0.07486266353
	7	−0.7637276192	−1.435698771	0.03153792368
	8	−2.365927912	10.23747478	0.02383135094
	9	−3.471488539	18.29227364	0.02495119003
	10	−4.121338028	23.02689134	0.02281378
	11	−4.360156085	24.76685146	0.0125558468
	b_2	c_3/c_1	c_4/c_1	z_e/c_1
$b_1 = 9$, $c_2 = 0$	2.1	−4.658365569	33.96048246	−0.8668081094
	3	−4.694338046	34.29222867	−0.5268554644
	4	−4.984904242	36.97189467	−0.2895477576
	5	−5.559588017	42.27175616	−0.1430927554
	6	−6.493059670	50.88043917	−0.05002614151
	7	−8.029209728	65.04715644	0.01558886642
	8	−11.18812141	94.17934182	0.08142263713
	9	−26.81749100	238.3168614	0.2999700352
	10	10.67225062	−107.4218668	−0.1928873688
	11	−0.1987488811	−7.167093665	−0.04900808771
	12	−2.900073295	17.74512038	−0.01594925495
	13	−4.226415295	29.97694102	−0.00271072203
	14	−5.011522285	37.21737220	0.00224000722
	15	−5.480719588	41.54441396	0.0021101149

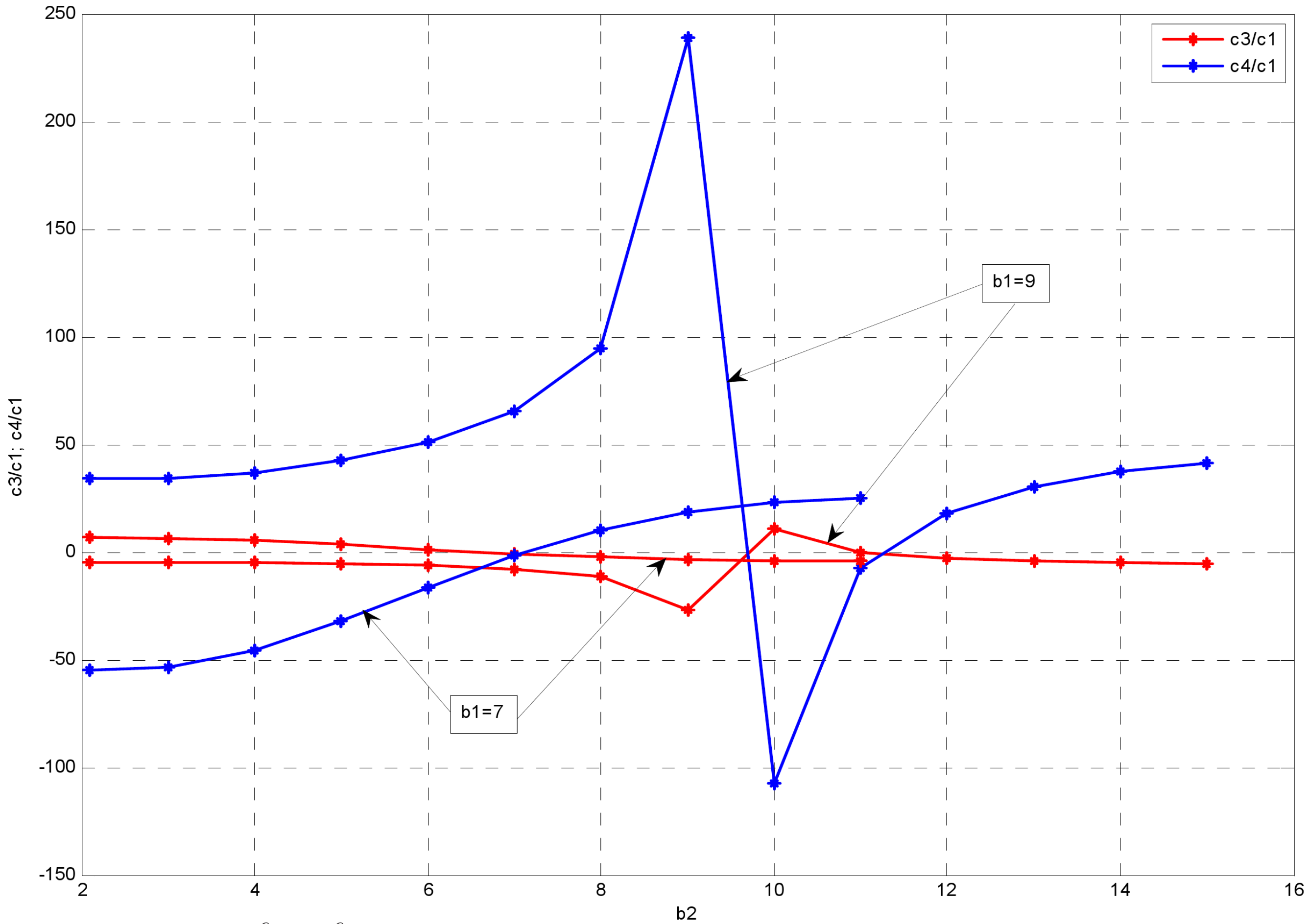

Fig. 4. Calculated values of $\frac{c_3}{c_1}$ and $\frac{c_4}{c_1}$ as a function of b_2 for desired $b_1 = \tau$ (the region of two real roots and one pair of complex-conjugate roots)

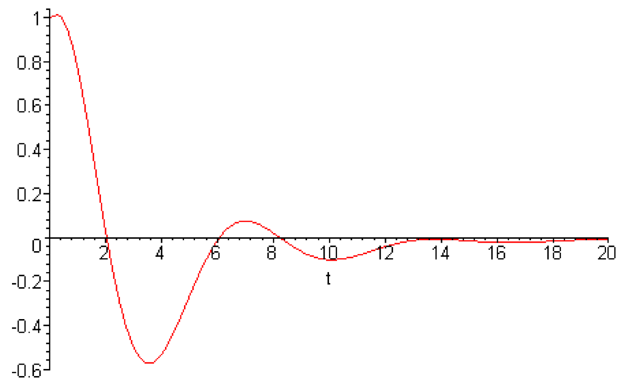

Fig. 5. The response of the system for $b_1 = \tau = 7$, $b_2 = 6$, $c_2 = 0$, $c_1 = 1$ and calculated c_3 and c_4 (the region of two real roots and one pair of complex-conjugate roots)

