

A Viscoelastic Frictionless Contact Problem with Adhesion

by

Arezki TOUZALINE

Presented by Bogdan BOJARSKI

Summary. We consider a mathematical model which describes the equilibrium between a viscoelastic body in frictionless contact with an obstacle. The contact is modelled with normal compliance, associated with Signorini's conditions and adhesion. The adhesion is modelled with a surface variable, the bonding field, whose evolution is described by a first-order differential equation. We establish a variational formulation of the mechanical problem and prove the existence and uniqueness of the weak solution. The proof is based on arguments of evolution equations with multivalued maximal monotone operators, differential equations and the Banach fixed point theorem.

1. Introduction. Contact problems involving deformable bodies are quite frequent in industry as well as in daily life and play an important role in structural and mechanical systems. Contact processes involve complicated surface phenomena, and are modelled with highly nonlinear initial boundary value problems. Taking into account various contact conditions associated with more and more complex behavior laws leads to the introduction of new and nonstandard models, expressed with the aid of evolution variational inequalities.

An early attempt to study contact problems within the framework of variational inequalities was made in [13]. Recently, a book [23] appeared that represents a broad insight into the theory of inclusions, hemivariational inequalities, and their applications to contact mechanics. The mathematical, mechanical and numerical state of the art can be found in [28] where we find detailed mathematical and numerical studies of adhesive contact problems.

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Adhesion may take place between parts of contacting surfaces. It may be intentional, when the surfaces are bonded with glue, or unintentional, as a seizure between very clean surfaces. The adhesion contact is modelled by a bonding field on the contact surface, denoted in this paper by β ; it describes the pointwise fractional density of active bonds on the contact surface, and is sometimes referred to as the intensity of adhesion. Following [16, 17], β satisfies $0 \leq \beta \leq 1$; when $\beta = 0$ all the bonds are severed and there are no active bonds, when $\beta = 1$ all the bonds are active; when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. Basic modelling can be found in [16, 18]. Recall that unilateral contact problems involving Signorini's condition with or without adhesion were studied by several authors (see for instance [1–3, 8–12, 14, 20, 21, 25, 26, 28, 29, 32, 33]).

Contact problems for elastic and viscoelastic bodies with adhesion and friction appear in many applications of solid mathematics such as the fiber-matrix interface of composite materials. A consistent model coupling unilateral contact, adhesion and friction was proposed in [25]. Several models for dynamic or quasistatic processes of adhesive contact between a deformable body and an obstacle have been studied in [5–7, 9, 10, 15, 18, 19, 22, 23, 25, 27, 28, 30–33] and the references therein.

The novelty in all the above papers is the introduction of a surface internal variable, the bonding field β . In [19], an adhesive contact problem for viscoelastic materials with long memory was studied. The authors obtain existence and uniqueness results for abstract inclusions and variational-hemivariational inequalities, which they apply to prove the existence of a unique weak solution to the contact problem.

The aim of this paper is to continue the study of the contact problem begun in [33]. The novelty of this paper is the study of a viscoelastic frictionless contact problem with unilateral constraint and adhesion. Recall that this type of contact was used for the first time in [20]. We establish a variational formulation of the mechanical problem, for which we prove the existence and uniqueness of solution. The proof is based on a general result on evolution equations with multivalued maximal monotone operators and fixed point arguments.

The paper is structured as follows. In Section 2 we list the assumptions on the data, derive the variational formulation and present our main existence and uniqueness result, Theorem 2.1. Its proof is provided in Section 3.

2. Problem statement and variational formulation. We consider a viscoelastic body which occupies a domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) and assume that its boundary Γ is regular and partitioned into three measurable and disjoint parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that $\text{meas}(\Gamma_1) > 0$. The body is acted upon by

a volume force of density f_1 in Ω and a surface traction of density f_2 on Γ_2 . On Γ_3 the body is in adhesive frictionless contact with an obstacle.

The classical formulation of this mechanical problem is as follows.

