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ANALYTICAL SOLUTION FOR CERTAIN NONLINEAR ELECTROMAGNETIC FIELD PROBLEMS

The paper presents a nonlinear electromagnetic field problem with cylindrical symmetry. A time-sinusoidal boundary condition is applied, which yields non-sinusoidal steady-state fields. The problem is brought forth to test an analytical-numerical method based on the method of small parameter. A distinctive feature of this method is that it requires symbolic calculations to be performed. These allow to reduce the problem to a system of nonlinear equations. In order to check the accuracy of the solution, several error criteria are introduced. Moreover, the power balance of a nonlinear conductive region is verified.

1. INTRODUCTION

The authors' work deals with electromagnetic field problems with components described by periodic time functions. A method was developed which deals with calculations of nonlinear conductive media. It is based on the small parameter expansion [1]. The need for symbolic calculations is a distinctive feature of the method and supplements its numerical part. For this purpose, an implementation of a specialized class for symbolic calculations was performed in C++. A series of papers focused on the advances of the method, which now includes higher harmonic terms and relatively good accuracy, which is proven in point 4 titled "Results and Error Calculations".

In the subsequent points: first, an exemplary problem is formulated so that the method can be explained by example. Secondly, a procedure is explained how to obtain the solution. Finally, the exemplary problem is solved and a thorough error analysis is performed.

2. PROBLEM FORMULATION

A fragment of a superconducting cable is presented (Fig. 1), which is a two-layer structure: a linear conductive region covered by a superconductor.

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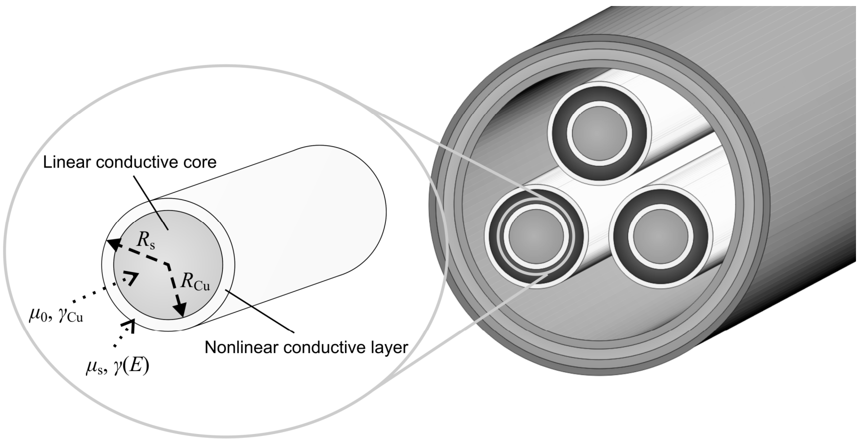

Fig. 1. Superconducting cable core (on the right: a three phase superconducting cable presented in [2])

The nonlinear conductivity of the superconductor can be described by a nonlinear relationship between current density and electric field strength [3, 4]:

$$J(E) = \sum_{k=1,3,5...}^{m} \gamma_k E^k. \tag{1}$$

It is assumed that the core structure carries a total current of I_c. An angularly independent and longitudinally uniform field (omitting exterior fields and edge effects) is formed which can be described by single E and H components. The field strength components are subject to the magnetic vector potential and its radial derivative (derived in a similar problem in [5]):

$$E_z(t,r) = E(t,r) = -\frac{\partial A(t,r)}{\partial t}, \tag{2}$$

$$H_\varphi(t,r) = H(t,r) = -\nu_s \frac{\partial A(t,r)}{\partial r}. \tag{3}$$

The total current constitutes the Neumann boundary condition:

$$H(t,R_s) = I_c(t)/(2\pi R_s), \tag{4}$$

which finalizes the formulation of an exemplary problem for the analytical-numerical method.

3. SOLUTION BY ANALTYICAL-NUMERICAL METHOD

The linear region electromagnetic field is determined by a linear differential equation with a known solution and therefore its verification is not presented in this paper.

