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Dedicated to Professor Leon Mikołajczyk
on the occasion of his 85th birthday.

NECESSARY AND SUFFICIENT CONDITIONS FOR A PARETO OPTIMAL ALLOCATION IN A DISCONTINUOUS GALE ECONOMIC MODEL

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Abstract. In this paper we examine the concept of Pareto optimality in a simplified Gale economic model without assuming continuity of the utility functions. We apply some existing results on higher-order optimality conditions to get necessary and sufficient conditions for a locally Pareto optimal allocation.

Keywords: Gale model, discontinuous functions, generalized directional derivatives, higher-order conditions, Pareto optimality.

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1. INTRODUCTION

In some economic models (see, for example, [2, 5]) one has to consider discontinuous functions. The existing research results on such models are focused on the existence of equilibria. In this paper, we present necessary and sufficient conditions for local Pareto optimality in the Gale model involving possibly discontinuous functions. These conditions are obtained by applying the recent results of [4] and are formulated in terms of generalized lower and upper directional derivatives of utility functions.

The results presented here concern Pareto optimal allocations as defined in [2, Definition 2.2]. Obtaining optimality conditions for equilibrium allocations [2, Definition 2.3] requires a different approach and will be the subject of further research.

2. FORMULATION OF THE MODEL

We now describe a simplified version of the Gale model [2]. Suppose we have n goods $G_1, \ldots, G_n$ and p economic agents $A_1, \ldots, A_p$. The set of goods includes all types of labor and services as well as material commodities. The economic agents may be thought of as either consumers or as producers.

The amount of goods $G_1, \dots, G_n$ supplied or consumed by an agent A_i in a certain fixed time interval is given by a vector

$$x_i = (x_{i,1}, \dots, x_{i,n}) \in \mathbb{R}^n. \tag{2.1}$$

The j-th coordinate $x_{i,j}$ represents the amount of the good G_j and is positive (respectively, negative) if G_j is supplied (respectively, consumed). Such a vector is called a *commodity bundle* of A_i. The set C_i of all possible commodity bundles (2.1) is called the *commodity set* or *technology set* of the agent A_i, $i = 1, \dots, p$.

In the Gale model it is assumed that the *balance inequalities* hold, i.e. the total amount of each good consumed by all agents must not exceed the total amount supplied:

$$\sum_{i=1}^{p} x_{i,j} \geq 0, \quad j = 1, \dots, n. \tag{2.2}$$

Definition 2.1. A vector system $\{x_1, \dots, x_p\}$ is called:

(i) a *feasible allocation* if $x_i \in C_i$, $i = 1, \dots, p$, and inequalities (2.2) hold;
(ii) a *feasible allocation without savings* if $x_i \in C_i$, $i = 1, \dots, p$, and

$$\sum_{i=1}^{p} x_{i,j} = 0, \quad j = 1, \dots, n. \tag{2.3}$$

Let us note that condition (2.2) (respectively, (2.3)) may be written down in an equivalent vector form $\sum_{i=1}^{p} x_i \geq 0$ (respectively, $\sum_{i=1}^{p} x_i = 0$).

We assume that, for each agent A_i, there exists a *utility function*

$$h_i : C_i \to \mathbb{R}, \tag{2.4}$$

$i = 1, \dots, p$. Each agent tends to maximize his utility function.

Definition 2.2. A feasible allocation $\{\bar{x}_1, \dots, \bar{x}_p\}$ is called a *Pareto optimal allocation* if, for every other feasible allocation $\{x_1, \dots, x_p\}$, we have either

$$h_i(x_i) = h_i(\bar{x}_i) \quad \text{for all} \quad i \in \{1, \dots, p\} \tag{2.5}$$

or

$$h_j(x_j) < h_j(\bar{x}_j) \quad \text{for some } j \in \{1, \dots, p\}. \tag{2.6}$$

3. A MULTIOBJECTIVE OPTIMIZATION PROBLEM

In this section we formulate the problem of finding a Pareto optimal allocation in the Gale model as a multiobjective optimization problem. Such a formulation will enable us to apply the results of [4] in this particular situation.

