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Comparison evolutionary algorithms with Metropolis-Hastings method exemplified by high-fatigue Wohler curve parameter identification

Keywords

Wohler curve, Metropolis-Hastings, evolutionary algorithms probabilistic fatigue

Abstract

In this work, evolutionary algorithms together with the Metropolis-Hastings sampling technique have been used for parameter identification of the Wohler curve of duraluminum alloy 2024-T3. An evolutionary algorithm is a subset of evolutionary computation, a generic population-based metaheuristic optimization algorithm. The Metropolis-Hasting algorithm is one of the most widespread Markov chain Monte Carlo methods for posterior distribution estimation. In this contribution, both algorithms have been presented to estimate the probability density functions using Wohler parameters as a case study. Results were shown in terms of distribution shape and parameter correlations and the differences, arising from applied algorithms, have been compared. The information about parameter distributions of Wohler equation is useful to prepare risk analyses based on statistical safe life approach. The safe life approach can be met, for instance, in assessing the reliability of an aircraft.

1. Introduction

High-fatigue diagram shape optimization is a problem without an analytical solution. The problem can be approached by optimization algorithms.
Particularly noteworthy are evolutionary algorithms, due to the ease of adjustment to the input data and relatively short time required to solve a given problem. These metaheuristic optimization algorithms search the space alternative solutions in order to find the best, or potentially the best solution.
Markov chain Monte Carlo (MCMC) methods, are different from other system identification methods that are based on maximum likelihood, in that they allow describing the probability density function (pdf) of the inferred parameters, without any assumption on their shapes.
The aim of this work was to apply the evolutionary algorithms and consolidated theory of MCMC methods to identify the parameters of fatigue curves, commonly defined as *S-N* curves or Wohler curves.
The results of the algorithms have been critical analysed and compare with consolidated nonlinear fitting methods. The efficacy of the algorithms application in order to simplify the fatigue testing and facilitate the diagnostic inference about durability of structural components has been demonstrated.

2. Evolutionary algorithms

An evolutionary algorithm is a subset of evolutionary computation, a generic population-based metaheuristic optimization algorithm [4]-[5].
In genetic algorithms, string populations are known as chromosomes or genotype, which encodes a set of potential solutions to the analyzed problem evolves towards the better solution. *Figure 1* shows the schematic diagram of gene construction:

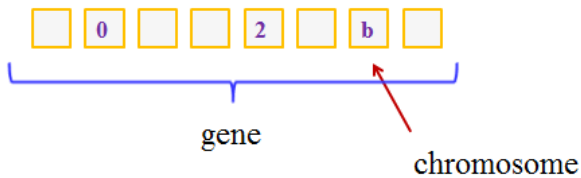

Figure 1. Example of gene

Development generally starts from a population of randomly generated individuals / solutions and continues in subsequent generations / iterations. In every generation, adaptation of each solution in the population is assessed then they are stochastically selected from the current population on the basis of

their suitability and modified (recombined and randomly mutated) to create a new generation of society, which is then used in the next iteration of the algorithm. The algorithm ends when the maximum number of generations has been reached, or when it reaches a satisfactory level of adaptation (the assumed level of solution precision). If the algorithm is terminated due to the maximum number of generations, a satisfactory solution may not be found. *Figure 2* illustrates flowchart of the evolutionary algorithm.