For problems with a nonlinear conductivity (where the magnetic vector potential is described by a single coordinate component) the wave equation is:

$$\nabla^2 A = \mu_s \gamma(E) \frac{\partial A}{\partial t}. \tag{5}$$

In order to begin the solution procedure, the small parameter expansion is used:

$$A = \sum_{i=1}^{n} \kappa^{i-1} A_i. \tag{6}$$

The differential equation (5), subject to a nonlinear conductivity (1) is divided into linear differential equations. For the analyzed problem, taking into account the (2) relationship, these are:

$$\nabla^2 A_i - \mu_s \gamma_1 \frac{\partial A_i}{\partial t} = f_i\left(\frac{\partial A_1}{\partial t}, \ldots, \frac{\partial A_{i-1}}{\partial t}\right), \tag{7}$$

where the right-hand side, which contains the nonlinear function f_i is evaluated in the following way:

- the κ parameter is chosen arbitrarily (e.g. $\mu_s \gamma_m$) and γ_k for $k < m$ are associated with κ so that the right-hand side of (7) is multiplied by κ,
- the right-hand side of (5) is expanded subject to (6) and (2),
- terms on both sides of the differential equation are compared, the ones multiplied by the same power of the κ parameter are placed into respective (7) equations with the same index i.

This method leaves residues of higher powers of κ in the differential equation hence it is expected that a more accurate result will be achieved for more n in (6) taken into account (which point 4 of this paper attempts to prove).

The (7) equations can be solved one after the other starting from i = 1 as the right-hand side always contains magnetic vector potential terms of smaller i. The solutions of equations (7) for single harmonic terms are presented in a previous paper [3].

The solution of (5) contains a number of unknowns that can be reduced to the number of non-zero harmonics: this can be obtained by taking into account the continuity of E and H on $r = R_{Cu}$ (derived in [3]). These unknowns determine the magnetic vector potential distribution.

Hence, the boundary condition values of each time harmonic have a nonlinear relationship with the unknowns. In the simplest way, they can be obtained by a shooting method, where the nonlinear system is evaluated each time the coefficients are guessed. However, the authors of this article have constructed a different method in which the relationship between the boundary condition and the unknowns of the field distribution is evaluated in a symbolic form and a system of nonlinear equations is formulated. For a Neumann boundary condition, this gives equations of the form:

$$\frac{\mathrm{d}\underline{A}_h(R_\mathrm{s})}{\mathrm{d}r} = -\mu_\mathrm{s}\underline{H}_h(R_\mathrm{s}) = -\mu_\mathrm{s}\underline{\lambda}_h, \tag{8}$$

relating to single non-zero time harmonics. The system of nonlinear equations can be solved by a numerical method. The authors have chosen a script using Newton's method.

4. RESULTS AND ERROR CALCULATIONS

According to the described method, the Neumann boundary condition (4), constitutes nonlinear equations of the (8) form. In this point, an electromagnetic field distribution is determined for a Neumann boundary condition:

$$\underline{\lambda}_h = \begin{cases} h = 1 & \rightarrow \quad 1.7\cdot 10^3 \text{ A/m}, \\ h > 1 & \rightarrow \quad 0. \end{cases} \tag{9}$$

The resulting electromagnetic field distribution (for $n = 10$) described by electric field strength, magnetic field strength and current density are presented on Fig. 2.