Below we reformulate some results from [3] and [4] to the forms where maximization instead of minimization is considered. Such versions can easily be obtained by substituting $-f$ for f in the original theorems. Let X and Y be normed spaces. We consider the following general multiobjective optimization problem:

$$\max f(x) \tag{3.1}$$

subject to

$$x \in S := \Big\{ z \in X : -g(z) \in D,\, z \in C \Big\}, \tag{3.2}$$

where $f : X \to \mathbb{R}^p$ and $g : X \to Y$. We assume that C and D are nonempty closed subsets of X and Y, respectively, and D is a convex cone, $D \neq Y$. The maximization in (3.1) is understood with respect to the natural partial order defined by

$$(x \leq y) \quad \text{if and only if} \quad (x_i \leq y_i \ \text{ for all } \ i \in \{1, \ldots, p\})$$

or, which is equivalent, with respect to the *positive cone* $\mathbb{R}^p_+ := [0, \infty)^p$.

We denote by $\mathcal{N}(x)$ the collection of all neighborhoods of x.

Definition 3.1 ([4, Definition 9(a)]). Let m be a positive integer, and let $\bar{x} \in S$. We say that $\bar{x}$ is a *strict local Pareto maximizer of order* m for problem (3.1)–(3.2) if there exist $\alpha > 0$ and $U \in \mathcal{N}(\bar{x})$ such that

$$(f(x) - \mathbb{R}^p_+) \cap B(f(\bar{x}), \alpha \left\| x - \bar{x} \right\|^m) = \emptyset \quad \text{for all} \quad x \in S \cap U \backslash \{\bar{x}\},$$

where $B(\bar{u}, \varepsilon) := \{u \in \mathbb{R}^p : \|u - \bar{u}\| < \varepsilon\}$ for $\bar{u} \in \mathbb{R}^p$ and $\varepsilon > 0$.

Proposition 3.2 ([3, Proposition 2.11]). *Let $\bar{x} \in S$. Then $\bar{x}$ is a strict local Pareto maximizer of order m for problem* (3.1)–(3.2) *if and only if there exist $\eta \in \operatorname{int}\mathbb{R}^p_+$ (i.e. $\eta = (\eta_1, \ldots, \eta_p)$ with $\eta_i > 0$, $i = 1, \ldots, p$) and $U \in \mathcal{N}(\bar{x})$ such that there is no $x \in S \cap U \backslash \{\bar{x}\}$ satisfying*

$$f_i(x) \geq f_i(\bar{x}) - \eta_i \left\| x - \bar{x} \right\|^m \quad \textit{for all } i \in \{1, \ldots, p\}, \tag{3.3}$$

$$f_j(x) > f_j(\bar{x}) - \eta_j \left\| x - \bar{x} \right\|^m \quad \textit{for some } j \in \{1, \ldots, p\}. \tag{3.4}$$

We now introduce the following m-th order lower and upper directional derivatives:

$$\underline{d}^m f(\bar{x}; y) := \liminf_{(t,v) \to (0^+, y)} \frac{f(\bar{x} + tv) - f(\bar{x})}{t^m},$$

$$\overline{d}^m f(\bar{x}; y) := \limsup_{(t,v) \to (0^+, y)} \frac{f(\bar{x} + tv) - f(\bar{x})}{t^m},$$

where the lower and upper limits are taken with respect to the natural partial order in $\bar{\mathbb{R}}^p$ (we denote by $\bar{\mathbb{R}}^p$ the Cartesian product of p copies of $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$; see [4] for details).

It is shown in [4] that

$$\begin{aligned} \underline{d}^m f(\bar{x}; y) &= (\underline{d}^m f_1(\bar{x}; y), \ldots, \underline{d}^m f_p(\bar{x}; y)), \\ \overline{d}^m f(\bar{x}; y) &= (\overline{d}^m f_1(\bar{x}; y), \ldots, \overline{d}^m f_p(\bar{x}; y)). \end{aligned} \tag{3.5}$$

We will also use the notation

$$dg(\bar{x}; y) := \lim_{(t,v)\to(0^+,y)} \frac{g(\bar{x} + tv) - g(\bar{x})}{t^m},$$

whenever this limit exists in Y.