PROBLEM P_1 . Find a displacement field $u : \Omega \times [0, T] \rightarrow \mathbb{R}^d$ and a bonding field $\beta : \Gamma_3 \times [0, T] \rightarrow [0, 1]$ such that

$$(2.1) \quad \operatorname{div} \sigma(u, \dot{u}) = -f_1 \quad \text{in } \Omega \times (0, T),$$

$$(2.2) \quad \sigma(u, \dot{u}) = \mathcal{A}\varepsilon(\dot{u}) + \mathcal{B}\varepsilon(u) \quad \text{in } \Omega \times (0, T),$$

$$(2.3) \quad u = 0 \quad \text{on } \Gamma_1 \times (0, T),$$

$$(2.4) \quad \sigma\nu = f_2 \quad \text{on } \Gamma_2 \times (0, T),$$

$$(2.5) \quad \left. \begin{aligned} u_\nu \leq g, \quad \sigma_\nu + p(u_\nu) - c_\nu \beta^2 R_\nu(u_\nu) \leq 0 \\ (u_\nu - g)(\sigma_\nu + p(u_\nu) - c_\nu \beta^2 R_\nu(u_\nu)) = 0, \quad \sigma_\tau = 0 \end{aligned} \right\} \text{ on } \Gamma_3 \times (0, T),$$

$$(2.6) \quad \dot{\beta} = -[\beta c_\nu (R_\nu(u_\nu))^2 - \varepsilon_a]_+ \quad \text{on } \Gamma_3 \times (0, T),$$

$$(2.7) \quad u(0) = u_0 \quad \text{in } \Omega,$$

$$(2.8) \quad \beta(0) = \beta_0 \quad \text{on } \Gamma_3.$$

Equation (2.1) represents the equilibrium equation where $\sigma = \sigma(u, \dot{u})$ denotes the stress tensor. Equation (2.2) is the viscoelastic constitutive law of the material in which \mathcal{A} and \mathcal{B} are the viscosity and elasticity operators, respectively, and $\varepsilon(u)$ is the small strain tensor. Here and below, a dot above a variable represents the time derivative.

Relations (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal vector on Γ and $\sigma\nu$ represents the Cauchy stress vector.

Condition (2.5) represents the unilateral and frictionless contact with adhesion on the contact surface Γ_3 where u_ν is the normal displacement, σ_ν is the normal constraint, σ_τ is the tangential constraint and p is a normal compliance function. Here $g \geq 0$ is a maximal value of the penetration of the viscoelastic body in the obstacle (see [20]); R_ν is a truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \leq s \leq 0, \\ 0 & \text{if } s > 0, \end{cases}$$

where $L > 0$ is a characteristic length of the bonds; and the parameter c_ν is an adhesion coefficient.

Equation (2.6) represents the ordinary differential equation which describes the evolution of the bonding field where ε_a is an adhesion coefficient and $[s]_+ = \max(s, 0)$ for $s \in \mathbb{R}$. This equation was already used in several papers (see for example [28]). Since $\dot{\beta} \leq 0$ on $\Gamma_3 \times (0, T)$, once debonding

occurs, bonding cannot be reestablished. Also we wish to make it clear that from [24] it follows that the model does not allow for a complete debonding field in finite time.

Finally, (2.7) and (2.8) represent respectively the initial displacement field and the initial bonding field.

We recall that the inner products and the corresponding norms on \mathbb{R}^d and S_d are given by

$$\begin{aligned} u \cdot v &= u_i v_i, & |v| &= (v \cdot v)^{1/2} \quad \forall u, v \in \mathbb{R}^d, \\ \sigma \cdot \tau &= \sigma_{ij} \tau_{ij}, & |\tau| &= (\tau \cdot \tau)^{1/2} \quad \forall \sigma, \tau \in S_d, \end{aligned}$$

where S_d is the space of second order symmetric tensors on \mathbb{R}^d ($d = 2, 3$). Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted.

Now, to proceed with the variational formulation, we need the following function spaces:

$$\begin{aligned} H &= (L^2(\Omega))^d, & H_1 &= (H^1(\Omega))^d, \\ Q &= \{\sigma = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}, \\ Q_1 &= \{\sigma \in Q : \operatorname{div} \sigma \in H\}. \end{aligned}$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products

$$(u, v)_H = \int_{\Omega} u_i v_i \, dx, \quad (\sigma, \tau)_Q = \int_{\Omega} \sigma_{ij} \tau_{ij} \, dx.$$

The strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \text{where} \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i});$$

$\operatorname{div} \sigma = (\sigma_{ij,j})$ is the divergence of σ . For every $v \in H_1$, we also write v for the trace of v on Γ and we denote by v_ν and v_τ the normal and the tangential components of v on the boundary Γ given by

$$v_\nu = v \cdot \nu, \quad v_\tau = v - v_\nu \nu.$$

Similarly, for a regular function $\sigma \in Q_1$, we define its normal and tangential components by

$$\sigma_\nu = (\sigma \nu) \cdot \nu, \quad \sigma_\tau = \sigma \nu - \sigma_\nu \nu$$

and we recall the Green formula

$$(\sigma, \varepsilon(v))_Q + (\operatorname{div} \sigma, v)_H = \int_{\Gamma} \sigma_\nu \cdot v \, da \quad \forall v \in H_1,$$

where da is the surface measure element.