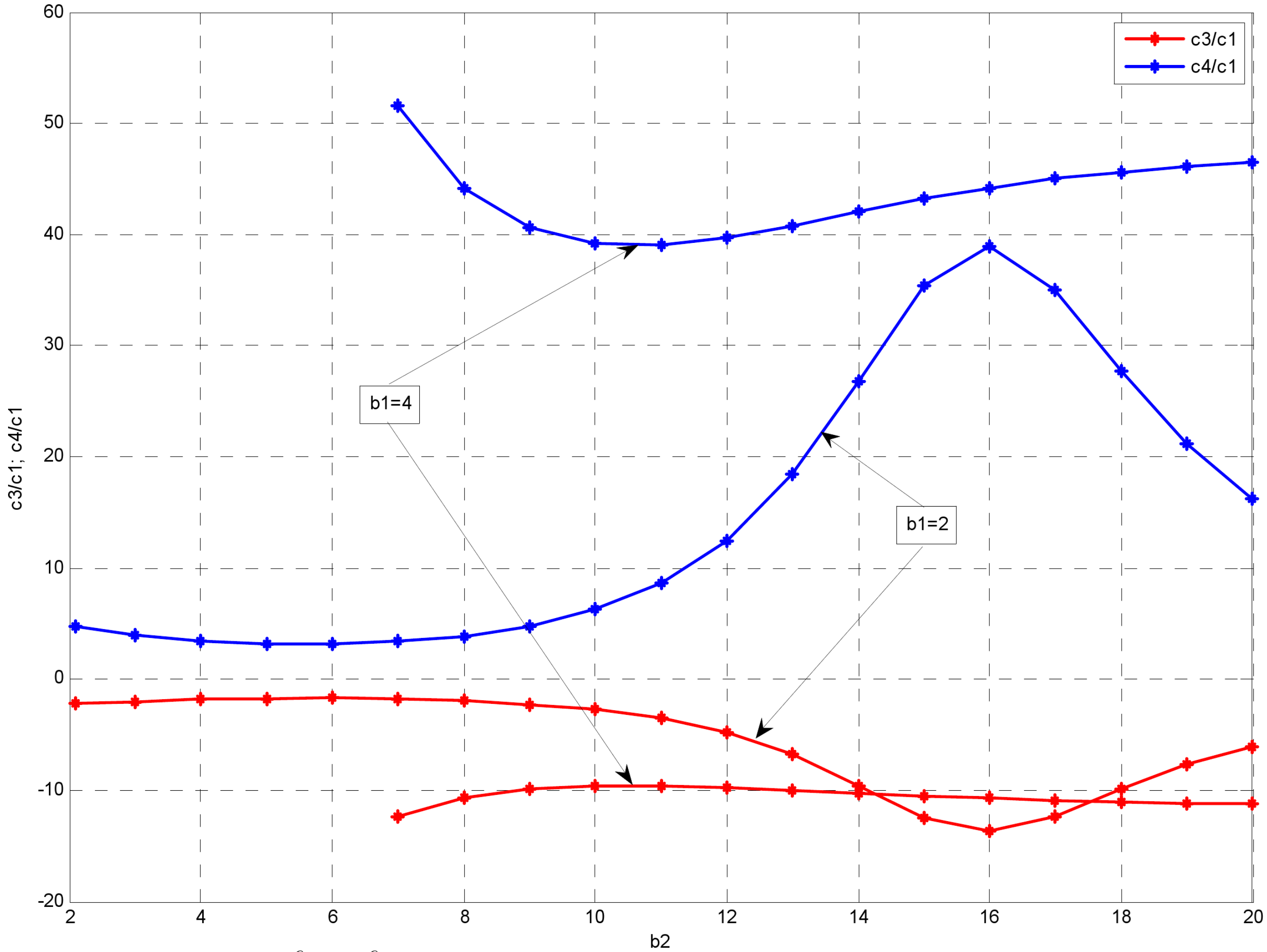

Fig. 6. Calculated values of $\frac{c_3}{c_1}$ and $\frac{c_4}{c_1}$ as a function of b_2 for desired $b_1 = \tau$ (the region of two pairs of complex-conjugate roots)

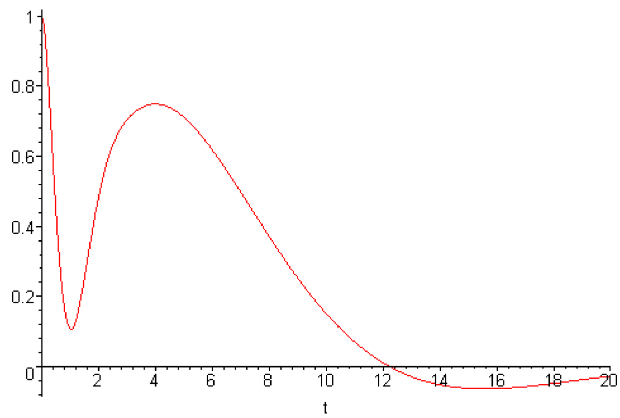

Fig. 7. The response of the system for $b_1 = \tau = 4$, $b_2 = 10$, $c_2 = 0$, $c_1 = 1$ and calculated c_3 and c_4 (the region of two pairs of complex-conjugate roots)

Finally, in Table 4 the relations are presented for the region of two pairs of complex-conjugate roots, which is illustrated in Fig. 6, and the representative example is shown in Fig. 7.

Table 4
Calculated values of $\frac{c_3}{c_1}$, $\frac{c_4}{c_1}$ and extremal value $\frac{z_e}{c_1}$ as a function of b_2, for desired $b_1 = \tau$ (the region of two pairs of complex-conjugate roots)

