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INVESTIGATION OF G-NETWORKS WITH RESTART AT A NON-STATIONARY MODE AND THEIR APPLICATION

Mikhail Matalytski[1], Victor Naumenko[2], Dmitry Kopats[2]

[1] *Institute of Mathematics, Czestochowa University of Technology Czestochowa, Poland*
[2] *Department of Fundamental and Applied Mathematics, Grodno State University of Yanka Kupala, Grodno, Belarus*

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Abstract. This article discusses the question of restarting the script, when restart is used by many users of the information network, which can be modelled as a G-network. Negative claims simulate the crash of the script and the re-sending of the request. Investigation of an open queuing network (QN) with several types of positive customers with the phase type of distribution of their service time and one type of negative customers have been carried out. Negative customers are signals whose effect is to restart one customers in a queue. A technique is proposed for finding the probability of states. It is based on the use of a modified method of successive approximations, combined with the method of a series. The successive approximations converge with time to a stationary distribution of state probabilities, the form of which is indicated in the article, and the sequence of approximations converges to the solution of the difference-differential equations (DDE) system. The uniqueness of this solution is proved. Any successive approximation is representable in the form of a convergent power series with an infinite radius of convergence, the coefficients of which satisfy recurrence relations, which is convenient for computer calculations. A model example illustrating the finding of time-dependent probabilities of network states using the proposed technique is also presented.

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1. Introduction

Restart is a method that can be used to improve response time in complex systems where the causes of delays can't be detected, recognized, or eliminated by the user. When restarting, the user cancels the current job (script) that is greater than the prescribed period, and immediately sends it to the system. In many common scenarios, this approach can shorten the response time that is given to the user.

In everyday life, there are many situations where an impatient client after a certain waiting time refuses to wait longer for his work to finish, cancels the task and restarts it. For example, downloading over the Internet is the most widely known situation in which restart can be useful. There are also many other cases. Although restarting is often a simple solution, it can also negatively affect the system to which it is applied, since restarting can actually mean increasing the load on the system, thereby exacerbating the problem that it must solve. In this regard, it is often necessary to carefully select the restart interval.

In recent years, some aspects of the restart problem have been studied. In [1, 2], a stochastic restart model is proposed to minimize the shutdown time. The probability of completing the job during a restart was maximized in [3]. In these works, the authors considered an individual user sending independent tasks that must be completed in accordance with some distribution of the completion time. Restart has therefore been well studied for scenarios where only one user applies this technique.

In this article, consider the case when a restart is applied by several users in one or more shared resources. As a model, we will use G-networks with signals. The signals in our model are restarted as they delete the random job in the queue. The restarted job is returned to the network system queue as other type of requests and processed with a different processing intensity during restart. In our model, we will use phase-type distributions (PH) for the distribution of service time [4] to be able to reflect the characteristics of real systems. Such a distribution is more general than the exponential distribution.

The queueing theory with signals has received considerable attention since the original article on positive and negative customers [5], published by E. Gelenbe 27 years ago. Traditional simulation systems of QN are used to represent competition among customers for some resources. The client tasks are moved from the server to the server, they wait for service, but do not interact with each other. Signals are used to change these rules. In a G-network with signals, clients are allowed to change signals at the end of their service and the signals interact when they arrive in the queue with client tasks already present in the queue. In addition, signals are never queued. They try to interact with the tasks of the clients and immediately disappear.

For the first time in [6], signals were introduced as a negative claim. A negative customer removes the positive upon its arrival in the queue. A negative customer is never placed in the queue. Positive customers are ordinary claims that are queued and received service or deleted by negative customers. Under typical assumptions for Markov queueing networks (incoming Poisson flow for both types of claims, exponential service time for positive claims, Markov routing of claims, open topology, independence) E. Gelenbe proved that in such a network the stationary states probabilities have a multiplicative form. Flow equations in G networks have some unusual properties: they are neither linear, as in closed queueing networks, nor compressed as in Jackson's networks. Therefore, the existence of the solution was proved in [7] by a new method from the theory of equations with a fixed point.

At present, the description proposed by Cheo and his co-authors in [8] looks acceptable for the analysis of queueing networks with signals. A completely different approach based on algebraic methods of analyzing random processes was proposed by Harrison in [9, 10]. These methods were used to study many new signals: triggers that redirect claims between queues, catastrophes that kill all claims in the queue [11, 12], reset [13], synchronize incoming threads in the queues [14], signals that change the type service requests [15].