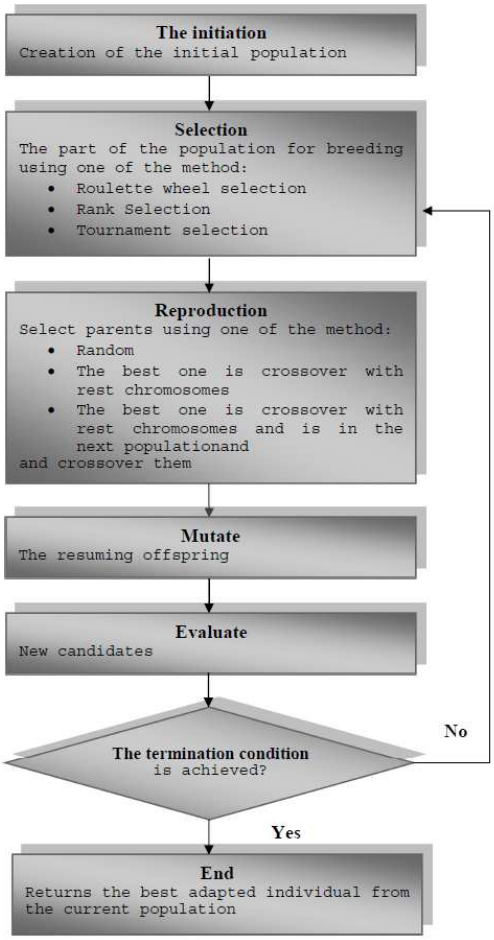

Figure 2. Flowchart illustrating an evolutionary algorithms model

3. Markov Chain model

According to [7], a First-order Markov Chain (FCM) with finite space E is a sequence of E-valued random variables ϑ_k (with k belonging to the set of natural numbers) such that the conditional distribution of ϑ_k (knowing all the discrete values ϑ_{k-m} with $m \geq 1$) is the same as the conditional distribution of ϑ_k given only ϑ_{k-1}, which can be described by:

$$P(\vartheta_k \mid \vartheta_{k-1}, \vartheta_{k-2}, ..., \vartheta_1) = P(\vartheta_k \mid \vartheta_{k-1}) \tag{1}$$

The transition kernel K drives the link between two subsequent random variables; it is a conditional probability satisfying the detailed-balance condition or reversibility condition of the chain:

$$K(\vartheta_k \mid \vartheta_{k+1}) \cdot \pi(\vartheta_{k+1}) = K(\vartheta_{k+1} \mid \vartheta_k) \cdot \pi(\vartheta_k) \tag{2}$$

where $\pi(\vartheta_k)$ is the stationary probability of the realization ϑ_k.

4. Metropolis-Hastings sampling technique for parameter identifications

Metropolis – Hasting algorithm is a generalization of the Gibs-sampler algorithm, which based on the theory of Markov chain Monte Carlo. This is the first historically and remains the most important MCMC algorithm. It is usually implemented to estimate parameter pdfs in presence of highly nonlinear functions and non-Gaussian distributions. Additionally, it estimates the noise associated to the observations given as input of the algorithm. It draws a series of dependent samples of the parameters according to the transition kernel K by the decomposition of the transition kernel itself.

Metropolis-Hastings algorithm is used to generate a reversible Markov chain. The aim is to draw dependent samples with transition kernel $K(\vartheta_{k+1}|\vartheta_k)$. According to the MH theory, the transition kernel $K(\vartheta_{k+1}|\vartheta_k)$ can be split into a proposal distribution q and an acceptance probability α that continue to satisfy (2). The splitting of the transition kernel becomes $K(\vartheta_{k+1}|\vartheta_k)=q(\vartheta_{k+1}|\vartheta_k)\alpha(\vartheta_{k+1}|\vartheta_k)$. In this way, the *detailed balance condition* change from (2) to (3), but the stationary distribution $\pi(\vartheta)$ is still unknown.

$$\begin{aligned} & q(\vartheta_k \mid \vartheta_{k+1}) \cdot \alpha(\vartheta_k \mid \vartheta_{k+1}) \cdot \pi(\vartheta_{k+1}) \\ & = q(\vartheta_{k+1} \mid \vartheta_k) \cdot \alpha(\vartheta_{k+1} \mid \vartheta_k) \cdot \pi(\vartheta_k) \end{aligned} \tag{3}$$

Let us assume a series of independent identically distributed observations y depending on the value of ϑ through a whatever nonlinear relationship $y = h(\vartheta)$. If the relation $h(\cdot)$ is known, the conditioned probability of ϑ given the observations y can be evaluated. According to Bayes' rule [1], the conditioned probability $\pi(\vartheta|y)$ follows the relation (4). The conditioned pdf can be evaluated by the substitution of $\pi(\vartheta|y)$ with the likelihood of ϑ given the observations $L(\vartheta|y)$.