In the study of Problem P_1 we assume that the viscosity operator \mathcal{A} and the elasticity operator \mathcal{B} satisfy the conditions:

$$(2.9) \quad \left\{ \begin{array}{l} \text{(a) } \mathcal{A} : \Omega \times S_d \rightarrow S_d; \\ \text{(b) } \mathcal{A}(x, \tau) = (a_{ijkl}(x)\tau_{ij}) \text{ for all } \tau \in S_d \text{ and a.e. } x \in \Omega; \\ \text{(c) } a_{ijkl} = a_{klij} = a_{jikl} \in L^\infty(\Omega); \\ \text{(d) there exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \quad a_{ijkl}\tau_{ij}\tau_{kl} \geq m_{\mathcal{A}}|\tau|^2 \text{ for all } \tau = (\tau_{ij}) \in S_d \text{ and a.e. in } \Omega, \end{array} \right.$$

and

$$(2.10) \quad \left\{ \begin{array}{l} \text{(a) } \mathcal{B} : \Omega \times S_d \rightarrow S_d; \\ \text{(b) there exists } L_{\mathcal{B}} > 0 \text{ such that} \\ \quad |\mathcal{B}(x, \varepsilon_1) - \mathcal{B}(x, \varepsilon_2)| \leq L_{\mathcal{B}}|\varepsilon_1 - \varepsilon_2| \\ \quad \text{for all } \varepsilon_1, \varepsilon_2 \text{ in } S_d \text{ and a.e. } x \in \Omega; \\ \text{(c) the mapping } x \mapsto \mathcal{B}(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega, \\ \quad \text{for any } \varepsilon \text{ in } S_d; \\ \text{(d) } x \mapsto \mathcal{B}(x, 0) \text{ is in } Q. \end{array} \right.$$

Let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1 : v = 0 \text{ on } \Gamma_1\}.$$

Since $\text{meas}(\Gamma_1) > 0$ and the viscosity tensor satisfies the assumption (2.9), it follows that V is a real Hilbert space endowed with the inner product

$$(u, v)_V = (\mathcal{A}\varepsilon(u), \varepsilon(v))_Q;$$

let $\|\cdot\|_V$ be the associated norm. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which depends only on the domain Ω , Γ_1 and Γ_3 such that

$$(2.11) \quad \|v\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \|v\|_V \quad \forall v \in V.$$

We suppose that the adhesion coefficients c_ν and ε_a satisfy

$$(2.12) \quad c_\nu \in L^\infty(\Gamma_3), \quad \varepsilon_a \in L^2(\Gamma_3) \quad c_\nu, \varepsilon_a \geq 0 \text{ a.e. on } \Gamma_3.$$

For the Signorini problem, we use the convex subset of admissible displacement fields given by

$$K = \{v \in V : v_\nu \leq g \text{ a.e. on } \Gamma_3\}.$$

We assume that the initial data satisfy

$$(2.13) \quad u_0 \in K,$$

$$(2.14) \quad \beta_0 \in L^2(\Gamma_3), \quad 0 \leq \beta_0 \leq 1 \quad \text{a.e. on } \Gamma_3.$$

Next, we define the functional $j : L^2(\Gamma_3) \times V \times V \rightarrow \mathbb{R}$ by

$$j(\beta, u, v) = \int_{\Gamma_3} (-c_\nu \beta^2 R_\nu(u_\nu) + p(u_\nu))v_\nu \, da, \quad \forall (\beta, u, v) \in L^2(\Gamma_3) \times V \times V,$$

where the normal compliance function $p : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}_+$ satisfies:

$$(2.15) \quad \left\{ \begin{array}{l} \text{(a) there exists } L_p > 0 \text{ such that} \\ \quad |p(x, r_1) - p(x, r_2)| \leq L_p |r_1 - r_2| \\ \quad \text{for all } r_1, r_2 \in \mathbb{R} \text{ and a.e. } x \in \Gamma_3; \\ \text{(b) } (p(x, r_1) - p(x, r_2))(r_1 - r_2) \geq 0 \\ \quad \text{for all } r_1, r_2 \in \mathbb{R} \text{ and a.e. } x \in \Gamma_3; \\ \text{(c) the mapping } x \mapsto p(x, r) \text{ is measurable on } \Gamma_3, \text{ for any } r \in \mathbb{R}; \\ \text{(d) } p(x, r) = 0 \text{ for all } r \leq 0 \text{ and a.e. } x \in \Gamma_3. \end{array} \right.$$