Here we present a different result for G-networks, when the effect of the signals restarts the service of claims. The service time according to the PH distributions, which depend on the type of the claim.

Note that, the results obtained in this article (with restart of a customer) is a special case of the G-network with resets that have published some years ago [16, 17].

2. Model and analysis

We will consider an open G-network with n queueing systems (QS). In the network serviced r types of positive customers and one type of negative customer (also referred to as signals, the effect of which is to restart one customer in the queue). Simple flow of customers of type c arrives to the i-th system from the external environment with the rate λ_{0ic}^{+}, additional simplest signal flow also arrives with the rate λ_{0i}^{-}, $i=\overline{1,\ n}$, $c=\overline{1,\ r}$. We assume that all customer flows arriving to QS are independent.

The service discipline of positive customers is processor sharing (PS). Suppose that service time is distributed by a phase basis. Let $\mu_{ic}^{(h)}$ the rate of service customer of type c in the i-th QS in phase h, $i=\overline{1,\ n}$, $c=\overline{1,\ r}$, $h=\overline{1,\ H}$. The matrix of probability transitions H_{ic} describe the phase changes of customers of type c in the i-th queue. Without loss of generality, we assume that phase-type distributions that describe the service time follows the rules: 1. the initial state has index 1; 2. the output state has index 0. Thus, servicing for the customer of type c in the i-th QS is a transition from state 1 to state 0 with the following matrix H_{ic}. We have:

$$\forall i,c,h \ \sum_{q=0}^{H} H_{ic}\left(h,q\right)=1 \qquad (1)$$

According to the Markov transition matrix a customer of type c at the moment of completing its service in the i-th QS (i.e. the transition from phase h to phase 0 in H_{ic}) can join the j-th as a positive customer of type s with the probability p_{icjs}^{+}. It can also leave the network with the probability p_{i0c}. Suppose that a customer cannot return to the queue that it just left: $p_{icis}^{+}=0$ for all i, c, and s. Naturally, that:

$$\forall i,c \ \sum_{j=1}^{n}\sum_{s=1}^{r} p_{icjs}^{+}+p_{i0c}=1. \qquad (2)$$

The network state descripted by the vector $\vec{k}(t)=(\vec{k}_1,\vec{k}_2,...,\vec{k}_n,t)$ (dimension $n\times r\times h$) where component $\vec{k}_i$ indicates the state of the i-th QS. The state of the i-th QS is given by the vector $\vec{k}_{ic}^{(h)}$ for all indexes of type c and index of phase h. In addition, note that $|\vec{k}_i|$ - the total number of customers in the i-th QS. It is clear that $\vec{k}(t)$ the Markov chain.

The signals do not stay in the network. Upon arrival in the queue, the signal interacts with the selected customer and then disappears instantly. If on arrival, the queue is already empty, it also instantly disappears. The selected customer is randomly selected according to the state-dependent distribution that simulates the PS service discipline. In the state $\vec{k}_i$, the probability of selecting of the customer is equal to $\frac{|\vec{k}_{ic}^{(h)}|}{|\vec{k}_i|}u(|\vec{k}_i|)$, and the signal is triggered with probability $\alpha_{ic}^{(h)}$. The result is a request restart: this customer (it has class c and phase h) is routed as a customer of type s in phase 1 with the probability β_{ics}. Suppose that for all c, $\beta_{icc}=0$. We have:

$$\forall c \sum_{s=1}^{r}\beta_{ics}=1. \tag{3}$$

Note that at the end of the service we do not allow the customer to become a signal. In our model this is not required because we do not want to represent a join restart of a group of customers.

We introduce some notation. Let $I_{ic}^{(h)}$ be a zero dimensional $n\times r\times h$ vector, with the exception of a component with the number $H(r(i-1)+c)+h$ which is equal to 1; $P(\vec{k},t)$ - nonstationary probability distribution of the network state $(\vec{k},t)$, if it exists at time t; $u(x)=\begin{cases}1, x>0;\\ 0, x\le 0.\end{cases}$ is the Heaviside function.