$$\pi(\vartheta_k \mid y) \propto \pi(y \mid \vartheta_k) \cdot \pi(\vartheta_k) \tag{4}$$

Starting from equation (3), the acceptance probability $\alpha(\vartheta_k|\vartheta_{k-1})$ is extracted:

$$\alpha\left(\tilde{\vartheta}_{k+1}\middle|\vartheta_k\right)=min\left(\frac{\pi\left(\tilde{\vartheta}_{k+1}\middle|y\right)\cdot q\left(\vartheta_k\middle|\tilde{\vartheta}_{k+1}\right)}{\pi\left(\vartheta_k\middle|y\right)\cdot q\left(\tilde{\vartheta}_{k+1}\middle|\vartheta_k\right)},1\right) \quad (5)$$

It represents the probability to accept the *k-th* sample of ϑ_{k+1} given the previous sample ϑ. If the prior probability $\pi(\vartheta)$ has a symmetric distribution, $\pi(\cdot)$ can be removed form eq.(5). For a normally-distributed random noise affecting the observations *y*, the likelihood of the *k-th* sample $L(\vartheta_k|y)$ leads:

$$L\left(\vartheta_k\mid y\right)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(\frac{-\varphi_y\left(\vartheta_k\right)}{2\sigma^2}\right) \quad (6)$$

where $\varphi_y(\vartheta_k)$ is the cost function representing the error between the observation *y* and the simulation of the system based on the parameter sample ϑ_k [8], that is $\varphi_y(\vartheta_k)=[y-f(\vartheta_k)]^2$, while σ is the standard deviation of the uncertainty or noise affecting the observations. It is important to consider that σ belongs to the vector of parameter ϑ_k, thus a sample of σ is generated at each step *k*. New state $\tilde{\vartheta}_{k+1}$ is accepted with probability $\alpha\left(\tilde{\vartheta}_{k+1}\mid\vartheta_k\right)$ if a random number sampled from a uniform distribution U[0, 1] is less than $\alpha(\vartheta_{k+1}|\vartheta_k)$; otherwise it is refused and the current sample remains equal to the previous one ϑ_{k-1}. At the end of the algorithm, the conditional mean and variance of ϑ can be evaluated

The algorithm can be written as follows:

1. start from arbitrary values for the unknown parameter vector ϑ_0,
2. draw a proposal sample $\tilde{\vartheta}_k$ from the proposal density $q\left(\vartheta_{k+1}\mid\vartheta_k\right)$,
3. evaluate α using (5),
4. accept the new state $\tilde{\vartheta}_{k+1}$,
5. repeat steps 2 to 4 until the desired chain length is reached (k>N),
6. Skip k_0 samples to avoid the *burn-in* period of the chain, save a state every n_t Markov chain steps (*thinning* process) and compute the required estimates using:

$$E\left(\vartheta\mid y\right)\approx\frac{1}{N-k_0}\sum_{k=k_0+1}^{N}\vartheta_k$$

where ϑ_k are the samples selected from the Markov chain, k_0 is the transient period and *N* is the number of selected steps of the Markov chain.

5. Wohler curve

contrast to the static strength fatigue strength is characterized by a large scatter of the results for sets of samples. In order to construct the Wohler curve the results of experimental data equation (7) were used [2], [6]. Generally, the relationship between the amplitude of cycles and the fatigue life is exponential but a specific form of mathematical functions may be varies.

$$S=S_{\inf}\left(1+A/\left(N+G\right)^m\right) \quad (7)$$

where *A is* numerical constant, *G is* numerical constant governing low endurances, *m* is numerical exponent, *S* represents stress, *N* is the number of load cycles, endurance, S_{inf} can represents stress at infinite life.

