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Dedicated to Prof. Dušan D. Repovš
on the occasion of his 65th birthday

FRACTIONAL $p\&q$-LAPLACIAN PROBLEMS WITH POTENTIALS VANISHING AT INFINITY

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Abstract. In this paper we prove the existence of a positive and a negative ground state weak solution for the following class of fractional $p\&q$-Laplacian problems

$$(-\Delta)_p^s u + (-\Delta)_q^s u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = K(x)f(u) \quad \text{in } \mathbb{R}^N,$$

where $s \in (0,1)$, $1 < p < q < \frac{N}{s}$, $V : \mathbb{R}^N \to \mathbb{R}$ and $K : \mathbb{R}^N \to \mathbb{R}$ are continuous, positive functions, allowed for vanishing behavior at infinity, f is a continuous function with quasicritical growth and the leading operator $(-\Delta)_t^s$, with $t \in \{p,q\}$, is the fractional t-Laplacian operator.

Keywords: fractional $p\&q$-Laplacian, vanishing potentials, ground state solution.

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1. INTRODUCTION

In this work we study the existence of least energy weak solutions for the following class of fractional $p\&q$-Laplacian problems

$$(-\Delta)_p^s u + (-\Delta)_q^s u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = K(x)f(u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $s \in (0,1)$, $1 < p < q < \frac{N}{s}$, $V : \mathbb{R}^N \to \mathbb{R}$ and $K : \mathbb{R}^N \to \mathbb{R}$ are positive functions and f is a continuous function with quasicritical growth. The main operator $(-\Delta)_t^s$, with $t \in \{p,q\}$, denotes the fractional t-Laplacian operator which, up to a normalizing constant, may be defined as

$$(-\Delta)_t^s u(x) := 2 \lim_{\varepsilon \to 0^+} \int\limits_{\mathbb{R}^N \setminus \mathcal{B}_\varepsilon(x)} \frac{|u(x)-u(y)|^{t-2}(u(x)-u(y))}{|x-y|^{N+st}}\, dy \quad (x \in \mathbb{R}^N)$$

for any $u \in \mathcal{C}_c^\infty(\mathbb{R}^N)$, where $\mathcal{B}_\varepsilon(x) = \{y \in \mathbb{R}^N \,:\, |x-y| < \varepsilon\}$; we refer to [23] for more motivations on this operator.

Throughout the paper we will assume that $V, K : \mathbb{R}^N \to \mathbb{R}$ are continuous functions and we say that $(V, K) \in \mathcal{K}$ if the following hypotheses are satisfied (see [2]):

(VK_1) $V(x), K(x) > 0$ for all $x \in \mathbb{R}^N$ and $K \in L^\infty(\mathbb{R}^N)$,

(VK_2) if $\{A_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ is a sequence of Borel sets such that the Lebesgue measure $|A_n| \le R$, for all $n \in \mathbb{N}$ and for some $R > 0$, then

$$\lim_{r\to\infty} \int\limits_{A_n \cap \mathcal{B}_\varrho^c(0)} K(x)\, dx = 0,$$

uniformly in $n \in \mathbb{N}$, where $\mathcal{B}_\varrho^c(0) := \mathbb{R}^N \setminus \mathcal{B}_\varrho(0)$.

Furthermore, one of the following conditions occurs

(VK_3) $\dfrac{K}{V} \in L^\infty(\mathbb{R}^N)$,

(VK_4) there exists $m \in (q, q_s^*)$ such that

$$\frac{K(x)}{V(x)^{\frac{q_s^*-m}{q_s^*-p}}} \to 0 \qquad \text{as } |x| \to \infty.$$

Remark 1.1. We stress that assumption (VK_2) is weaker than any one of the following conditions:

(i) there are $r \ge 1$ and $\rho \ge 0$ such that $K \in L^r(\mathcal{B}_\rho^c(0))$,

(ii) $K(x) \to 0$ as $|x| \to \infty$,

(iii) $K(x) = K_1(x) + K_2(x)$, with K_1 and K_2 fulfilling (i) and (ii) respectively.

Let us point out that the hypotheses on the functions V and K characterize problem (1.1) as *zero mass* problem.