Let $(\vec{k}_i+I_{ic}^{(h)})$ (respectively $(\vec{k}_i-I_{ic}^{(h)})$ the state of the i-th QS, obtained by adding (correspondingly decreasing) one customer of type c at the maintenance phase h. Note that $M_{ic}^{(h)}(\vec{k}_i)$ the intensity of service of customers of type c at phase h in the queue i. Since the servicing discipline under consideration is a processor sharing $M_{ic}^{(h)}(\vec{k}_i)$ can be written as a function $\mu_{ic}^{(h)}$:

$$M_{ic}^{(h)}(\vec{k}_i)=\mu_{ic}^{(h)}\frac{|\vec{k}_{ic}^{(h)}|}{|\vec{k}_i|}u(|\vec{k}_i|). \tag{4}$$

Since the selection of customers according to the PS discipline, the probability of restarting a customer of type c in phase h, when the QS is in the state $\vec{k}_i$ is equal to:

$$Q_{ic}^{(h)}\left(\vec{k}_i\right)=\alpha_{ic}^{(h)}\frac{\left|\vec{k}_{ic}^{(h)}\right|}{\left|\vec{k}_i\right|}u\left(\left|\vec{k}_i\right|\right). \tag{5}$$

The nonstationary state probabilities of the considered network in this case will satisfy the Kolmogorov different-difference equations (DDE) system

$$\begin{aligned}\frac{dP\left(\vec{k},t\right)}{dt}=&-\sum_{i=1}^{n}\sum_{c=1}^{r}\left[\lambda_{oic}^{+}+\sum_{h=1}^{H}M_{ic}^{(h)}\left(\vec{k}_i\right)u\left(\left|\vec{k}_{ic}^{(h)}\right|\right)+\sum_{h=1}^{H}\lambda_{0i}^{-}Q_{ic}^{(h)}\left(\vec{k}_i\right)u\left(\left|\vec{k}_{ic}^{(h)}\right|\right)\right]P\left(\vec{k},t\right)+\\
&+\sum_{i=1}^{n}\sum_{c=1}^{r}\lambda_{0ic}^{+}u\left(\left|k_{ic}^{(1)}\right|\right)P\left(\vec{k}_i-I_{ic}^{(1)},t\right)+\\
&+\sum_{i=1}^{n}\sum_{c=1}^{r}\sum_{h=1}^{H}M_{ic}^{(h)}\left(\vec{k}_i+I_{ic}^{(h)}\right)p_{i0c}H_{ic}(h,0)P\left(\vec{k}_i+I_{ic}^{(h)},t\right)+\\
&+\sum_{i=1}^{n}\sum_{c=1}^{r}\sum_{h=1}^{H}\sum_{q=1}^{H}M_{ic}^{(h)}\left(\vec{k}_i+I_{ic}^{(h)}-I_{ic}^{(q)}\right)H_{ic}(h,q)u\left(\left|\vec{k}_{ic}^{(q)}\right|\right)P\left(\vec{k}_i+I_{ic}^{(h)}-I_{ic}^{(q)},t\right)+\\
&+\sum_{i=1}^{n}\sum_{c=1}^{r}\sum_{j=1}^{n}\sum_{s=1}^{r}\sum_{h=1}^{H}M_{ic}^{(h)}\left(\vec{k}_i+I_{ic}^{(h)}\right)p_{icjs}^{+}H_{ic}(h,0)u\left(\left|\vec{k}_{js}^{(1)}\right|\right)P\left(\vec{k}_i+I_{ic}^{(h)}-I_{js}^{(1)},t\right)+\\
&+\sum_{i=1}^{n}\sum_{c=1}^{r}\sum_{h=1}^{H}\lambda_{0i}^{-}Q_{ic}^{(h)}\left(\vec{k}_i+I_{ic}^{(h)}\right)\sum_{s=1}^{r}\beta_{ics}u\left(\left|\vec{k}_{is}^{(1)}\right|\right)P\left(\vec{k}_i+I_{ic}^{(h)}-I_{1s}^{(1)},t\right).\end{aligned} \tag{6}$$

Let us explain the right-hand side of this equation. The second sum corresponds to the external arrival of customers. The third summa is used to describe the end of service and leaving of customers, while the fourth is related to the completion of the phase h of servicing. With the fifth sum, the case is considered when the customer of type c leaves the i-th QS and passes to j-th QS as a customer of type s. The last sum is associated with a restart: the signal arriving to the i-th QS restarts the request (customer) of type c and of phase h, which appends queue i to the customer of type s in step 1.