We suppose that the body forces and surface tractions have the regularity

$$(2.16) \quad f_1 \in W^{1,1}(0, T; H), \quad f_2 \in W^{1,1}(0, T; (L^2(\Gamma_2))^d),$$

and we use Riesz's representation to define a function $F : [0, T] \rightarrow V$ by

$$(2.17) \quad (F(t), v)_V = \int_{\Omega} f_1(t) \cdot v \, dx + \int_{\Gamma_2} f_2(t) \cdot v \, da \quad \forall v \in V, t \in [0, T].$$

We see that (2.16) and (2.17) imply

$$F \in W^{1,1}(0, T; V).$$

We also need to introduce the following set for the bonding field:

$$B = \{\theta : [0, T] \rightarrow L^2(\Gamma_3) : 0 \leq \theta(t) \leq 1 \, \forall t \in [0, T], \text{ a.e. on } \Gamma_3\}.$$

Finally, for $p \in [1, \infty]$, we use the usual notation for the Lebesgue spaces $L^p(0, T; V)$ and Sobolev spaces $W^{k,\infty}(0, T; V)$, $k = 1, 2, \dots$.

For every real Banach space $(X, \|\cdot\|_X)$ and $T > 0$ we write $C([0, T]; X)$ for the space of continuous functions from $[0, T]$ to X ; recall that $C([0, T]; X)$ is a real Banach space with the norm

$$\|x\|_{C([0, T]; X)} = \max_{t \in [0, T]} \|x(t)\|_X.$$

Now assuming that the solution is sufficiently regular, by using Green's formula we find that Problem P_1 has the following variational formulation.

PROBLEM P_2 . Find a displacement field $u : [0, T] \rightarrow V$ and a bonding field $\beta : [0, T] \rightarrow L^2(\Gamma_3)$ such that

$$(3.1) \quad \begin{aligned} & u(t) \in K, \\ & (\mathcal{A}\varepsilon(\dot{u}(t)), \varepsilon(v) - \varepsilon(u(t)))_Q + (\mathcal{B}\varepsilon(u(t)), \varepsilon(v) - \varepsilon(u(t)))_Q \\ & + j(\beta(t), u(t), v - u(t)) \geq (F(t), v - u(t))_V, \quad \forall v \in K, \text{ a.e. } t \in (0, T), \end{aligned}$$

$$(3.2) \quad \dot{\beta}(t) = -[\beta(t)c_\nu(R_\nu(u_\nu(t)))^2 - \varepsilon_a]_+, \quad \text{a.e. } t \in (0, T),$$

$$(3.3) \quad u(0) = u_0,$$

$$(3.4) \quad \beta(0) = \beta_0.$$

The main result of this section, to be proved in the next one, is the following theorem.

THEOREM 2.1. *Assume that (2.9), (2.10) and (2.12)–(2.16) hold. Then there exists a unique solution to Problem P_2 which satisfies*

$$u \in W^{1,\infty}(0, T; V), \quad \beta \in W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap B.$$

3. Proof of Theorem 2.1. As in [3], the proof of Theorem 2.1 will be carried out in several steps and we need the following abstract results. Let X be a Hilbert space with the inner product $(\cdot, \cdot)_X$ and the associated norm $\|\cdot\|_X$. The domain $D(A)$ of a multivalued operator $A : D(A) \subset X \rightarrow 2^X$ is defined as

$$D(A) = \{x \in X : Ax \neq \emptyset\},$$

where 2^X represents the set of all subsets of X . We say that the operator A is *monotone* if

$$(u_1 - u_2, v_1 - v_2)_X \geq 0 \quad \forall v_1 \in Au_1, v_2 \in Au_2, \forall u_1, u_2 \in D(A).$$

We say that A is *maximal monotone* if there exists no monotone multivalued operator $B : D(B) \subset X \rightarrow 2^X$ that is a proper extension of A .

For a function $\phi : X \rightarrow]-\infty, \infty]$ we denote the *subdifferential* of ϕ at $u \in X$ as

$$\partial\phi(u) = \{f \in X : \phi(v) - \phi(u) \geq (f, v - u)_X \quad \forall v \in X\}.$$

It can be shown that if $\phi : X \rightarrow]-\infty, \infty]$ is a proper, convex, and lower semicontinuous function, then $\partial\phi$ is a maximal monotone operator. It can also be shown that if $A_1 : D(A_1) \subset X \rightarrow 2^X$ is a maximal monotone operator and $A_2 : X \rightarrow X$ is a single-valued, monotone, and Lipschitz continuous operator, then $A_1 + A_2$ is a maximal monotone operator. The proofs of these results and of the theorem below can be found in [4].