	b_2	c_3/c_1	c_4/c_1	z_e/c_1
$b_1 = 2$, $c_2 = 0$	2.1	−2.248809837	4.74642951	0.1306296305
	3	−2	4	0.268705265
	4	−1.826567898	3.479703694	0.3866830853
	5	−1.737303398	3.212210193	0.4812627463
	6	−1.724068127	3.172204378	0.5613624496
	7	−1.790842614	3.372527841	0.6331581293
	8	−1.955172845	3.865518534	0.701648558
	9	−2.252290764	4.756872292	0.7716972316
	10	−2.745134664	6.235403992	0.8489488114
	11	−3.541401788	8.624205358	0.9407498264
	12	−4.814787823	12.44436346	1.05659048
	13	−6.79547724	18.38643172	1.205240265
	14	−9.57534339	26.72603018	1.379285011
	15	−12.45804420	35.37413261	1.51897341
	16	−13.63492921	38.90478765	1.521078675
	17	−12.31576928	34.94730784	1.382102952
	18	−9.90443416	27.71330248	1.213283112
	19	−7.704187413	21.11256225	1.083726842
	20	−6.056868333	16.17060500	0.999142181
	b_2	c_3/c_1	c_4/c_1	z_e/c_1
$b_1 = 4$, $c_2 = 0$	7	−12.35381503	51.59216757	1.274878821
	8	−10.69818549	44.14183468	0.9634778725
	9	−9.912760664	40.60742298	0.8166015685
	10	−9.599016296	39.19557334	0.7489064027
	11	−9.569617962	39.06328083	0.7227051559
	12	−9.713497686	39.71073957	0.7184386495
	13	−9.95167804	40.78255117	0.7247878465
	14	−10.22353183	42.00589328	0.7349235474
	15	−10.48493376	43.18220196	0.7449514457
	16	−10.70956552	44.19304482	0.7530939317
	17	−10.8884772	44.99814739	0.7590375048
	18	−11.02638808	45.61874638	0.7633093813
	19	−11.13593097	46.11168936	0.7667443072
	20	−11.2319728	46.54387757	0.770119612

5. Practical example

In Fig. 8, there is shown a simple model of the suspension of the car (one wheel) [5].

The state matrix A is equal to

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1 + k_2}{m} & \frac{-d_2}{m} & \frac{k_2}{m} & \frac{d_2}{m} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{M} & \frac{d_2}{M} & \frac{-k_2}{M} & \frac{-d_2}{M} \end{bmatrix}. \tag{77}$$

The state dynamics is represented by the differential equation

$$\frac{dx(t)}{dt} = Ax(t) \tag{78}$$

with initial conditions $x(0) = c_1$, $x^{(1)}(0) = c_2$, $x^{(2)}(0) = c_3$, $x^{(3)}(0) = c_4$ where

$$x = \left[x, x^{(1)}, x^{(2)}, x^{(3)}\right]^T. \tag{79}$$

The characteristic equation is equal

$$|sI - A| = 0 \tag{80}$$

which after calculation of the determinant (80) is

$$s^4 + s^3\left(\frac{d_2}{M} + \frac{d_2}{m}\right) + s^2\left(\frac{k_2}{M} + \frac{k_1 + k_2}{m}\right) + s\left(\frac{k_1 d_2}{mM}\right) + \frac{k_1 k_2}{mM} = 0, \tag{81}$$

where k_1 – is the elasticity coefficient of the tire, k_2 – is the coefficient of spring carriage, d_2 – is the attenuation coefficient, m – mass of the wheel, M – mass of the car.

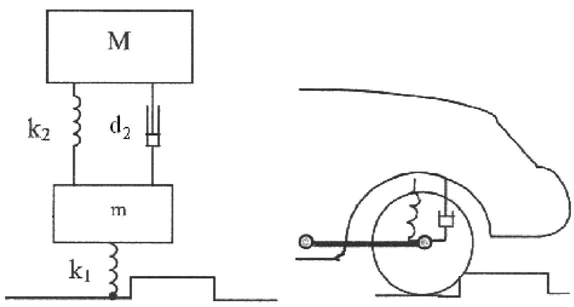

Fig. 8. Model of the suspension system

We want to choose the coefficients k_1, k_2, d_2, m and M. Putting

$$s = \sqrt[4]{\frac{k_1 k_2}{mM}} z \tag{82}$$

and

$$\frac{k_1 d_2}{mM} = \left(\frac{d_2}{m} + \frac{d_2}{M}\right)\sqrt{\frac{k_1 k_2}{mM}} \tag{83}$$

we obtain the symmetric equation

$$z^4 + b_1 z^3 + b_2 z^2 + b_1 z + 1 = 0, \tag{84}$$

where

$$b_1 = \frac{\frac{k_1 d_2}{mM}}{\sqrt[4]{\left(\frac{k_1 k_2}{mM}\right)^3}} = \tau, \tag{85}$$

$$b_2 = \frac{\frac{k_2}{M} + \frac{k_1 + k_2}{m}}{\sqrt{\left(\frac{k_1 k_2}{mM}\right)}}. \tag{86}$$

For determination of the optimal values of the parameters k_1, k_2, d_2, m and M we have the following relations (45), (75), (76), (83), (85), (86).