3. The determination of the network state probabilities by the method of successive approximations, combined with the method of series

The DDE system (6) can be represented as:

$$\frac{dP\left(\vec{k},t\right)}{dt}=-\Lambda\left(\vec{k}\right)P\left(\vec{k},t\right)+\sum_{i,j=0}^{n}\sum_{s,c=0}^{r}\sum_{h,q=0}^{H}\mathrm{T}_{ichjsq}\left(\vec{k}\right)P\left(\vec{k}+I_{ic}^{(h)}-I_{js}^{(q)},t\right), \tag{7}$$

$$\Lambda\left(\vec{k}\right)=-\sum_{i=1}^{n}\sum_{c=1}^{r}\left[\lambda_{oic}^{+}+\sum_{h=1}^{H}M_{ic}^{(h)}\left(\vec{k}_i\right)u\left(\left|\vec{k}_{ic}^{(h)}\right|\right)+\sum_{h=1}^{H}\lambda_{0i}^{-}Q_{ic}^{(h)}\left(\vec{k}_i\right)u\left(\left|\vec{k}_{ic}^{(h)}\right|\right)\right],$$

$$\mathrm{T}_{ichjsq}\left(\vec{k}\right)=\delta_{0i}\delta_{0c}\delta_{q1}\delta_{h0}\lambda_{0js}^{+}u\left(\left|k_{js}^{(1)}\right|\right)+\delta_{j0}\delta_{s0}\delta_{q0}M_{ic}^{(h)}\left(\vec{k}_i+I_{ic}^{(h)}\right)p_{i0c}H_{ic}\left(h,0\right)+$$
$$+\delta_{ji}\delta_{sc}\delta_{qh}M_{ic}^{(h)}\left(\vec{k}_i+I_{ic}^{(h)}-I_{ic}^{(q)}\right)H_{ic}\left(h,q\right)u\left(\left|\vec{k}_{ic}^{(q)}\right|\right)+$$
$$+\delta_{q1}M_{ic}^{(h)}\left(\vec{k}_i+I_{ic}^{(h)}\right)p_{icjs}^{+}H_{ic}\left(h,0\right)u\left(\left|\vec{k}_{js}^{(1)}\right|\right)+\delta_{j1}\delta_{q1}\lambda_{0i}^{-}Q_{ic}^{(h)}\left(\vec{k}_i+I_{ic}^{(h)}\right)\beta_{ics}u\left(\left|\vec{k}_{is}^{(1)}\right|\right).$$

It follows from (7) that

$$P\left(\vec{k},t\right)=e^{-\Lambda\left(\vec{k}\right)t}\left(P\left(\vec{k},0\right)+\right.$$
$$\left.+\int_0^t e^{\Lambda\left(\vec{k}\right)x}\sum_{i,j=0}^{n}\sum_{s,c=0}^{r}\sum_{h,q=0}^{H}\mathrm{T}_{ichjsq}\left(\vec{k}\right)P\left(\vec{k}+I_{ic}^{(h)}-I_{js}^{(q)},x\right)dx\right). \quad (8)$$

Let $P_q\left(\vec{k},t\right)$ - approximation $P\left(\vec{k},t\right)$ on the q-th iteration, $P_{q+1}\left(k,t\right)$ - solution of the system (7) obtained by the method of successive approximations. Then it follows from (8) that

$$P_{q+1}\left(\vec{k},t\right)=e^{-\Lambda\left(\vec{k}\right)t}\left(P\left(\vec{k},0\right)+\right.$$
$$\left.+\int_0^t e^{\Lambda\left(\vec{k}\right)x}\sum_{i,j=0}^{n}\sum_{s,c=0}^{r}\sum_{h,q=0}^{H}\mathrm{T}_{ichjsq}\left(\vec{k}\right)P_q\left(\vec{k}+I_{ic}^{(h)}-I_{js}^{(q)},x\right)dx\right). \quad (9)$$

As an initial approximation, we take the stationary distribution $P_0\left(k,t\right)=P\left(k\right)=\lim_{t\to\infty}P\left(k,t\right)$, which satisfies relation

$$\Lambda\left(\vec{k}\right)P\left(\vec{k}\right)=\sum_{i,j=0}^{n}\sum_{s,c=0}^{r}\sum_{h,q=0}^{H}\mathrm{T}_{ichjsq}\left(\vec{k}\right)P\left(\vec{k}+I_{ic}^{(h)}-I_{js}^{(q)}\right). \quad (10)$$

The following assertions are valid for successive approximations.