THEOREM 3.1. *Let X be a Hilbert space and let $A : D(A) \subset X \rightarrow 2^X$ be a multivalued operator such that $A + \omega I_X$ is a maximal operator for some real ω . Then, for every $f \in W^{1,1}(0, T; X)$ and $u_0 \in D(A)$, there exists a unique function $u \in W^{1,\infty}(0, T; V)$ which satisfies*

$$\begin{aligned} \dot{u}(t) + Au(t) \ni f(t), \quad \text{a.e. } t \in (0, T), \\ u(0) = u_0. \end{aligned}$$

Next, we use this result to prove Theorem 2.1.

Proof of Theorem 2.1. In the first step, for a given $\eta \in W^{1,\infty}(0, T; V)$ we consider the following variational problem.

PROBLEM P_η . Find $u_\eta : [0, T] \rightarrow V$ such that

$$\begin{aligned} u_\eta(t) &\in K, \\ (\mathcal{A}\varepsilon(\dot{u}_\eta(t)), \varepsilon(v) - \varepsilon(u_\eta(t)))_Q &+ (\mathcal{B}\varepsilon(u_\eta(t)), \varepsilon(v) - \varepsilon(u_\eta(t)))_Q \\ &+ (\eta(t), v - u_\eta(t))_V \geq (F(t), v - u_\eta(t))_V \quad \forall v \in K, \text{ a.e. } t \in (0, T), \\ u_\eta(0) &= u_0. \end{aligned}$$

We show the following result.

LEMMA 3.2. *Problem P_η has a unique solution which satisfies $u_\eta \in W^{1,\infty}(0, T; V)$.*

Proof. We see that Problem P_η is equivalent to Problem Q_η defined below.

PROBLEM Q_η . Find $u_\eta : [0, T] \rightarrow V$ such that

$$\begin{aligned} (3.5) \quad &u_\eta(t) \in K, \\ &(\mathcal{A}\varepsilon(\dot{u}_\eta(t)), \varepsilon(v) - \varepsilon(u_\eta(t)))_Q + (\mathcal{B}\varepsilon(u_\eta(t)), \varepsilon(v) - \varepsilon(u_\eta(t)))_Q \\ &\geq (F_\eta(t), v - u_\eta(t))_V \quad \forall v \in K, \text{ a.e. } t \in (0, T), \end{aligned}$$

$$(3.6) \quad u_\eta(0) = u_0,$$

where $F_\eta \in W^{1,1}(0, T; V)$ is defined as $F_\eta = F - \eta$.

Next, by the Riesz representation theorem we can define an operator $C : V \rightarrow V$ by

$$(Cu, v)_V = (\mathcal{B}\varepsilon(u), \varepsilon(v))_Q \quad \forall u, v \in V.$$

It follows from the assumptions (2.9) and (2.10) that

$$(3.7) \quad \|Cu_1 - Cu_2\|_V \leq \frac{L_{\mathcal{B}}}{m_{\mathcal{A}}} \|u_1 - u_2\|_V \quad \forall u_1, u_2 \in V,$$

i.e., C is a Lipschitz continuous.

For $V_1 \subset V$, we denote by $I_{V_1} : V \rightarrow]-\infty, \infty]$ the indicator function of the set V_1 , defined by

$$I_{V_1}(v) = \begin{cases} 0 & \text{if } v \in V_1, \\ \infty & \text{if } v \notin V_1. \end{cases}$$

Since K is a nonempty convex closed subset of V , it follows that ∂I_K is a maximal monotone operator on V and $D(\partial I_K) = K$.

Using (3.7), we see that the operator

$$C + \frac{L_{\mathcal{B}}}{m_{\mathcal{A}}} I_V : V \rightarrow V$$

is monotone and Lipschitz continuous. Therefore,

$$\partial I_K + C + \frac{L_{\mathcal{B}}}{m_{\mathcal{A}}} I_V : K \subset V \rightarrow 2^V$$

is a maximal monotone operator. Conditions (2.12) and (3.6) allow us to apply Theorem 3.1 with $X = V$ endowed with the inner product $(\cdot, \cdot)_{\mathcal{A}}$, $A = \partial I_K + C$, $D(A) = K \subset V$, and $\omega = L_{\mathcal{B}}/m_{\mathcal{A}}$. We deduce that there exists a unique element $u_{\eta} \in W^{1,\infty}(0, T; V)$ such that

$$(3.8) \quad \dot{u}_{\eta}(t) + \partial I_K(u_{\eta}(t)) + Cu_{\eta}(t) \ni F_{\eta}(t), \quad \text{a.e. } t \in (0, T),$$

$$(3.9) \quad u_{\eta}(0) = u_0.$$

We recall that for each $u, z \in V$ we have the following equivalence:

$$z \in \partial I_K(u) \Leftrightarrow u \in K, (z, v - u)_V \leq 0 \quad \forall v \in K.$$

