ISSN 1232-3829
tts 10/2013
Technika Transportu Szynowego koleje ▪ tramwaje ▪ metro

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A MATLAB-BASED SIMULATION OF POISSON PROCESS IN MASS TRANSPORTATION PROBLEMS

Abstract

The paper presents a method of solving mass transportation problems using Matlab. Matlab codes for simulation of the Poisson process sample paths were designed and applied in a probabilistic mass transportation problem FIFO. The simulated sample paths can be augmented with simulated paths of Brownian motion to numerically solve the problems driven by the Levy-type processes.

INTRODUCTION

A crucial role in queueing theory as well as in traffic generation models is being played by the Poisson process [9]. The wide variety of its applications includes routing and scheduling problems [2], queuing models of airport mass transit (cabs, busses) [1], modeling of traffic flows [10]. The more sophisticated models based on stochastic differential equations can require a generalization of the Poisson process, that is the Levy process [8, 9, 11]. A theorem on decomposition of the Levy process guarantees that the Levy process can be decomposed onto three elements: linear drift, Brownian motion and jump process (e. g. Poisson) [4]. This decomposition combined with numerical methods of discretization allows to analyze numerically solutions to the Levy-driven stochastic differential equations [11]. The simulation method is priceless in the cases when the analytical solution is difficult or even impossible. This work is a step forward the numerical solving of such equations. The stochastic differential equations driven by the Brownian motion has already been worked out [5,6]. The method is general and can be employed in various technical and financial problems [7].

The Poisson process usually is defined as follows [3]:

Definition

A stochastic process N is called Poisson if the conditions are all satisfied:

1. $N_0 = 0$ with probability 1,
2. the increments of the process are independent,
3. for any nonnegative s and positive t the increment $N_{s+t} - N_s$ has the Poisson distribution with parameter $\lambda t, \lambda > 0$, that is
$$P(N_{s+t} - N_s = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t},$$
for $k = 0, 1, 2, \ldots$,
4. the trajectories of the process are nondecreasing, left-continuous step functions with jumps equal to 1.

By the definition it can easily be shown (see [9], [11]) that the random variables $T_0, T_1, \ldots$ denoting the waiting time for the consecutive jumps are independent and have the exponential distribution with parameter λ, that is, the density of T_i is of the form:

$$f(x) = \begin{cases} 0, & x < 0 \\ \lambda e^{-\lambda x}, & x \geq 0. \end{cases}$$

The expectation of the distribution can be evaluated using by parts integration

$$\int_{-\infty}^{+\infty} x f(x) dx = \int_{0}^{\infty} \lambda x e^{-\lambda x} dx = \frac{1}{\lambda}.$$

These properties will be used in Matlab simulations.

1. MATLAB SIMULATION OF SAMPLE PATHS

The first simulation of the Poisson process trajectory consisting of fixed number of stairs will be performed in a loop containing two steps, after setting the parameter n – the number of stairs [7,9,12]:

– generation of n random variables T_i according to the exponential distribution, using the built-in random number generator,

– assigning a proper constant value (consecutive natural numbers starting from 0) over the interval $(T_{i-1}, T_i]$.

The following code in Matlab generates and plots a single trajectory of the Poisson process with given λ=1, number of stairs n=10.

Procedure 1

```
n=10
T=exprnd(1,n,1)
M=sum(T);
t=0:0.01:M;
st=0;
f=ones(size(t));
it=t<=T(1);
f(it)=0;
for j= 1:(n-1)
    st=st+T(j);
    it=st<t&t<=st+T(j+1);
    f(it)=j;
end
stairs(t, f)
```

The result of the procedure is shown in Fig. 1.

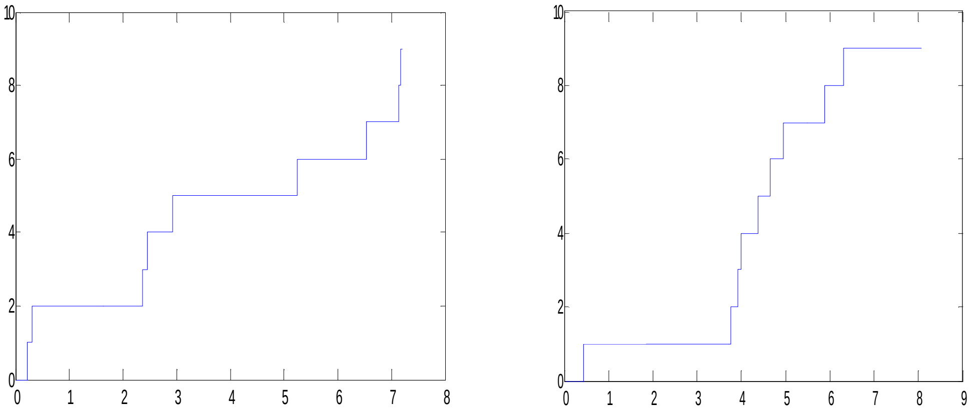

Fig 1. Two simulated trajectories of the Poisson process, given the number of stairs.

The code presented above can be used when the number of impulses is fixed, given at the beginning of the experiment. On the other hand, it is often necessary to observe the trajectory over a fixed interval – in this case the number of steps is usually unknown. Therefore, the approach must be slightly changed. Fix the interval [0,T]. The following steps will now be followed [7,9,12]:

– generation of random variables T_i according to the exponential distribution, using the built-in random number generator, until their sum exceeds the specified T,

– assigning a proper constant value (consecutive natural numbers starting from 0) over the interval $(T_{i-1}, T_i]$.