Theorem 1. *The sequential approximations* $P_q\left(\vec{k},t\right)$, $q=0,1,2,...$, *converge when* $t\to\infty$ *to the stationary solution of the system of equations (7).*

Proof. It is obvious that the expression for the first approximation, according to (9) and (10), has the form

$$P_1\left(\vec{k},t\right)=e^{-\Lambda\left(\vec{k}\right)t}\left(P\left(\vec{k},0\right)+\int_0^t e^{\Lambda\left(\vec{k}\right)x}\sum_{i,j=0}^{n}\sum_{s,c=0}^{r}\sum_{h,q=0}^{H}\mathrm{T}_{ichjsq}\left(\vec{k}\right)P_0\left(\vec{k}+I_{ic}^{(h)}-I_{js}^{(q)},x\right)dx\right)=$$
$$=e^{-\Lambda\left(\vec{k}\right)t}\left(P\left(\vec{k},0\right)+\int_0^t e^{\Lambda\left(\vec{k}\right)x}\sum_{i,j=0}^{n}\sum_{s,c=0}^{r}\sum_{h,q=0}^{H}\mathrm{T}_{ichjsq}\left(\vec{k}\right)P\left(\vec{k}+I_{ic}^{(h)}-I_{js}^{(q)}\right)dx\right)=$$
$$=e^{-\Lambda\left(\vec{k}\right)t}\left(P\left(\vec{k},0\right)+\frac{1}{\Lambda\left(\vec{k}\right)}\left(e^{\Lambda\left(\vec{k}\right)t}-1\right)\sum_{i,j=0}^{n}\sum_{s,c=0}^{r}\sum_{h,q=0}^{H}\mathrm{T}_{ichjsq}\left(\vec{k}\right)P\left(\vec{k}+I_{ic}^{(h)}-I_{js}^{(q)}\right)\right)=$$

$$= e^{-\Lambda(\vec{k})t} P(\vec{k},0) + \frac{1}{\Lambda(\vec{k})}\left(1 - e^{-\Lambda(\vec{k})t}\right) \sum_{i,j=0}^{n} \sum_{s,c=0}^{r} \sum_{h,q=0}^{H} \mathrm{T}_{ichjsq}(\vec{k}) P\left(\vec{k} + I_{ic}^{(h)} - I_{js}^{(q)}\right) \xrightarrow[t\to\infty]{}$$

$$\xrightarrow[t\to\infty]{} \frac{1}{\Lambda(\vec{k})} \sum_{i,j=0}^{n} \sum_{s,c=0}^{r} \sum_{h,q=0}^{H} \mathrm{T}_{ichjsq}(\vec{k}) P\left(\vec{k} + I_{ic}^{(h)} - I_{js}^{(q)}\right).$$

It follows that when $q = 1$ the theorem is satisfied. Suppose that the theorem is true until q-th iteration. Then from (9), (10) and L'Hospital's rule we have:

$$\lim_{t\to\infty} P_{q+1}(\vec{k},t) = \lim_{t\to\infty} \frac{P(\vec{k},0) + \int_0^t e^{\Lambda(\vec{k})x} \sum_{i,j=0}^{n} \sum_{s,c=0}^{r} \sum_{h,q=0}^{H} \mathrm{T}_{ichjsq}(\vec{k}) P_q\left(\vec{k} + I_{ic}^{(h)} - I_{js}^{(q)}, x\right) dx}{e^{\Lambda(\vec{k})t}} =$$

$$= \lim_{t\to\infty} \frac{e^{\Lambda(\vec{k})t} \sum_{i,j=0}^{n} \sum_{s,c=0}^{r} \sum_{h,q=0}^{H} \mathrm{T}_{ichjsq}(\vec{k}) P_q\left(\vec{k} + I_{ic}^{(h)} - I_{js}^{(q)}, t\right)}{\Lambda(\vec{k}) e^{\Lambda(\vec{k})t}} =$$

$$= \lim_{t\to\infty} \frac{\sum_{i,j=0}^{n} \sum_{s,c=0}^{r} \sum_{h,q=0}^{H} \mathrm{T}_{ichjsq}(\vec{k}) P_q\left(\vec{k} + I_{ic}^{(h)} - I_{js}^{(q)}, t\right)}{\Lambda(\vec{k})} = P(\vec{k}).$$

Hence the theorem is also valid for q+1. Therefore, using the method of mathematical induction, we obtain the assertion of the theorem.