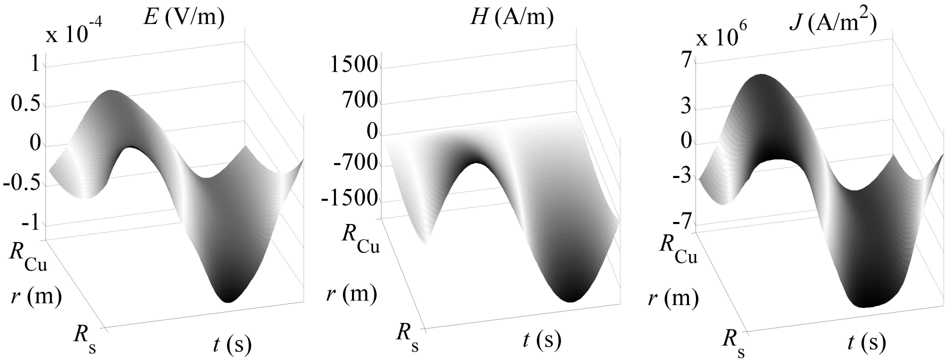

Fig. 2. Distribution of electromagnetic field components in the superconducting layer displayed as time functions (from the left respectively: electric field strength, magnetic field strength, current density) for an imposed Neumann boundary condition

In order to verify the nonlinear problem solution, the authors use a number of differential equation errors. These include errors for amplitude and phase of time harmonics: an integral error (describing a total error in a region) and a maximum error. The formulae for these errors are presented in Table 1. An auxiliary expression can be used to calculate the errors – it is a ratio of the complex values of chosen right- and left-hand sides of the differential equation (5), for harmonic h:

$$\underline{w}_h(r) = \frac{-\mu_\mathrm{s}\underline{J}_h(r)}{\dfrac{\mathrm{d}^2\underline{A}_h(r)}{\mathrm{d}r^2} + \dfrac{1}{r}\dfrac{\mathrm{d}\underline{A}_h(r)}{\mathrm{d}r}}, \tag{10}$$

Table 1. Defined errors used to ascertain the accuracy of the solution in the nonlinear region

Differential equation errors (helpful for calculating integral and maximum errors)	
$e_{\mathrm{amp.}\,h}(r) = \left\|1 - \left\|\underline{w}_h(r)\right\|\right\| \cdot 100\%,$	$e_{\mathrm{ph.}\,h}(r) = \left\|\frac{1}{\pi}\arg(\underline{w}(r))\right\| \cdot 100\%,$
Integral errors	
$e_{\int \mathrm{amp.}\,h} = \frac{1}{R_\mathrm{s} - R_\mathrm{Cu}} \int_{R_\mathrm{Cu}}^{R_\mathrm{s}} e_{\mathrm{amp.}\,h}(r)\,\mathrm{d}r,$	$e_{\int \mathrm{ph.}\,h} = \frac{1}{R_\mathrm{s} - R_\mathrm{Cu}} \int_{R_\mathrm{Cu}}^{R_\mathrm{s}} e_{\mathrm{ph.}\,h}(r)\,\mathrm{d}r,$
Maximum errors	
$e_{\max\,\mathrm{amp.}\,h} = \max\limits_{r \in [R_\mathrm{Cu}, R_\mathrm{s}]} e_{\mathrm{amp.}\,h}(r),$	$e_{\max\,\mathrm{ph.}\,h} = \max\limits_{r \in [R_\mathrm{Cu}, R_\mathrm{s}]} e_{\mathrm{ph.}\,h}(r).$

The evaluation of these errors for time harmonics up to $h = 9$ are presented on bar graphs in Figures 3 and 4.