We denote by $K(C, \bar{x})$ the *contingent cone* to C at $\bar{x}$:

$$K(C, \bar{x}) := \{y \in X : \text{ there exists } (t_n, y_n) \to (0^+, y) \text{ such that } \bar{x}+t_n y_n \in C \text{ for all } n\}.$$

We also introduce the notation $\bar{\mathbb{R}}^p_+ := [0, \infty]^p$.

Theorem 3.3 ([4, Theorem 11(a)]). *Let $\bar{x} \in S$ be a strict local Pareto maximizer of order m for problem* (3.1)–(3.2). *Suppose that* $\mathrm{int}D \neq \emptyset$ *and $dg(\bar{x}; y)$ exists for all $y \in X$. Then there exists $\beta > 0$ such that*

$$\begin{aligned} &\underline{d}^m f(\bar{x}; y) \notin B(0, \beta \|y\|^m) + \bar{\mathbb{R}}^p_+ \\ &\textit{for all } y \in K(C, \bar{x}) \cap \{u \in X : dg(\bar{x}; u) \in -\mathrm{int}D\}. \end{aligned} \tag{3.6}$$

In the next theorem we shall use the following notation for the closure of the cone generated by $D + g(\bar{x})$:

$$D_{g(\bar{x})} := \mathrm{cl\ cone}(D + g(\bar{x})). \tag{3.7}$$

It follows from the convexity of D that $D_{g(\bar{x})}$ is a closed convex cone.

Theorem 3.4 ([4, Theorem 15]). *Let* $\dim X < \infty$*, and let $\bar{x} \in S$. Suppose that $dg(\bar{x}; y)$ exists for all $y \in X$. If*

$$\overline{d}^m f(\bar{x}; y) \notin \bar{\mathbb{R}}^p_+, \ \textit{ for all } y \in K(C, \bar{x}) \cap \Big\{u \in X : dg(\bar{x}; u) \in -D_{g(\bar{x})}\Big\} \backslash \{0\}, \tag{3.8}$$

then $\bar{x}$ is a strict local Pareto maximizer of order m for problem (3.1)–(3.2).

4. APPLICATION TO THE GALE MODEL

Let us note that Definitions 2.2 and 3.1 are uncomparable, i.e. no one of them implies the other. However, later we shall compare Definition 3.1 with the following local version of Definition 2.2.

Definition 4.1. A feasible allocation $\{\bar{x}_1, \dots, \bar{x}_p\}$ is called a *locally Pareto optimal allocation* if there exists $U \in \mathcal{N}(\bar{x})$, where $\bar{x} = (\bar{x}_1, \dots, \bar{x}_p)$, such that for every other feasible allocation $\{x_1, \dots, x_p\}$ with $x = (x_1, \dots, x_p) \in U$, we have either (2.5) or (2.6).

To apply the general framework presented in Section 3 to the Gale model, we consider the following spaces:

$$X = Y := \mathbb{R}^{np} = \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{p \text{ times}},$$

and define the sets C, D and the mapping g in (3.2) as follows:

$$C := C_1 \times \ldots \times C_p, \quad D := \Big\{x \in X : \sum_{i=1}^{p} x_i \geq 0\Big\}, \quad g(x) := -x. \tag{4.1}$$

We assume that the sets $C_1, \dots, C_p$ are convex and closed. We also assume that each utility function (2.4) has an extension $\bar{h}_i$ to the whole space $\mathbb{R}^n$, that is, there exist functions $\bar{h}_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, p$, such that

$$\bar{h}_i(x_i) = h_i(x_i) \;\text{ if } x_i \in C_i. \tag{4.2}$$

This allows us to define the function f in (3.1) by $f = (f_1, \dots, f_p)$ where

$$f_i(x) = f_i(x_1, \dots, x_p) := \bar{h}_i(x_i), \;\text{ for all } x \in X,\, i = 1, \dots, p. \tag{4.3}$$