6. Calculation and results

6.1. Input parameters

Samples of unnotched 2024-T3 aluminum alloy for various wrought products at longitudinal direction were obtained from [3], presented in *Table 1*:

Table 1. Data for Wohler curve

No. of samples	No of cycles	Maximum stress [MPa]	No. of samples	No of cycles	Maximum stress [MPa]
1	2500	234.4	11	28800	164.8
2	2970	234.4	12	64200	153.8
3	3530	241.3	13	70600	137.2
4	6500	206.2	14	235000	123.4
5	7710	206.2	15	210000	102.0
6	10100	206.2	16	409000	102.0
7	10100	193.1	17	505000	102.0
8	11700	191.7	18	754000	102.0
9	17500	191.7	19	3280000	88.9
10	17500	171.7	20	5290000	81.4

It was assumed that the individual gene is expressed in the form of a 4-element array containing the desired parameters of the equation (7) *A, G, m,* S_{inf}. Population size has been accepted as 1000 individuals. Selection of parameters of equation (7) was based on minimizing the objective function describing the tracking error δ calculated from the equation (8):

$$\delta=\sum_i\left(\frac{\sigma_i}{\sigma^*}-1\right)\cdot\left(1-\frac{\sigma^*}{\sigma_i}\right) \quad (6)$$

where:
σ_i - stress for the *i*-th sample calculated on the basis of the calculated parameters of the equation,
σ^* - stress the *i*-th sample of the fatigue tests.
Under these conditions, the algorithm calculation ended after 100 iterations to provide the best results with the required accuracy – this is a sufficient number of operations for such a large population in order to achieve algorithms convergence and the result sought. Over 2000 simulation of the evolutionary algorithm were carried out.
Table 2 shows the interested parameters and the corresponding distributions for evolutionary algorithm; the subscripts *low* and *up* mean *lower limit* and *upper limit*, respectively.

Table 2. Parameter distributions

ϑ	prior pdf
A	$U([A_{low},A_{up}])=U([20,120])$
G	$U([G_{low},G_{up}])=U([0,12000])$
m	$U([m_{low},m_{up}])=U([0,0.45])$
S_0	$U([S_{0low},S_{0up}])=U([50,75])$

The length of the M-H chain has been heuristically selected as 50000 samples. Starting parameters ϑ_0 for Metropolis-Hastings algorithm are described in *Table 3*.

Table 3. Starting parameters ϑ_0 of Wohler curve

A_0	120
G_0	40 000
m_0	0.4
$S_{inf\,0}$	24
σ_0	0

New samples $\tilde{\vartheta}_{k+1}$ are drawing from proposal probability q for Wohler curve parameters, which was selected as follows:

Table 4. Proposal probability q

$\tilde{A}_{k+1}$	$q(x)=\frac{1}{10\sqrt{2\pi}}\exp(\frac{-(x-A_k)^2}{2\cdot 10^2})$
$\tilde{G}_{k+1}$	$q(x)=\frac{1}{500\sqrt{2\pi}}\exp(\frac{-(x-G_k)^2}{2\cdot 500^2})$
$\tilde{m}_{k+1}$	$q(x)=\frac{1}{0.01\sqrt{2\pi}}\exp(\frac{-(x-m_k)^2}{2\cdot 0.01^2})$
$\tilde{S}_{\inf k+1}$	$q(x)=\frac{1}{1.4\sqrt{2\pi}}\exp(\frac{-(x-S_{\inf k})^2}{2\cdot 10^2})$
$\tilde{\sigma}_{k+1}$	$q(x)=\exp\left[\frac{1}{x\cdot 0.3\sqrt{2\pi}}\exp(\frac{-(\ln x-\sigma_k)^2}{2\cdot 0.3^2})\right]$

Prior distribution π for parameters is presented in *Table 5*.

Table 5. Prior distribution π

A_{k+1}	$\pi(x)=\frac{1}{20\sqrt{2\pi}}\exp(\frac{-(x-60)^2}{2\cdot 20^2})$
G_{k+1}	$\pi(x)=\frac{1}{3000\sqrt{2\pi}}\exp(\frac{-(x-4000)^2}{2\cdot 3000^2})$
m_{k+1}	$\pi(x)=\frac{1}{0.07\sqrt{2\pi}}\exp(\frac{-(x-0.3)^2}{2\cdot 0.07^2})$
$S_{inf\,k+1}$	$\pi(x)=\frac{1}{4\sqrt{2\pi}}\exp(\frac{-(x-60)^2}{2\cdot 4^2})$
σ_{k+1}	$\pi(x)=\frac{1}{x\cdot 0.5\sqrt{2\pi}}\exp(\frac{-(\ln x-0)^2}{2\cdot 0.5^2})$

6.2. Results

Figures 1-2 show the difference between the results from non-lineral regression, mean, mode and median value of the resulting parameters for both algorithms. *Figures 3-4* show randomly chosen results.