Regarding the nonlinearity f, we assume that $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and fulfills the following growth conditions in the origin and at infinity:

(f_1) $\displaystyle\lim_{|t|\to 0} \frac{f(t)}{|t|^{p-1}} = 0$ if (VK_3) holds,

$(\tilde{f}_1)$ $\displaystyle\lim_{|t|\to 0} \frac{f(t)}{|t|^{m-1}} = 0$ if (VK_4) holds, with $m \in (q, q_s^*)$ defined in (VK_4),

(f_2) $\displaystyle\lim_{|t|\to \infty} \frac{f(t)}{|t|^{q_s^*-1}} = 0.$

We suppose that the function f satisfies the Ambrosetti–Rabinowitz condition:

(f_3) there exists $\vartheta \in (q, q_s^*)$ such that

$$0 < \vartheta F(t) \le f(t)t \quad \text{for all } |t| > 0, \text{ where } F(t) := \int_0^t f(\tau)\, d\tau,$$

and furthermore we assume that

(f_4) the map $t \mapsto \dfrac{f(t)}{|t|^{q-1}}$ is strictly increasing for all $|t| > 0$.

Remark 1.2. As a model of nonlinearity satisfying the above assumptions we can take

$$f(t) = (t^+)^m, \text{ where } t^+ = \max\{t, 0\},$$

also, the function

$$f(t) = \begin{cases} \log 2(t^+)^m & \text{if } t \leq 1, \\ t\log(1+t) & \text{if } t > 1 \end{cases}$$

for some $m \in (q, q_s^*)$.

When $s = 1$, problem (1.1) boils down to a $p\&q$ elliptic problem of the type

$$-\Delta_p u - \Delta_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = K(x)f(u) \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

As underlined in [26], this equation appears in a lot of applications such as biophysics, plasma physics and chemical reaction design. We point out that classical $p\&q$ Laplacian problems in bounded or unbounded domains have been studied by several authors; see for instance [15, 16, 30, 32, 36, 37, 42] and references therein.

In the past years there has been a great attention on the existence of nontrivial solutions for (1.2) in the special case $p = q = 2$, that is the classical nonlinear Schrödinger equation

$$-\Delta u + V(x)u = K(x)f(u) \quad \text{in } \mathbb{R}^N,$$

where the potentials $V, K \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ are allowed for vanishing behavior at infinity and $f : \mathbb{R} \to \mathbb{R}$ is a nonlinearity satisfying suitable growth assumptions in the origin and at infinity. Such class of *zero mass* problem has been investigated by many authors using several variational methods; we refer the interested reader to [1, 4, 17, 18, 20] and references therein.

On the other hand, in the last decade nonlinear problems involving nonlocal operators have received a great interest from the mathematical community thanks to their intriguing structure and in view of their great application in several contexts such as obstacle problem, optimization, finance, phase transition, material science, anomalous diffusion, soft thin films, multiple scattering, quasi-geostrophic flows, water waves, and so on. For more details we refer to [28, 40].

Equation (1.1) with $p = q = 2$ appears in the study of standing wave solutions $\psi(x,t) = u(x)e^{-\imath\omega t}$ to the following fractional Schrödinger equation

$$\imath\hbar\frac{\partial\psi}{\partial t} = \hbar^2(-\Delta)^s\psi + W(x)\psi - f(|\psi|) \quad \text{in } \mathbb{R}^N,$$

where $\hbar$ is the Planck constant, $W : \mathbb{R}^N \to \mathbb{R}$ is an external potential and f is a suitable nonlinearity. The fractional Schrödinger equation is one of the most important objects of the fractional quantum mechanics because it appears in problems involving nonlinear optics, plasma physics and condensed matter physics. This equation has been introduced for the first time by Laskin [35] as a result of expanding the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. Lately, the study of fractional Schrödinger equations has attracted the attention of many mathematicians and several papers appeared studying existence, multiplicity, regularity and asymptotic behavior of solutions to fractional Schrödinger equations assuming different conditions on the potential and considering nonlinearities with subcritical or critical growth; see [7, 9, 13, 21, 29, 33, 34, 39, 43].