Now, problem (3.1)–(3.2) can be rewritten as

$$\max(f_1(x), \dots, f_p(x)) \tag{4.4}$$

subject to

$$x \in S := \Big\{z \in X : z_i \in C_i,\, i = 1, \dots, p;\; \sum_{i=1}^{p} z_i \geq 0\Big\}. \tag{4.5}$$

Proposition 4.2. *If $\bar{x} = (\bar{x}_1, \dots, \bar{x}_p)$ is a strict local Pareto maximizer of order m for problem* (4.4)–(4.5)*, then $\{\bar{x}_1, \dots, \bar{x}_p\}$ is a locally Pareto optimal allocation. Moreover, condition* (2.6) *holds for each $x \neq \bar{x}$.*

Proof. Let $\bar{x}$ be a strict local Pareto maximizer of order m for (4.4)–(4.5). It follows from Proposition 3.2 that there exist $\eta \in \operatorname{int}\mathbb{R}^p_+$ and $U \in \mathcal{N}(\bar{x})$ such that there is no $x \in S \cap U \backslash \{\bar{x}\}$ satisfying (3.3)–(3.4). Since $\bar{x}$ is feasible for (4.4)–(4.5), we have $\bar{x}_i \in C_i$, $i = 1, \dots, p$, and so, by (4.2) and (4.3),

$$f_i(\bar{x}) = \bar{h}_i(\bar{x}_i) = h_i(\bar{x}_i), \quad i = 1, \dots, p.$$

Hence, there is no $x \in S \cap U \backslash \{\bar{x}\}$ satisfying

$$h_i(x_i) \geq h_i(\bar{x}_i) - \eta_i \left\|x - \bar{x}\right\|^m \;\text{ for all } i \in \{1, \dots, p\}, \tag{4.6}$$

$$h_j(x_j) > h_j(\bar{x}_j) - \eta_j \left\|x - \bar{x}\right\|^m \;\text{ for some } j \in \{1, \dots, p\}. \tag{4.7}$$

Now, take any other feasible allocation $\{x_1, \ldots, x_p\}$ such that $x = (x_1, \ldots, x_p) \in U$. If $x = \bar{x}$, then obviously (2.5) holds. Suppose that $x \neq \bar{x}$, which implies $x \in S \cap U \backslash \{\bar{x}\}$ by (4.5). As (4.6) and (4.7) cannot hold simultaneously, we have either

$$h_i(x_i) = h_i(\bar{x}_i) - \eta_i \|x - \bar{x}\|^m \quad \text{for all} \quad i \in \{1, \ldots, p\}, \tag{4.8}$$

or there exists an index $l \in \{1, \ldots, p\}$, depending on x, such that

$$h_l(x_l) < h_l(\bar{x}_l) - \eta_l \|x - \bar{x}\|^m . \tag{4.9}$$

Both (4.8) and (4.9) imply condition (2.6). □

Theorem 4.3 (necessary conditions)**.** *Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_p)$ be a strict local Pareto maximizer of order m for problem (4.4)–(4.5). Then, for each $y = (y_1, \ldots, y_p)$ such that*

$$y_i \in K(C_i, \bar{x}_i),\ i = 1, \ldots, p;\quad \sum_{i=1}^{p} y_i > 0, \tag{4.10}$$

there exists $j \in \{1, \ldots, p\}$ such that $\underline{d}^m \bar{h}_j(\bar{x}_j; y_j) < 0$.