Thus, the differential inclusion (3.8) is equivalent to the following variational inequality:

$$(3.10) \quad \begin{aligned} u_{\eta}(t) \in K, \\ (\dot{u}_{\eta}(t), v - u_{\eta}(t))_V + (Cu_{\eta}(t), v - u_{\eta}(t))_V \\ \geq (F_{\eta}(t), v - u_{\eta}(t))_V \quad \forall v \in K, \text{ a.e. } t \in (0, T). \end{aligned}$$

It follows from (3.10) that u_{η} satisfies the variational inequality

$$(3.11) \quad \begin{aligned} u_{\eta}(t) \in K, \\ (\mathcal{A}\varepsilon(\dot{u}_{\eta}(t)), \varepsilon(v) - \varepsilon(u_{\eta}(t)))_Q + (\mathcal{B}\varepsilon(u_{\eta}(t)), \varepsilon(v) - \varepsilon(u_{\eta}(t)))_Q \\ \geq (F_{\eta}(t), v - u_{\eta}(t))_V \quad \forall v \in K, \text{ a.e. } t \in (0, T). \end{aligned}$$

Therefore, we deduce from (3.11) and (3.9) the existence part of Theorem 2.1. The uniqueness of solution which satisfies (3.5) and (3.6) is guaranteed by Theorem 3.1.

In the second step we consider the following initial value problem.

PROBLEM R_{η} . Find a bonding field $\beta_{\eta} : [0, T] \rightarrow L^2(\Gamma_3)$ such that

$$(3.12) \quad \dot{\beta}_{\eta}(t) = -[\beta_{\eta}(t)c_{\nu}(R_{\nu}(u_{\eta\nu}(t)))^2 - \varepsilon_a]_+, \quad \text{a.e. } t \in (0, T),$$

$$(3.13) \quad \beta_{\eta}(0) = \beta_0.$$

We have the following result.

LEMMA 3.3. *There exists a unique solution to Problem R_{η} and it satisfies*

$$\beta_{\eta} \in W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap B.$$

Proof. Consider the mapping $F(t, \theta) : [0, T] \times L^2(\Gamma_3) \rightarrow L^2(\Gamma_3)$ defined by

$$F(t, \theta) = -[c_{\nu}\theta(R_{\nu}(u_{\eta\nu}(t)))^2 - \varepsilon_a]_+.$$

It follows from the properties of the truncation operator R that F is Lipschitz continuous with respect to the second argument, uniformly in time. Moreover, for any $\theta \in L^2(\Gamma_3)$, the mapping $t \mapsto F(t, \theta)$ belongs to $L^{\infty}(0, T; L^2(\Gamma_3))$. Then, from a version of the Cauchy–Lipschitz theorem (see [28]), we deduce the existence of a unique function $\beta_{\eta} \in W^{1,\infty}(0, T; L^2(\Gamma_3))$ which satisfies

(3.12), (3.13). The regularity $\beta_\eta \in B$ follows from (3.12), (3.13) and (2.14), (see also [28]). ■

Next, in the third step for $\eta \in W^{1,\infty}(0, T; V)$ we denote by u_η and β_η the solutions of Problem P_η and R_η respectively. Using Riesz's representation theorem we define the function

$$A\eta : [0, T] \rightarrow V$$

by

$$(3.14) \quad (A\eta(t), v)_V = j(\beta_\eta(t), u_\eta(t), v) \quad \forall v \in V, t \in [0, T].$$

We have the lemma below.