Recently, a great attention has been devoted to the study of fractional p-Laplacian operator because both nonlocal and nonlinear phenomena appear in it. We refer the interested reader to [6, 10, 12, 27, 31, 38, 41] for some interesting existence, multiplicity and regularity results involving this nonlocal operator.

Differently from the local case, only few papers deal with fractional $p\&q$-Laplacian problems. Chen and Bao [25] obtained the existence, nonexistence and multiplicity of solutions to the following fractional $p\&q$-Laplacian equation

$$\begin{aligned}&(-\Delta)^s_p u + a(x)|u|^{p-2}u + (-\Delta)^s_q u + b(x)|u|^{q-2}u + \mu(x)|u|^{r-2}u\\&= \lambda h(x)|u|^{m-2}u \text{ in } \mathbb{R}^N\end{aligned}$$

where $\lambda \in \mathbb{R}$, $0 < s < 1 < q < p$, $r > 1$, $sp < N$, the functions $a(x), b(x), \mu(x)$ and $h(x)$ are nonnegative in $\mathbb{R}^N$, and the following three cases on p, q, r, m are considered: $p < m < r < p^*_s$, $\max\{p, r\} < m < p^*_s$, and $1 < m < q < r < p^*_s$. Using variational arguments and concentration-compactness lemma, in [8] the author established the existence of a nontrivial non-negative solution to

$$(-\Delta)^s_p u + (-\Delta)^s_q u + |u|^{p-2}u + |u|^{q-2}u = \lambda h(x) f(u) + |u|^{q^*_s-2}u \quad \text{in } \mathbb{R}^N,$$

where $s \in (0, 1)$, $1 < p < q < \frac{N}{s}$, $\lambda > 0$ is a parameter, h is a nontrivial bounded perturbation and f is a superlinear continuous function with subcritical growth. Subsequently, [19] obtained the existence of infinitely many nontrivial solutions for the class of (p, q) fractional elliptic equations involving concave-critical nonlinearities in bounded domains. Very recently, in [3] the authors studied the following class of problems

$$(-\Delta)^s_p u + (-\Delta)^s_q u + V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = f(u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

with $s \in (0, 1)$ and $1 < p < q < \frac{N}{s}$. Under suitable assumptions on the potential and the nonlinearity, but without requiring the Ambrosetti–Rabinowitz condition, the authors proved the existence of a ground state solution to (1.3) that concentrates around a minimum point of the potential V. Furthermore, a multiplicity result is established by using the Lyustenick–Schnirelmann category theory and the boundedness of solutions to (1.3). We also mention [11, 14] for problems in bounded domains.

Motivated by the above papers, in this work we are interested in the existence of nontrivial solutions for a fractional $p\&q$-Laplacian problem involving potentials allowed for vanishing behavior at infinity.

Our main result can be stated as follows:

Theorem 1.3. *Assume that* $(V,K)\in\mathcal{K}$ *and* f *satisfies* (f_1) *or* $(\tilde{f}_1)$ *and* (f_2)–(f_4). *Then problem* (1.1) *possesses a positive and a negative ground state weak solution.*

It is worth pointing out that problem (1.1) involves the fractional t-Laplacian $(-\Delta)^s_t$, with $s\in(0,1)$ and $t\in\{p,q\}$, which is not linear when $t\neq 2$, so we can not benefit of the s-harmonic extension due to Caffarelli-Silvestre [24]. Furthermore, a more accurate inspection will be needed with respect to the classical framework due to the non-Hilbertian structure of the involved fractional Sobolev spaces $W^{s,t}$, and some ideas contained in [9,13,16,34] will play a fundamental role to achieve the desired result. The proof is based on variational arguments and a key role is played by the mountain pass theorem [5].

The paper is organized as follows: in Section 2 we recall some useful lemmas which will be used along the paper. In Section 3 we show the existence of a positive and a negative ground state weak solution by means of mountain pass theorem.