Proof. It follows from (4.1) that

$$\text{int}D = \left\{x \in X : \sum_{i=1}^{p} x_i > 0\right\} \neq \emptyset \;\text{ and }\; dg(\bar{x}; y) = -y \;\text{ for all } y \in X. \tag{4.11}$$

Hence, we can apply Theorem 3.3. Instead of (3.6), we only need the weaker condition that

$$\underline{d}^m f(\bar{x}; y) \notin \bar{\mathbb{R}}^p_+ \tag{4.12}$$

for all y satisfying

$$y \in K(C, \bar{x}) \cap \left\{u \in X : dg(\bar{x}; u) \in -\text{int}D\right\}. \tag{4.13}$$

Since the sets $C_1, \ldots, C_p$ are convex, we have, by formula (46) in [1, Chapter 4],

$$K(C, \bar{x}) = K(C_1, \bar{x}_1) \times \ldots \times K(C_p, \bar{x}_p). \tag{4.14}$$

Now, let $y = (y_1, \ldots, y_p)$ be such that (4.10) holds. Then the condition $\sum_{i=1}^p y_i > 0$ is equivalent, by (4.11), to $dg(\bar{x}; y) \in -\text{int}D$. From this and (4.14), we have that y satisfies (4.13). Therefore, (4.12) also holds for y, which means that at least one component of $\underline{d}^m f(\bar{x}; y)$ is strictly negative. However, it is easy to see that the j-th component of $\underline{d}^m f(\bar{x}; y)$ is equal, by (3.5) and (4.3), to

$$\begin{aligned} \underline{d}^m f_j(\bar{x}; y) &= \liminf_{(t,v)\to(0^+,y)} \frac{f_j(\bar{x} + tv) - f_j(\bar{x})}{t^m} \\ &= \liminf_{(t,v)\to(0^+,y)} \frac{\bar{h}_j(\bar{x}_j + tv_j) - \bar{h}_j(\bar{x}_j)}{t^m} = \underline{d}^m \bar{h}_j(\bar{x}_j; y_j). \end{aligned}$$

Hence, the conclusion of the theorem holds. □

Theorem 4.4 (sufficient conditions)**.** *Suppose that one of the following two conditions holds:*

(*a*) $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_p)$ *is a feasible allocation without savings and, for each* $y = (y_1, \ldots, y_p) \neq 0$ *satisfying*

$$y_i \in K(C_i, \bar{x}_i),\ i = 1, \ldots, p, \quad \sum_{i=1}^{p} y_i \geq 0, \tag{4.15}$$

there exists $j \in \{1, \ldots, p\}$ *such that* $\overline{d}^m \bar{h}_j(\bar{x}_j; y_j) < 0$.

(*b*) $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_p)$ *is a feasible allocation and, for each* $y = (y_1, \ldots, y_p) \neq 0$ *satisfying*

$$y_i \in K(C_i, \bar{x}_i), \quad i = 1, \ldots, p, \tag{4.16}$$

there exists $j \in \{1, \ldots, p\}$ *such that* $\overline{d}^m \bar{h}_j(\bar{x}_j; y_j) < 0$.

Then $\bar{x}$ *is a strict local Pareto maximizer of order* m *for problem* (4.4)–(4.5).

Proof. (a) By assumption, $\bar{x}$ is a feasible allocation without savings, hence by Definition 2.1(ii), we have that $\sum_{i=1}^{p} \bar{x}_i = 0$. Therefore,

$$\begin{aligned} D + g(\bar{x}) = D - \bar{x} &= \Big\{ x - \bar{x} : \sum_{i=1}^{p} x_i \geq 0 \Big\} \\ &= \Big\{ u : \sum_{i=1}^{p} (u_i + \bar{x}_i) \geq 0 \Big\} = \Big\{ u : \sum_{i=1}^{p} u_i \geq 0 \Big\}. \end{aligned}$$

Since this set is a closed cone, we obtain by (3.7) that

$$D_{g(\bar{x})} = \Big\{ u : \sum_{i=1}^{p} u_i \geq 0 \Big\}. \tag{4.17}$$

We shall prove that assumption (3.8) of Theorem 3.4 is satisfied. Using (4.11), (4.14) and (4.17), we can reformulate it as follows:

$$\overline{d}^m f(\bar{x}; y) \notin \bar{\mathbb{R}}^p_+ \ \text{ for all } y \neq 0 \text{ satisfying (4.15)}. \tag{4.18}$$

Now, take any $y \neq 0$ for which (4.15) holds. By assumption (a), we have that

$$\overline{d}^m f_j(\bar{x}; y) = \overline{d}^m \bar{h}_j(\bar{x}_j; y_j) < 0 \quad \text{for some } j \in \{1, \ldots, p\}.$$

Therefore at least one component of $\overline{d}^m f(\bar{x}; y)$ is strictly negative. We have thus verified condition (4.18). Applying Theorem 3.4, we conclude that $\bar{x}$ is a strict local Pareto maximizer of order m for (4.4)–(4.5).