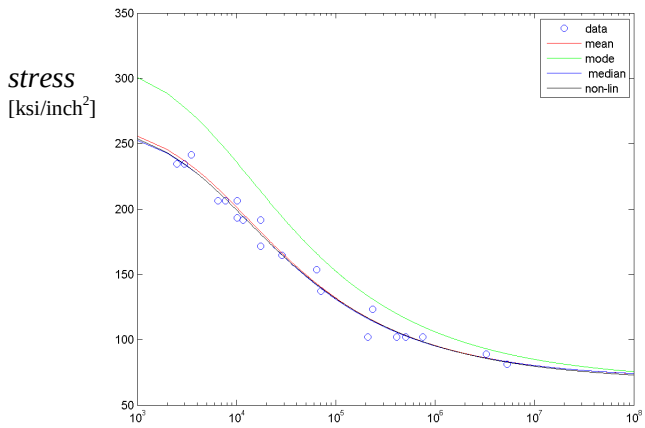

Figure 1. Evolutionary algorithm results

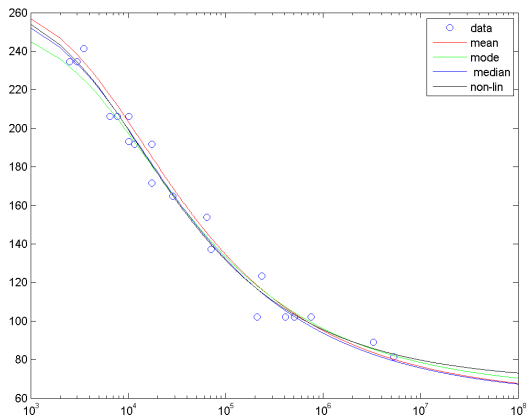

Figure 2. M-H results

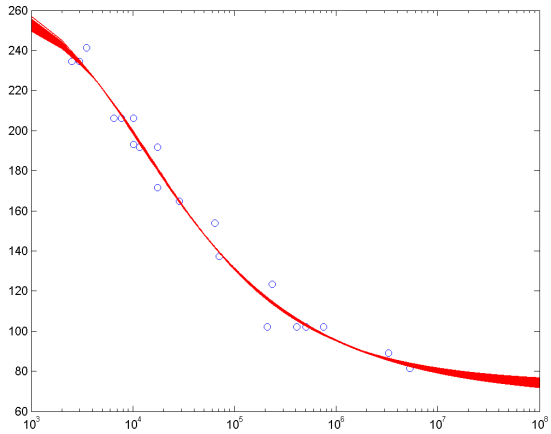

Figure 3. Randomly chosen evolutionary algorithm results

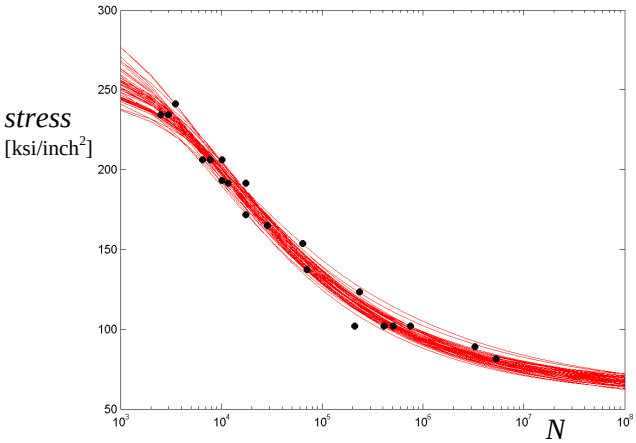

Figure 4. Randomly chosen M-H results

Figure 5 shows the estimation of *A*, *G*, *m*, S_{inf} and σ over time. The parameters governing the Wohler curve are well-estimated by the M-H algorithm.