2. VARIATIONAL FRAMEWORK

Let $1\leq r\leq\infty$ and $A\subset\mathbb{R}^N$. We denote by $|u|_{L^r(A)}$ the $L^r(A)$-norm of $u:\mathbb{R}^N\to\mathbb{R}$ belonging to $L^r(A)$. When $A=\mathbb{R}^N$, we will write $|u|_r$.

For $s\in(0,1)$, let $\mathcal{D}^{s,r}(\mathbb{R}^N)$ be the closure of $\mathcal{C}^\infty_c(\mathbb{R}^N)$ with respect to

$$[u]^r_{s,r}=\iint\limits_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^r}{|x-y|^{N+sr}}\,dxdy.$$

We define $W^{s,r}(\mathbb{R}^N)$ as the set of functions $u\in L^r(\mathbb{R}^N)$ such that $[u]_{s,r}<\infty$, endowed with the norm

$$\|u\|^r_{s,r}:=[u]^r_{s,r}+|u|^r_r.$$

Let us introduce the space

$$\mathbb{X}=\left\{u\in\mathcal{D}^{s,p}(\mathbb{R}^N)\cap\mathcal{D}^{s,q}(\mathbb{R}^N)\,:\,\int\limits_{\mathbb{R}^N}V(x)(|u|^p+|u|^q)\,dx<\infty\right\}$$

endowed with the norm

$$\|u\|:=\|u\|_{V,p}+\|u\|_{V,q},$$

where

$$\|u\|^t_{V,t}:=[u]^t_{s,t}+\int\limits_{\mathbb{R}^N}V(x)|u|^t\,dx,\quad t\in\{p,q\}.$$

We recall the following embedding:

Theorem 2.1. *Let $s \in (0,1)$ and $N > sp$. Then there exists a constant $S_* > 0$ such that for any $u \in \mathcal{D}^{s,p}(\mathbb{R}^N)$*

$$|u|_{p_s^*}^p \leq S_*^{-1}[u]_{s,p}^p.$$

Moreover, $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^r(\mathbb{R}^N)$ for any $r \in [p, p_s^*]$ and compactly in $L^r(\mathcal{B}_R(0))$ for all $R > 0$ and for any $r \in [1, p_s^*)$.

Let us define the Lebesgue space

$$L_K^r(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \,:\, u \text{ is measurable and } \int\limits_{\mathbb{R}^N} K(x)|u|^r\,dx < \infty \right\}$$

endowed with the norm

$$\|u\|_{L_K^r(\mathbb{R}^N)} = \left(\int\limits_{\mathbb{R}^N} K(x)|u|^r\,dx \right)^{\frac{1}{r}}.$$

Now we prove the following continuous and compactness results, whose proofs can be obtained adapting the arguments in [2, 9]. For the reader's convenience we give the proofs.

Lemma 2.2. *Assume that $(V,K) \in \mathcal{K}$.*

- (i) *If (VK_3) holds true, then $\mathbb{X}$ is continuously embedded in $L_K^r(\mathbb{R}^N)$ for every $r \in [q, q_s^*]$.*
- (ii) *If (VK_4) holds, then $\mathbb{X}$ is continuously embedded in $L_K^m(\mathbb{R}^N)$.*

Proof. (i) Let $r \in (q, q_s^*)$ and let $\nu = \frac{q_s^* - r}{q_s^* - p}$. Then, using Hölder and Sobolev inequality, and recalling that $K \in L^\infty(\mathbb{R}^N)$ and $\frac{K}{V} \in L^\infty(\mathbb{R}^N)$, we get

$$\begin{aligned}
\|u\|_{L_K^r(\mathbb{R}^N)}^r &= \int\limits_{\mathbb{R}^N} K(x)|u|^{\nu p}|u|^{(1-\nu)q_s^*}\,dx \\
&\leq \left(\int\limits_{\mathbb{R}^N} |K(x)|^{\frac{1}{\nu}}|u|^p\,dx \right)^{\nu} \left(\int\limits_{\mathbb{R}^N} |u|^{q_s^*}\,dx \right)^{1-\nu} \\
&\leq \left(\sup_{x\in\mathbb{R}^N} \frac{|K(x)|}{|V(x)|^\nu} \right) \left(\int\limits_{\mathbb{R}^N} V(x)|u|^p\,dx \right)^{\nu} \left(\int\limits_{\mathbb{R}^N} |u|^{q_s^*}\,dx \right)^{1-\nu} \\
&\leq C \left(\sup_{x\in\mathbb{R}^N} \frac{|K(x)|}{|V(x)|^\nu} \right) \|u\|^{\nu p}\|u\|^{(1-\nu)q_s^*} \\
&= C \left(\sup_{x\in\mathbb{R}^N} \frac{|K(x)|}{|V(x)|^\nu} \right) \|u\|^r,
\end{aligned}$$

from which we deduce the thesis.