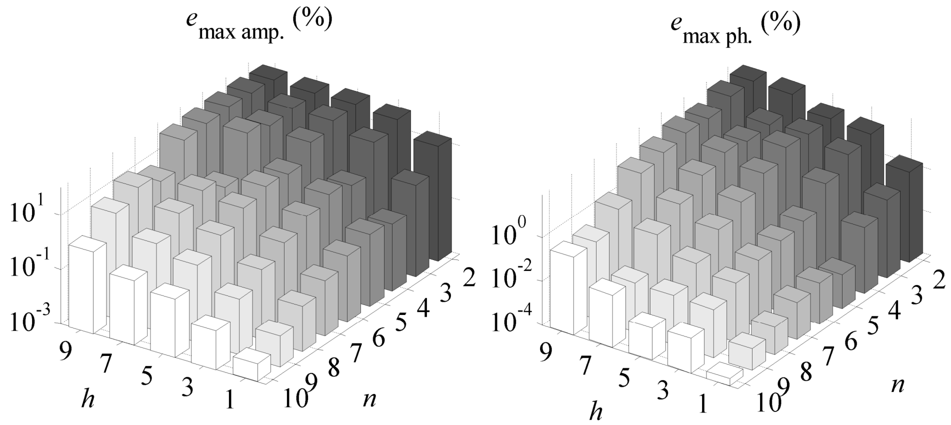

Fig. 3. Nonlinear region calculations' maximum errors for various n (on the left: amplitude maximum error, on the right: phase maximum error) and each evaluated time-harmonic

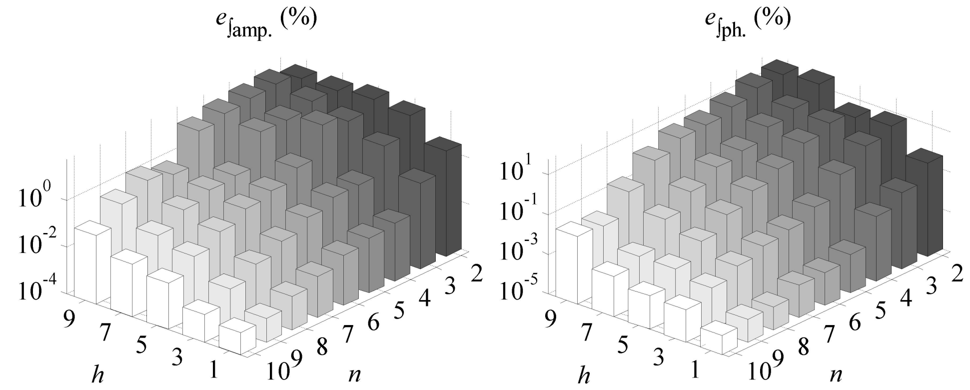

Fig. 4. Nonlinear region calculations' integral errors for various n (on the left: amplitude integral error, on the right: phase integral error) and each evaluated time-harmonic

Additionally, an important criterion is that a sufficient number of harmonic terms be included in the solution (first harmonic errors might be small although a potentially omitted third harmonic amplitude might be significant in relation to the first). To verify that the solution contains sufficient amount of harmonic terms, a residual error is calculated, which relates to the highest harmonic amplitude of the residue of J evaluated in (10). It is defined as a ratio according to the dominant harmonic amplitude as $J_{\mathrm{A}h}/J_{\mathrm{A}d}$ (in this case $d=1$).

Harmonic terms up to $h=9$ were used to obtain the solution presented at the beginning of this point, which is why the residual error is evaluated in the following way (for $n=10$):

$$\max_{r\in[R_{\mathrm{Cu}},R_{\mathrm{s}}]} \frac{J_{\mathrm{A}11}(r)}{J_{\mathrm{A}1}(r)} = 1.36\cdot 10^{-2}\,\%. \tag{11}$$

Such a low value of the remainder error proves that a sufficient amount of harmonic terms have been taken into account in the solution.

The power balance in the nonlinear region can be verified by evaluating the instantaneous power according to the Poynting method:

$$p_{\mathrm{P}}(t) = -\oint\!\!\oint_{\Omega} \left(\vec{E}(t,\mathbf{X}_{\Omega})\times\vec{H}(t,\mathbf{X}_{\Omega})\right)\mathrm{d}\vec{\Omega}, \tag{12}$$

and using volume integrals:

$$\begin{aligned} p_{V}(t) &= \iiint_{V} \left(\vec{J}(t,\mathbf{X}_{V})\cdot\vec{E}(t,\mathbf{X}_{V})\right)\mathrm{d}V + \\ &+ \frac{1}{2}\frac{\partial}{\partial t}\left(\iiint_{V}\mu_{\mathrm{s}}\vec{H}(t,\mathbf{X}_{V})^{2}\,\mathrm{d}V + \iiint_{V}\varepsilon_{\mathrm{s}}\vec{E}(t,\mathbf{X}_{V})^{2}\,\mathrm{d}V\right). \end{aligned} \tag{13}$$