Theorem 2. *The sequence* $\left\{P_q(\vec{k},t)\right\}, q = 0,1,2,...,$ *constructed according to the scheme (9), for any zeroth approximation* $P_0(\vec{k},t)$ *that is bounded with respect to t,* $0 \le P_0(\vec{k},t) \le 1$, *converges when* $m \to \infty$ *to the unique solution of system (7).*

Proof. As $P_0(\vec{k},t)$ bounded with respect to t, then by (9) $P_1(\vec{k},t)$ is also bounded, so

$$\left|P_1(\vec{k},t) - P_0(\vec{k},t)\right| \le C(\vec{k}). \tag{11}$$

We show that inequality

$$\left|P_q(\vec{k},t) - P_{q-1}(\vec{k},t)\right| \le C^* \alpha_1^{q-1} \frac{t^{q-1}}{(q-1)!}, \tag{12}$$

where

$$\max_{\vec{k}} \alpha_1(\vec{k}) = \alpha_1^*,\ \alpha_1(\vec{k}) = \sum_{i,j=1}^{n} \sum_{c,s=1}^{r} \sum_{h,q=1}^{H} \mathrm{T}_{ichjsq}(\vec{k}),\ \max_{\vec{k}} C(\vec{k}) = C^*. \tag{13}$$

When $q = 1$, according to (6) this inequality holds. Suppose that it holds for $q = N$, and we show, using (4), its validity when $q = N+1$. We have:

$$\left|P_{N+1}\left(\vec{k},t\right)-P_N\left(\vec{k},t\right)\right| = \left|e^{-\Lambda\left(\vec{k}\right)t}\left(P\left(\vec{k},0\right)+\right.\right.$$

$$\left.+\int_0^t e^{\Lambda\left(\vec{k}\right)x}\sum_{i,j=0}^{n}\sum_{s,c=0}^{r}\sum_{h,q=0}^{H}\mathrm{T}_{ichjsq}\left(\vec{k}\right)P_N\left(\vec{k}+I_{ic}^{(h)}-I_{js}^{(q)},x\right)dx\right)-$$

$$\left.-e^{-\Lambda\left(\vec{k}\right)t}\left(P\left(\vec{k},0\right)+\int_0^t e^{\Lambda\left(\vec{k}\right)x}\sum_{i,j=0}^{n}\sum_{s,c=0}^{r}\sum_{h,q=0}^{H}\mathrm{T}_{ichjsq}\left(\vec{k}\right)P_{N-1}\left(\vec{k}+I_{ic}^{(h)}-I_{js}^{(q)},x\right)dx\right)\right|\le$$

$$\le e^{-\Lambda\left(\vec{k}\right)t}\int_0^t e^{\Lambda\left(\vec{k}\right)x}\sum_{i,j=0}^{n}\sum_{s,c=0}^{r}\sum_{h,q=0}^{H}\mathrm{T}_{ichjsq}\left(\vec{k}\right)\times$$

$$\times\left|P_N\left(\vec{k}+I_{ic}^{(h)}-I_{js}^{(q)},x\right)-P_{N-1}\left(\vec{k}+I_{ic}^{(h)}-I_{js}^{(q)},x\right)\right|dx\le$$

$$\le C^*\alpha_1^{N-1}e^{-\Lambda\left(\vec{k}\right)t}\int_0^t e^{\Lambda\left(\vec{k}\right)x}\sum_{i,j=0}^{n}\sum_{s,c=0}^{r}\sum_{h,q=0}^{H}\mathrm{T}_{ichjsq}\left(\vec{k}\right)\frac{x^{N-1}}{(N-1)!}dx\le C^*\alpha_1^N e^{-\Lambda\left(\vec{k}\right)t}\times$$

$$\times\int_0^t e^{\Lambda\left(\vec{k}\right)x}\frac{x^{N-1}}{(N-1)!}dx.$$

Because the $e^{-\Lambda\left(\vec{k}\right)t}e^{\Lambda\left(\vec{k}\right)x}\le 1$ when $x\in[0,t]$, then

$$e^{-\Lambda\left(\vec{k}\right)t}\int_0^t e^{\Lambda\left(\vec{k}\right)x}\frac{x^{N-1}}{(N-1)!}dx\le\int_0^t\frac{x^{N-1}}{(N-1)!}dx=\frac{t^N}{N!}. \tag{14}$$