(ii) Let us define $\nu = \frac{q_s^*-m}{q_s^*-p}$, so that $m = \nu p + (1-\nu)q_s^*$. Then, using Hölder and Sobolev inequality and the fact that $\frac{K(x)}{V(x)^{\frac{q_s^*-m}{q_s^*-p}}} \in L^\infty(\mathbb{R}^N)$, we can infer

$$\begin{aligned}
\|u\|^m_{L^m_K(\mathbb{R}^N)} &\leq \left(\int\limits_{\mathbb{R}^N} |K(x)|^{\frac{1}{\nu}}|u|^p\,dx\right)^\nu \left(\int\limits_{\mathbb{R}^N} |u|^{q_s^*}\,dx\right)^{1-\nu} \\
&\leq \left(\sup_{x\in\mathbb{R}^N} \frac{|K(x)|}{|V(x)|^\nu}\right)\left(\int\limits_{\mathbb{R}^N} V(x)|u|^p\,dx\right)^\nu \left(\int\limits_{\mathbb{R}^N} |u|^{q_s^*}\,dx\right)^{1-\nu} \\
&\leq C\left(\sup_{x\in\mathbb{R}^N} \frac{|K(x)|}{|V(x)|^\nu}\right)\|u\|^{\nu p}\|u\|^{(1-\nu)q_s^*} = C\left(\sup_{x\in\mathbb{R}^N} \frac{|K(x)|}{|V(x)|^\nu}\right)\|u\|^m.
\end{aligned}$$

□

Lemma 2.3. *Assume that $(V,K) \in \mathcal{K}$.*

(i) *If (VK_3) holds true, then $\mathbb{X}$ is compactly embedded in $L^r_K(\mathbb{R}^N)$ for every $r \in (q, q_s^*)$.*

(ii) *If (VK_4) holds, then $\mathbb{X}$ is compactly embedded in $L^m_K(\mathbb{R}^N)$.*

Proof. (i) Our aim is to prove that

$$u_n \to u \quad \text{in } L^r_K(\mathbb{R}^N), \text{ for any } r \in (q, q_s^*),$$

that is

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} K(x)|u_n|^r\,dx = \int\limits_{\mathbb{R}^N} K(x)|u|^r\,dx.$$

Note that, for $R > 0$

$$\int\limits_{\mathbb{R}^N} K(x)|u_n|^r\,dx = \int\limits_{\mathcal{B}_R(0)} K(x)|u_n|^r\,dx + \int\limits_{\mathcal{B}^c_R(0)} K(x)|u_n|^r\,dx,$$

and recalling that $r \in (q, q_s^*)$, $K \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ and using Sobolev embedding, we have

$$\lim_{n\to\infty} \int\limits_{\mathcal{B}_R(0)} K(x)|u_n|^r\,dx = \int\limits_{\mathcal{B}_R(0)} K(x)|u|^r\,dx.$$

Now, let $\varepsilon > 0$. Then we can find $\xi_0, \xi_1 > 0$ with $\xi_0 < \xi_1$ and a positive constant C such that for all $\xi \in \mathbb{R}$

$$K(x)|\xi|^r \leq \varepsilon\, C\left[V(x)|\xi|^q + |\xi|^{q_s^*}\right] + CK(x)\chi_{[\xi_0,\xi_1]}(|\xi|)|\xi|^{q_s^*}. \tag{2.1}$$

Let us define

$$\mathcal{A} = \left\{x \in \mathbb{R}^N \,:\, \xi_0 \leq |u(x)| \leq \xi_1\right\}.$$