The following code in Matlab generates and plots a single trajectory of the Poisson process with given λ=1, over the specified interval [0,5].

Procedure 2

```
T0=5;
T(1)=0;
i=1;

while sum(T)<=T0
T(i)=exprnd(1,1,1);
i=i+1;
end
t=0:0.01:sum(T);
st=0;
n=i-1;
f=ones(size(t));
it=t<=T(1);
f(it)=0;
for j= 1:(n-1)
    st=st+T(j);
    it=st<t&t<=st+T(j+1);
    f(it)=j;
end
stairs(t, f);
axis([0 T0 0 n])
```

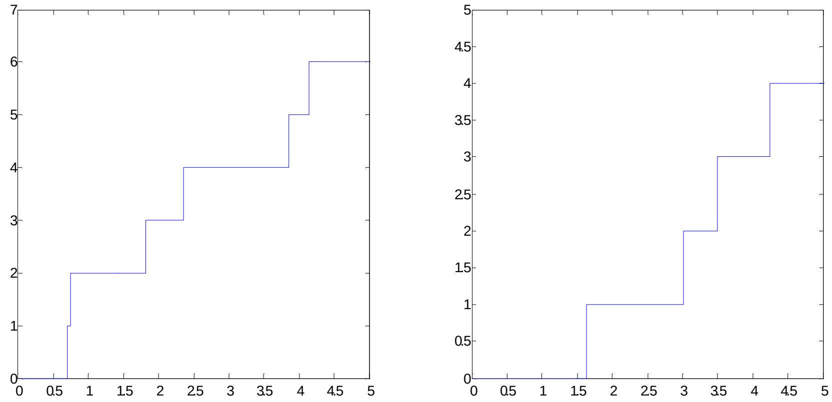

Fig.2. Two simulated trajectories of the Poisson process, given the interval [0,5].

Note that the sample paths presented in Fig. 2 consist of various number of steps: the first one 7 and the other 5.

2. APPLICATION IN MASS SERVICE PROBLEM

Consider the following problem, following the usual first in-first out (FIFO) scheme, inspired by [1], formulated and solved analytically in [11]:

A group taxis are waiting for passengers at the railway station. Passengers for those taxis arrive according to a Poisson process with an average of 20 passengers per hour. A taxi departs as soon as four passengers have been collected or ten minutes have expired since the first passenger got in the taxi. What is the probability that the first passenger has to wait ten minutes until the departure of the taxi?

As mentioned before the problem can be solved analytically:

$$P(N_{10} < 3) = \sum_{k=0}^{2} P(N_{10} = k) = \sum_{k=0}^{2} \frac{e^{-\frac{10}{3}} \left(\frac{10}{3}\right)^k}{k!} \cong 0.35.$$

That is why this problem was selected to illustrate the numerical approach using the Matlab simulation.

First, we take minute, as the unit of time. This implies the intensity of the process is equal to $\lambda = 1/3$, hence the Matlab parameter mu (the expectation), of the exponential distribution equals 3. We shall employ Procedure 2 in a loop to simulate a specified, large number of sample paths.

Procedure 3

```
T0=10;
counter=0;
for p=1:100
    T=zeros;
    i=1;
    while sum(T)<=T0
        T(i)=exprnd(3,1,1);
        i=i+1;
    end
    t=0:0.01:sum(T);
    st=0;
    n=i-1;
```

```
    f=ones(size(t));
    it=t<=T(1);
    f(it)=0;
    for j= 1:(n-1)
        st=st+T(j);
        it=st<t&t<=st+T(j+1);
        f(it)=j;
    end

    if f(size(t))<3 counter=counter+1; end;
end
counter
```

As the unbiased estimator of the derived probability one can take the maximum likelihood estimator equal to:

$$\hat{p} = \frac{n_0}{n},$$

where n denotes the sample size and n_0 is the number of the conducive elements in the sample.

After simulation 100 sample paths it turned out that the variable counter (n_0) was equal to 34. This denotes that 34 trajectories (out of 100) took value smaller than 3 at their terminal points. In terms of the example it means that in 34 (out of 100) cases one has to wait 10 minutes for the departure. Therefore, the probability is the specified event equals 0.34. The analytical solution is 0.35. One can conclude that the numerical result is very good.

CONCLUSIONS

Matlab package is a good tool in Monte Carlo simulations. In particular it can be easily employed in mass transportation and queueing problems. In the paper two Matlab codes for simulation of the Poisson sample paths were designed. One of this procedures was then run in a loop to assess numerically the probability of the specified waiting time in a FIFO problem. The numerical result was then compared with the one obtained analytically. It was concluded that the numerical result was highly satisfactory.

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SYMULACJE PROCESU POISSONA Z UŻYCIEM PAKIETU MATLAB W ZASTOSOWANIACH ZWIĄZANYCH Z MASOWĄ OBSŁUGĄ

Streszczenie

W artykule przedstawiono metodę rozwiązywania problemów z dziedziny teorii obsługi, opartą na symulacjach z użyciem pakietu Matlab. Przedstawiono procedury numeryczne służące do symulowania ścieżek procesu Poissona. Jedną z zaprezentowanych procedur zastosowano w szacowaniu prawdopodobieństwa oczekiwania w modelu FIFO.

Opracowane procedury można zsumować z symulacjami procesu Wienera otrzymując dyskretyzację bardziej wyrafinowanego procesu Levy'ego. Oznacza to możliwość rozwiązywania np. stochastycznych równań różniczkowych, w których komponenta stochastyczna jest właśnie procesem Levy'ego.

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