As $\lim\limits_{q\to\infty}P_q\left(\vec{k},t\right)=\lim\limits_{q\to\infty}\left(P_0\left(\vec{k},t\right)+\sum\limits_{n=0}^{m-1}\left(P_{q+1}\left(\vec{k},t\right)-P_q\left(\vec{k},t\right)\right)\right)=P_0\left(\vec{k},t\right)+$ $+\sum\limits_{q=0}^{\infty}\left(P_{q+1}\left(\vec{k},t\right)-P_q\left(\vec{k},t\right)\right)\le P_0\left(\vec{k},t\right)+C^*\sum\limits_{q=0}^{\infty}\frac{\left(\alpha_1^*t\right)^q}{q!}=P_0(k,t)+C^*e^{\alpha_1^*t}$, i.e. limit of the sequence $\left\{P_q\left(\vec{k},t\right)\right\}, q=0,1,2,...$, exists, we denote it by $P_\infty\left(\vec{k},t\right)$. If we substitute $P_\infty\left(\vec{k},t\right)$ in (4) instead of $P\left(\vec{k},t\right)$, it is clear that $P_\infty\left(\vec{k},t\right)$ is a solution of the system of equations (2) satisfying the initial conditions $P_\infty\left(\vec{k},0\right)=P\left(\vec{k},0\right)$ according to the conditions of the preceding theorem.

The uniqueness of the solution of the system of equations (7) is proved similarly, as in [18].

Theorem 3. *Any approximation* $P_q\left(\vec{k},t\right)$, $q \ge 1$ *is representable in the form of a convergent power series*

$$P_q\left(\vec{k},t\right)=\sum_{l=0}^{\infty} d_{ql}^{+-}\left(\vec{k}\right)t^l, \tag{15}$$

whose coefficients satisfy the recurrence relations:

$$d_{q+1l}^{+-}\left(\vec{k}\right)=\frac{-\Lambda\left(\vec{k}\right)^l}{l!}\left\{P\left(k,0\right)+\sum_{u=0}^{l-1}\frac{\left(-1\right)^{u+1}u!}{\Lambda\left(\vec{k}\right)^{u+1}}D_{qu}^{+-}\left(\vec{k}\right)\right\}, l \ge 0,$$

$$D_{ql}^{+-}\left(\vec{k}\right)=\sum_{i,j=1}^{n}\left[\sum_{s,c=1}^{r}\sum_{h,q=1}^{H}T_{icqjsh}\left(\vec{k}\right)d_{ql}^{+-}\left(\vec{k}+I_{ic}^{(h)}-I_{js}^{(q)}\right)\right]. \tag{16}$$

Proof. Let us prove that the coefficients of the power series (15) satisfy the recurrence relations (16). We substitute the successive approximations (15) in (9). Then, taking into account that

$$e^{-\Lambda\left(\vec{k}\right)t}\int_0^t e^{\Lambda\left(\vec{k}\right)x}x^l dx=\left[\frac{1}{\Lambda\left(\vec{k}\right)}\right]^{l+1} l!\sum_{j=l+1}^{\infty}\frac{\left[-\Lambda\left(\vec{k}\right)\right]^l}{j!}, l=0,1,2,...,$$

$$\sum_{l=0}^{\infty}d_{ql}^{+-}\left(\vec{k}\right)t^l=e^{-\Lambda\left(\vec{k}\right)t}P\left(\vec{k},0\right)+\sum_{l=0}^{\infty}\sum_{i,j=1}^{n}\left[\sum_{c,s=1}^{r}\sum_{h,q=1}^{H}T_{ichjq}\left(\vec{k}\right)d_{ql}^{+-}\left(\vec{k}+I_{ic}^{(h)}-I_{js}^{(q)}\right)\right].$$