The field components in the superconducting cable core are described by a series of odd harmonic terms. The instantaneous power can be therefore expressed as a series of even harmonics and a constant term. Hence, the power calculated by Poynting method is expressed as a Fourier series:

$$p_{\mathrm{P}}(t) = p_{\mathrm{P}0} + \sum_{h=2,4,6...}^{h_{p\max}} p_{\mathrm{P}h} \cos\left(h\omega_0 t + \phi_{\mathrm{P}h}\right), \tag{14}$$

and so is the instantaneous power calculated by volume integrals:

$$p_V(t) = p_{V0} + \sum_{h=2,4,6...}^{h_{p\max}} p_{Vh} \cos\left(h\omega_0 t + \phi_{Vh}\right). \tag{15}$$

In order to verify the balance of certain terms of instantaneous power, a relative error is defined:

$$e_{ph} = \left|1 - \frac{p_{Vh}}{p_{\mathrm{P}h}}\right| \cdot 100\%. \tag{16}$$

This error was evaluated up to h = 8 with various numbers of magnetic vector potential terms taken into account (Fig. 5).

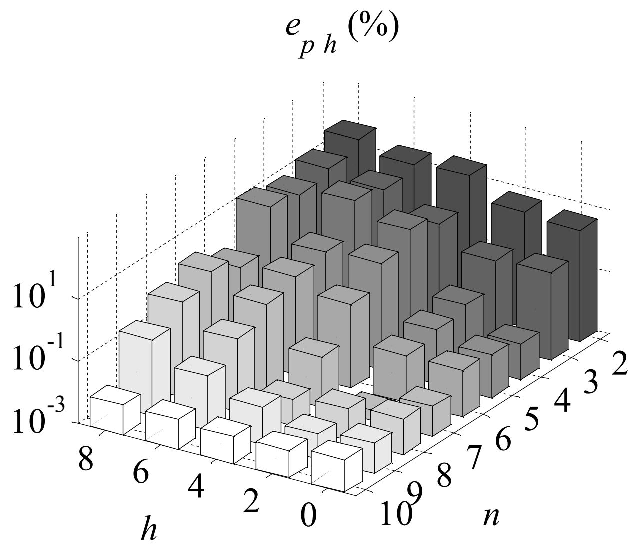

Fig. 5. Nonlinear region instantaneous power errors for various n and even harmonics up to h = 8

A tendency of the error decreasing for a greater amount of magnetic vector potential terms taken into account can be noticed once again. Relatively low error values can be observed in Figures 3, 4 and 5, which prove that a very accurate solution has been obtained.

5. CONCLUSIONS

A nonlinear problem of the electromagnetic field was formulated. The problem comprises of a fragment of a superconducting cable core.

A nonlinear differential equation was solved with the use of an analytical-numerical method with symbolic calculations. The method includes the expansion of the magnetic vector potential (6). A Neumann boundary condition was formulated and the electromagnetic field distribution was determined.

Furthermore a thorough error analysis was performed. Several differential equation errors were defined and evaluated for the obtained solution. Additionally, the balance of instantaneous power was verified.

A substantial amount of error evaluations for various numbers of magnetic vector potential terms clearly exhibits a precision improvement with an increase of n in (6). For $n = 10$ differential equation errors for harmonics up to 9 take values less than 1% (some even about 10^{-4} %). Additionally, the power balance errors calculated up to $h = 8$, reach values less than 10^{-2} % for $n = 10$.

The model solution obtained may now be used in the authors' future papers. It is planned that other analytical-numerical methods will be presented to solve problems with nonlinear conductivity. Then, the solutions obtained thereby can be compared with the one presented in this paper.

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