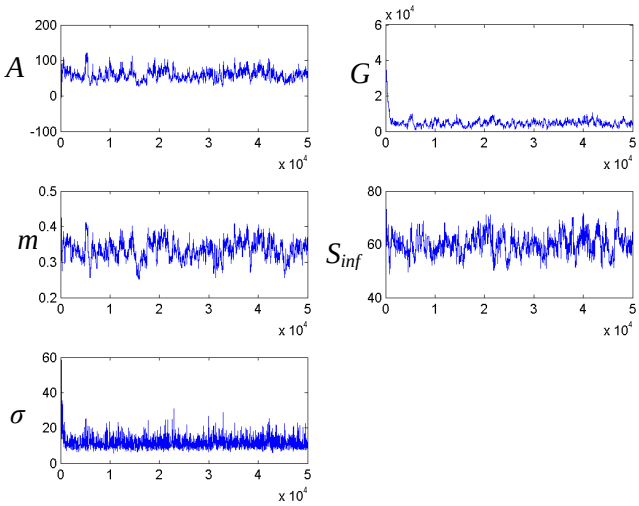

Figure 5. Parameters estimation through

Figures 6-9 show the posterior distribution of the parameters (*A*, *G*, *m*, S_{inf}) of both algorithms.

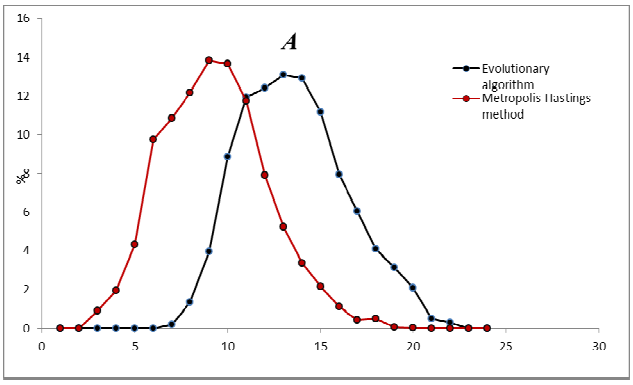

Figure 6. Distribution of parameter *A*, *G*, *m*, S_{inf}

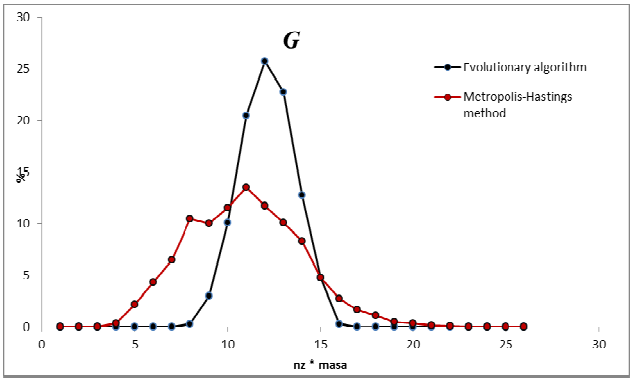

Figure 7. Distribution of parameter *A*, *G*, *m*, S_{inf}

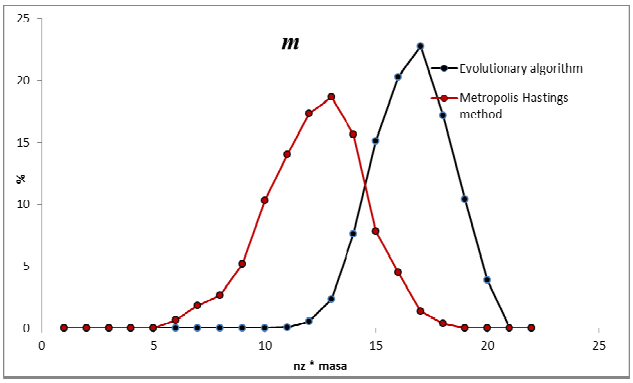

Figure 8. Distribution of parameter *A*, *G*, *m*, S_{inf}

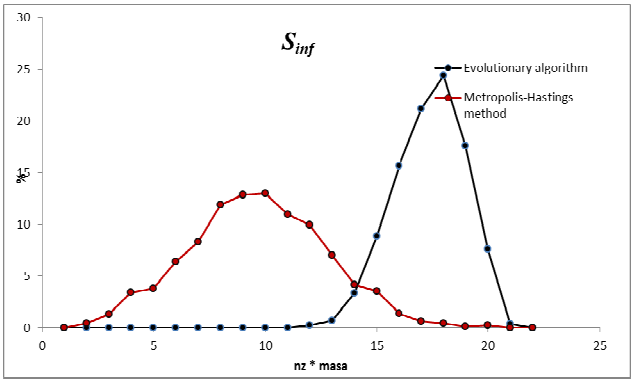

Figure 9. Distribution of parameter *A*, *G*, *m*, S_{inf}

The most common fitting technique is the nonlinear regression based on Levenberg-Marquardt algorithm. It has the restricting hypothesis of normally distributed parameters (like the hypothesis made in the calculation presented above), because it is not able to evaluate the shape of the probability density function parameters. The nonlinear fitting subroutine available in *Matlab*© has been used hereafter to analyze the results of the algorithms, which are shown in *Table 6*.

Table 6. Expected values of ϑ using a nonlinear fitting algorithm, evolutionary algorithms and MH algorithm

ϑ	μ_ϑ Nonlinear fitting	μ_ϑ Evolutionary algorithm	μ_ϑ Metropolis-Hastings
A	67.	80.35	59.08
G [cycles]	4796	5316	4698
m	0.3708	0.3893	0.3347
S_0 [MPa]	68.05	69.83	60.21
σ [MPa]	-	-	11.6

7. Conclusion

The possibility to use evolutionary algorithms and Metropolis-Hastings sampling technique for parameter identifications has been assessed in this work.

Both algorithms allows to draw samples from the posterior density function (the probability of model parameters conditioned on available data), thus representing both a statistical tool for the estimation of parameter uncertainty as well as a valid method for the updating of the prior knowledge on model parameters.