Using the notation (16), this series can be rewritten in the form

$$\sum_{l=0}^{\infty}d_{ql}^{+-}\left(\vec{k}\right)t^l=e^{-\Lambda\left(\vec{k}\right)t}P\left(\vec{k},0\right)+\sum_{l=0}^{\infty}D_{ql}^{+-}\left(\vec{k}\right)\left[\frac{1}{\Lambda\left(\vec{k}\right)}\right]^{l+1} l!\sum_{u=l+1}^{\infty}\frac{\left[-\Lambda\left(\vec{k}\right)\right]^u}{u!}t^u.$$

After interchanging the summation indices and expanding $e^{-\Lambda\left(\vec{k}\right)t}$ in a series in powers of t, we have

$$\sum_{l=0}^{\infty}d_{ql}^{+-}\left(\vec{k}\right)t^l=\sum_{l=0}^{\infty}\frac{\left[-\Lambda\left(\vec{k}\right)\right]^l}{l!}\left\{P\left(\vec{k},0\right)+\sum_{u=0}^{l-1}\frac{(-1)^{u+1}u!}{\left[\Lambda\left(\vec{k}\right)\right]^{u+1}}D_{qu}^{+-}\left(\vec{k}\right)\right\}t^l. \tag{17}$$

Equating the coefficients of t^l in the left and right sides of expression (17), we obtain the relations (16) for the coefficients of the series (15).

The radius of convergence of the power series (15) is equal to $+\infty$. This can be shown similarly, as in [18].

Example. Consider the network of $n = 3$ QS with $r = 2$ types of positive customers with $H = 2$ phases of service. The intensity of receipt of positive customers at each phase is equal to $\lambda_{0ic}^{+} = 25,\ i,c = \overline{1,3}$, and negative – $\lambda_{0i}^{-} = 15,\ i = \overline{1,3}$. The intensity of service customers for each phase is equal to $\mu_{ic}^{(h)} = \dfrac{60}{2h+1},\ i,\ c = \overline{1,3},\ h = \overline{0,2}$. The transition probabilities for customers at each phase are equal to $H_{ic}(h,q) = 0{,}5,\ h \neq q$. Suppose also that $\beta_{ics} = \dfrac{1}{2};\ \alpha_{ic}^{(h)} = 0{,}1,\ c = \overline{1,2},\ h = \overline{0,2},\ p_{icjs}^{+} = p_{ic0} = 0{,}1,\ i,j,s,c = \overline{1,2}$. Let us find the state probability $\vec{k} = (1,2,3,2,3,1,1,2,3,2,3,1)$, if the state $(1,1,1,1,1,1,1,1,1,1,1,1)$ was the initial.

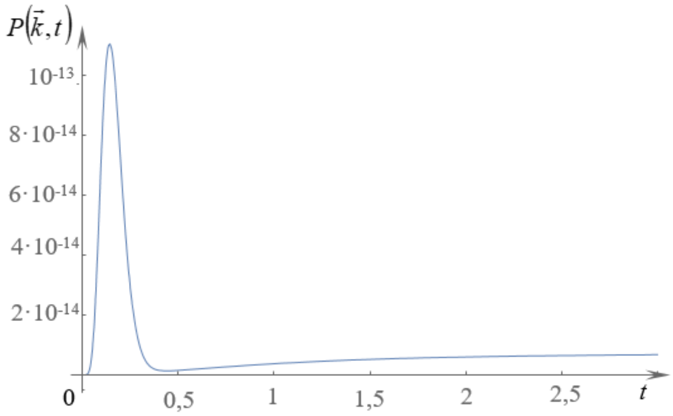

Fig. 1. State probability $(\vec{k},t)$ in the time interval $[0;6]$

4. Conclusions

In this paper, an investigation of the G-network with several types of positive customers with the phase type of distribution of their service time and one type of negative customers was conducted in the nonstationary mode. In our case, negative claims (customers) are signals. And the impact of signals is to restart one positive customer (claim) in the queue. The obtained results can be used to evaluate the restart effects in service-oriented systems. They can be used as a developer or operator of such systems, and to a limited extent by the user of services. For example, when a developer or a system operator may be wondering whether the system will be stable on restart and on what range the restart stability is maintained.

Further research in this area will be related to the calculation of the average characteristics of the network under investigation as well as the average response time for the query and the search for the optimal restart interval.

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