In particular from the relation (83) we have

$$k_1 = \frac{(m + M)^2}{mM} k_2. \tag{87}$$

From (86) using (87) we obtain

$$b_2 = \frac{2m + M}{m}. \tag{88}$$

Similarly we get

$$b_1 = d_2\sqrt{\frac{m + M}{k_2 m M}} \tag{89}$$

or

$$\frac{\tau}{d_2} = \sqrt{\frac{m + M}{k_2 m M}} \tag{90}$$

and finally

$$k_2 = \left(\frac{d_2}{\tau}\right)^2 \frac{m + M}{mM}. \tag{91}$$

Assuming $\frac{\tau}{d_2}$ we can calculate k_2 and then, from the relation (87), the coefficient k_1.

In general the problem of the location poles and zeroes is in [6].

6. Conclusions

Using the method of the symmetrical equations, analytical results are obtained. In particular, all the possible cases of the different roots and the extremal time τ and the extremal value of $x(\tau)$ for the differential equation of the 4-th order have been considered. The extension to the equations of higher order can be obtained immediately as shown in the paper.

Remark 1

It is also possible to enlarge the formula (22) on the system with time-delay using the method described in [7–10].

Let us consider a differential equation with time delay $h > 0$.

We assume that the observable and controllable conditions are fulfilled [8].

$$ax(t) + bx^{(1)}(t) + x(t - h) = 0. \tag{R1}$$

With the points initial conditions

$$\left.\begin{aligned} x(0) &= c_1 \\ x^{(1)}(0) &= c_2 \\ x(t-h) &= 0 \quad \text{for} \quad t < h \end{aligned}\right\} \tag{R2}$$

and a, b – constant parameters.

The characteristic equation of (R1) is

$$F(s) = a + bs + e^{-sh} = 0. \tag{R3}$$

After premultiplying by e^{sh} it is evident that the main term exist and is equal bse^{sh}.

In consequence the necessary condition is fulfilled.

We apply the Theorem 3 proved in [7].

The relation between coefficients and the roots of the quasipolynomial equations of the type (R1) is given by the following formula:

$$\sum_{k=1}^{\infty} \frac{1}{s_k} = \frac{1}{2}\left[\frac{F^{(1)}(s)}{F(s)}\right]_{s=-\infty} - \left[\frac{F^{(1)}(s)}{F(s)}\right]_{s=0}. \tag{R4}$$

We calculate first derivative with respect to s

$$F^{(1)}(s) = b - he^{-sh}. \tag{R5}$$

We have that

$$\left[\frac{F^{(1)}(s)}{F(s)}\right]_{s=0} = \frac{b - h}{a + 1} \tag{R6}$$

and

$$\frac{1}{2}\left[\frac{F^{(1)}(s)}{F(s)}\right]_{s=-\infty} = \frac{1}{2}\frac{b - he^{-sh}}{a + bs + e^{-sh}}\bigg|_{s=-\infty} = -\frac{h}{2}. \tag{R7}$$

Finally

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{s_k} &= \frac{1}{2}\left[\frac{F^{(1)}(s)}{F(s)}\right]_{s=-\infty} - \left[\frac{F^{(1)}(s)}{F(s)}\right]_{s=0} \\ &= -\frac{h}{2} - \frac{b-h}{a+1}. \end{aligned} \tag{R8}$$

The formula (R8) represents generalization the formula (22) in the case of the infinite number of the roots.

Remark 2

Investigation of the extremal time τ as the function of the initial conditions were presented in [11]. A solution of the problem of the extremal $\tau(s)$ in the case of one multiple roots may be found in [12].

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