LEMMA 3.4. *For each $\eta \in W^{1,\infty}(0, T; V)$ the function $A\eta$ belongs to $W^{1,\infty}(0, T; V)$. Moreover, there exists a unique $\eta^* \in W^{1,\infty}(0, T; V)$ such that $A\eta^* = \eta^*$.*

Proof. Let $\eta \in W^{1,\infty}(0, T; V)$ and $t_1, t_2 \in [0, T]$. Using (3.14), it follows that there exists a constant $c_1 > 0$ such that

$$\begin{aligned} \|A\eta(t_1) - A\eta(t_2)\|_V &\leq c_1 (\|\beta_\eta^2(t_1)R_\nu(u_{\eta\nu}(t_1)) - \beta_\eta^2(t_2)R_\nu(u_{\eta\nu}(t_2))\|_{L^2(\Gamma_3)} \\ &\quad + \|p(u_\nu(t_1)) - p(u_\nu(t_2))\|_{L^2(\Gamma_3)}). \end{aligned}$$

Now, keeping in mind (2.14), the inequalities $0 \leq \beta_\eta(t) \leq 1$ for all $t \in [0, T]$ and the properties of the truncation operator R_ν , we find that

$$(3.15) \quad \begin{aligned} \|A\eta(t_1) - A\eta(t_2)\|_V \\ \leq c_2 (\|u_\eta(t_1) - u_\eta(t_2)\|_V + \|\beta_\eta(t_1) - \beta_\eta(t_2)\|_{L^2(\Gamma_3)}) \end{aligned}$$

with some $c_2 > 0$. Since $u_\eta \in W^{1,\infty}(0, T; V)$ and $\beta_\eta \in W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap B$, we deduce from (3.15) that $A\eta \in W^{1,\infty}(0, T; V)$.

Let now $\eta_1, \eta_2 \in W^{1,\infty}(0, T; V)$. For $t \in [0, T]$ we integrate (3.12) with the initial condition (3.13) to obtain

$$\beta_{\eta_i}(t) = \beta_0 - \int_0^t [\beta_{\eta_i}(s)c_\nu(R_\nu(u_{\eta_i\nu}(s)))^2 - \varepsilon_a]_+ da.$$

Using the definition of R_ν , the inequality $|R_\nu(u_{\eta_i\nu})| \leq L$, (2.11) and writing $\beta_{\eta_1} = \beta_{\eta_1} - \beta_{\eta_2} + \beta_{\eta_2}$, we see that for some constant $c_3 > 0$,

$$\begin{aligned} \|\beta_{\eta_1}(t) - \beta_{\eta_2}(t)\|_{L^2(\Gamma_3)} \\ \leq c_3 \left(\int_0^t \|\beta_{\eta_1}(s) - \beta_{\eta_2}(s)\|_{L^2(\Gamma_3)} + d_\Omega \int_0^t \|u_{\eta_1}(s) - u_{\eta_2}(s)\|_V ds \right). \end{aligned}$$

By a Gronwall-type inequality, it follows that

$$(3.16) \quad \|\beta_{\eta_1}(t) - \beta_{\eta_2}(t)\|_{L^2(\Gamma_3)} \leq c_4 \int_0^t \|u_{\eta_1}(s) - u_{\eta_2}(s)\|_V ds$$

with some $c_4 > 0$. On the other hand, using arguments similar to those in the proof of (3.15), we find that there exists a constant $c_5 > 0$ such that

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V \leq c_5 (\|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V + \|\beta_{\eta_1}(t) - \beta_{\eta_2}(t)\|_{L^2(\Gamma_3)}).$$

Then, by (3.16) we have

$$(3.17) \quad \|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V \leq c_5 \left(\|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V + c_4 \int_0^t \|u_{\eta_1}(s) - u_{\eta_2}(s)\|_V ds \right).$$

Next, we use (3.11) to find that

$$\begin{aligned} (\dot{u}_{\eta_1}(t) - \dot{u}_{\eta_2}(t), u_{\eta_1}(t) - u_{\eta_2}(t))_V &\leq (\eta_2(t) - \eta_1(t), u_{\eta_1}(t) - u_{\eta_2}(t))_V \\ &\quad + (\mathcal{B}u_{\eta_1}(t) - \mathcal{B}u_{\eta_2}(t), u_{\eta_1}(t) - u_{\eta_2}(t))_V. \end{aligned}$$

Using the Cauchy-Schwarz inequality and (3.7) we obtain

$$\begin{aligned} (\dot{u}_{\eta_1}(t) - \dot{u}_{\eta_2}(t), u_{\eta_1}(t) - u_{\eta_2}(t))_V &\leq (\eta_2(t) - \eta_1(t), u_{\eta_1}(t) - u_{\eta_2}(t))_V \\ &\quad + \frac{L_{\mathcal{B}}}{m_{\mathcal{A}}} \|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V^2. \end{aligned}$$