(b) In this case, we cannot guarantee equality (4.17) (it may happen that $D + g(\bar{x}) = X$). Therefore, we need to replace (4.15) by (4.16). □

Example 4.5. Consider the case with two agents and one good ($p = 2$, $n = 1$). Let x_1 and x_2 be the amounts of the good for the first and second agent, respectively, and let $x_i \in C_i := [-10, 10]$, $i = 1, 2$. We shall assume that there exist utility functions

$$\bar{h}_i(x_i) = \begin{cases} x_i & \text{for} \quad x_i \leq 5 \\ x_i - 1 & \text{for} \quad x_i > 5 \end{cases}, \quad i = 1, 2.$$

After exceeding a certain level of production (in our case this level is 5), the agent loses some benefits, which he had, so his utility suddenly decreases. This is the reason why the utility functions in our case are discontinuous.

We shall prove that the point $\bar{x} = (\bar{x}_1, \bar{x}_2) = (5, 5)$ is a strict local Pareto maximizer of order one for the problem

$$\max(f_1(x), f_2(x)) \tag{4.19}$$

subject to

$$x \in S := \{x \in \mathbb{R}^2 : x_i \in C_i, \ i = 1, 2; \quad x_1 + x_2 \geq 0\}, \tag{4.20}$$

where $f_i(x) = \bar{h}_i(x_i)$.

Since $\bar{x}_i \in \mathrm{int} C_i$, we have $K(C_i, \bar{x}_i) = \mathbb{R}$ for $i = 1, 2$. Now, take any direction $y \neq 0$. We compute

$$\begin{aligned} \bar{d}^1 \bar{h}_i(\bar{x}_i; y) &= \limsup_{(t,v) \to (0^+, y)} \frac{\bar{h}_i(\bar{x}_i + tv) - \bar{h}_i(\bar{x}_i)}{t} \\ &= \begin{cases} \limsup\limits_{(t,v) \to (0^+, y)} \frac{tv}{t} & \text{for} \quad y < 0 \\ \limsup\limits_{(t,v) \to (0^+, y)} \frac{tv - 1}{t} & \text{for} \quad y > 0 \end{cases} \\ &= \begin{cases} y & \text{for} \quad y < 0 \\ -\infty & \text{for} \quad y > 0 \end{cases}, \quad i = 1, 2. \end{aligned}$$

This means that $\bar{d}^1 \bar{h}_i(\bar{x}_i; y) < 0$ for all $y \neq 0$ and $i = 1, 2$. It follows from Theorem 4.4(b) that $\bar{x}$ is a strict local Pareto maximizer of order one for problem (4.19)–(4.20).

REFERENCES

[1] J.-P. Aubin, *Optima and Equilibria: An Introduction to Nonlinear Analysis*, Springer-Verlag, Berlin, 1998.

[2] I. Bula, *Discontinuous functions in Gale economic model*, Math. Model. Anal. **8** (2003) 2, 93–102.

[3] A. Gupta, D. Bhatia, A. Mehra, *Higher order efficiency, saddle point optimality, and duality for vector optimization problems*, Numer. Func. Anal. Opt. **28** (2007) 3–4, 339–352.

[4] E.-D. Rahmo, M. Studniarski, *Higher-order conditions for strict local Pareto minima in terms of generalized lower and upper directional derivatives.* J. Math. Anal. Appl. **393** (2012), 212–221.

[5] G. Tian, *Existence of equilibrium in abstract economies with discontinuous payoffs and non-compact choice spaces*, J. Math. Econ. **21** (1992), 379–388.

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