Evolutionary algorithms more accurately locate approximation of the experimental data to the Wohler curve.

The validity of the fitting has been proven by the comparison with the consolidated nonlinear fitting procedure based on the Levenberg-Marquardt algorithm.

The information about parameter distributions of the Wohler equation is useful to prepare risk analyses based on statistical safe life approach. The safe life approach can be used, for instance, in assessing the reliability of an aircraft.

References

[1] Ayyub, B. M. (2011). *Vulnerability, Uncertainty, and Risk - Analysis, Modeling, and Management.* American Society of Civil Engineers (ASCE).

[2] DEF STAN 00-970, Part 1/3 Section Leaflet 35 *Fatigue. Safe-life substantiation*.

[3] DOT/FAA/AR-MMPDS-01. (2003). *Metallic Materials Properties Development and Standardization* (*MMPDS*). U.S. Department of Transportation, Federal Aviation Administration.

[4] Gwiazda, T. (2007). *Algorytmy genetyczne. Kompendium.* Warszawa, Wydawnictwo Naukowe PWN SA.

[5] Jędrzejewski, T. (2007). *Algorytmy ewolucyjne*. Wakacyjne Warsztaty Wielodyscyplinarne.

[6] Leski, A., et. al. (2009). *Oszacowanie indywidualnego zużycia zmęczeniowego struktur określonej populacji statków powietrznych.* ITWL, Warsaw.

[7] Mattrand, C., Bourinet, J.M. & Theret, D. (2011). *Analysis of Fatigue Crack Growth under Random Load Sequences Derived from Military In-flight Load Data.* 26th ICAF Symposium, Montreal.

[8] Sbarufatti, C. (2013). *Fatigue crack monitoring of helicopter fuselages and life evaluation through sensor network.* Politecnico di Milano, Dipartimento di Meccanica, PhD thesis.