We integrate the inequality above with respect to time to find that

$$\begin{aligned} \|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V^2 &\leq 2 \int_0^t \|\eta_2(s) - \eta_1(s)\|_V \|u_{\eta_1}(s) - u_{\eta_2}(s)\|_V ds \\ &\quad + 2 \frac{L_{\mathcal{B}}}{m_{\mathcal{A}}} \int_0^t \|u_{\eta_1}(s) - u_{\eta_2}(s)\|_V^2 ds. \end{aligned}$$

Applying the inequality

$$2ab \leq a^2 + b^2 \quad \forall a, b \in \mathbb{R},$$

we obtain

$$\begin{aligned} \|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V^2 &\leq 2 \int_0^t \|\eta_1(s) - \eta_2(s)\|_V^2 ds \\ &\quad + \left(2 \frac{L_{\mathcal{B}}}{m_{\mathcal{A}}} + 1 \right) \int_0^t \|u_{\eta_1}(s) - u_{\eta_2}(s)\|_V^2 ds. \end{aligned}$$

Using now a Gronwall-type inequality we get

$$(3.18) \quad \|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V^2 \leq c_6 \int_0^t \|\eta_1(s) - \eta_2(s)\|_V^2 ds$$

for some constant $c_6 > 0$. Using (3.17) after some calculations we find that there exists a constant $c_7 > 0$ such that

$$\begin{aligned} & \|A\eta_1(t) - A\eta_2(t)\|_V^2 \\ & \leq 2c_7^2 \left(\|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V^2 + \left(\int_0^t \|u_{\eta_1}(s) - u_{\eta_2}(s)\|_V ds \right)^2 \right). \end{aligned}$$

Then, using the Cauchy–Schwarz inequality we find

$$(3.19) \quad \begin{aligned} & \|A\eta_1(t) - A\eta_2(t)\|_V^2 \\ & \leq 2c_8^2 \left(\|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V^2 + \int_0^t \|u_{\eta_1}(s) - u_{\eta_2}(s)\|_V^2 ds \right) \end{aligned}$$

for some positive constant c_8 . Therefore by (3.18) and (3.19), it follows that there exists a constant $c_9 > 0$ such that

$$(3.20) \quad \|A\eta_1(t) - A\eta_2(t)\|_V^2 \leq c_9 \int_0^t \|\eta_1(s) - \eta_2(s)\|_V^2 ds.$$

Let now $\alpha > 0$, and denote

$$\|\eta\|_\alpha = \sup_{t \in [0, T]} e^{-\alpha t} \|\eta(t)\|_V \quad \forall \eta \in C([0, T]; V).$$

Clearly $\|\cdot\|_\alpha$ defines a norm on the space $C([0, T]; V)$ which is equivalent to the standard norm $\|\cdot\|_{C([0, T]; V)}$. Using (3.20), we get

$$\|A\eta_1 - A\eta_2\|_\alpha \leq \frac{c_{10}}{\sqrt{2\alpha}} \|\eta_1 - \eta_2\|_\alpha \quad \forall \eta_1, \eta_2 \in C([0, T]; V),$$

for some $c_{10} > 0$. So for α sufficiently large, the operator A is a contraction on the space $C([0, T]; V)$ endowed with the norm $\|\cdot\|_\alpha$. Then by the Banach fixed point theorem it follows that A has a unique fixed point $\eta^* \in C([0, T]; V)$ such that $A\eta^* = \eta^*$. The regularity $\eta^* \in W^{1, \infty}(0, T; V)$ follows from the regularity $A\eta^* \in W^{1, \infty}(0, T; V)$, which concludes the proof. ■

Now, we have all the ingredients to prove Theorem 2.1.

Proof of Theorem 2.1. Existence. Let $\eta^* \in W^{1, \infty}(0, T; V)$ be the fixed point of A and let u, β be the functions defined in Lemmas 3.3 and 3.4, respectively, for $\eta = \eta^*$, i.e. $u = u_{\eta^*}$, $\beta = \beta^*$. Clearly equalities (3.2) and (3.4) hold from Lemma 3.3. Moreover, since $A\eta^* = \eta^*$, it follows from (3.5) and (3.6) that (3.1) and (3.3) hold too. The regularity of solution to Problem P_2 follows from Lemmas 3.2 and 3.3.

Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed point of A and the uniqueness parts of Lemmas 3.3 and 3.4. ■

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Arezki Touzaline
Faculté de Mathématiques, USTHB
Laboratoire de Systèmes Dynamiques
BP 32 El Alia
Bab-Ezzouar 16111, Algeria
E-mail: ttouzaline@yahoo.fr

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