Then, integrating (2.1) over $\mathcal{B}_R^c(0)$ we have

$$\begin{aligned}\int_{\mathcal{B}_R^c(0)} K(x)|u|^r\,dx &\le \varepsilon\, C \int_{\mathcal{B}_R^c(0)} \left[V(x)|u|^q + |u|^{q_s^*}\right]\,dx + C \int_{\mathcal{B}_R^c(0)\cap\mathcal{A}} K(x)|u|^{q_s^*} \\ &\le \varepsilon\, C\mathfrak{L}(u) + C\xi_1^{q_s^*} \int_{\mathcal{B}_R^c(0)\cap\mathcal{A}} K(x)\,dx, \end{aligned} \tag{2.2}$$

where

$$\mathfrak{L}(u) = \int_{\mathcal{B}_R^c(0)} \left[V(x)|u|^q + |u|^{q_s^*}\right]\,dx.$$

Assume that $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{X}$ is a sequence such that $u_n \rightharpoonup u$ in $\mathbb{X}$. Then, $\{\mathfrak{L}(u_n)\}_{n\in\mathbb{N}}$ is bounded above by a positive constant, say M. Especially,

$$\int_{\mathbb{R}^N} |u_n|^{q_s^*}\,dx \le M \qquad \text{for any } n\in\mathbb{N}.$$

Denoting by $\mathcal{A}_n = \left\{x\in\mathbb{R}^N : \xi_0 \le |u_n(x)| \le \xi_1\right\}$, we have that $\sup_{n\in\mathbb{N}} |\mathcal{A}_n| < \infty$ in view of

$$\xi_0^{q_s^*}|\mathcal{A}_n| \le \int_{\mathcal{A}_n} |u_n|^{q_s^*}\,dx \le M, \qquad \text{for any } n\in\mathbb{N}.$$

Using assumption (VK_2) there exists $R>0$ large enough such that

$$\int_{\mathcal{B}_R^c(0)\cap\mathcal{A}_n} K(x)\,dx \le \frac{\varepsilon}{\xi_1^{q_s^*}}, \qquad \text{for any } n\in\mathbb{N}. \tag{2.3}$$

Combining (2.2) with (2.3) we get

$$\int_{\mathcal{B}_R^c(0)} K(x)|u|^r\,dx \le (CM+C)\,\varepsilon\,.$$

(ii) We prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K(x)|u_n|^m\,dx = \int_{\mathbb{R}^N} K(x)|u|^m\,dx.$$

Note that, for $R>0$

$$\int_{\mathbb{R}^N} K(x)|u_n|^m\,dx = \int_{\mathcal{B}_R(0)} K(x)|u_n|^m\,dx + \int_{\mathcal{B}_R^c(0)} K(x)|u_n|^m\,dx,$$

and recalling that $m \in (q, q_s^*)$, $K \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ and using the Sobolev embedding, we have

$$\lim_{n\to\infty} \int\limits_{\mathcal{B}_R(0)} K(x)|u_n|^m\,dx = \int\limits_{\mathcal{B}_R(0)} K(x)|u|^m\,dx. \tag{2.4}$$

Fixed $x \in \mathbb{R}^N$, let us consider the function

$$g(\xi) := \frac{V(x)}{\xi^{m-p}} + \xi^{q_s^*-m} \qquad \text{for every } \xi > 0.$$

Then we have that $g'(\xi) = 0$ for $\xi = \left(\frac{m-p}{q_s^*-m}V(x)\right)^{\frac{1}{q_s^*-p}} =: \xi_2$, and $g'(\xi) > 0$ for $\xi > \xi_2$. Hence $g(\xi)$ has its minimum value in

$$C_m V(x)^{\frac{q_s^*-m}{q_s^*-p}}, \quad C_m := \left(\frac{m-p}{q_s^*-m}\right)^{\frac{q_s^*-m}{q_s^*-p}} \left(\frac{q_s^*-p}{m-p}\right).$$

This combined with (VK_4) implies that for any $\varepsilon > 0$ there is a radius $R > 0$ sufficiently large such that

$$K(x)|\xi|^m \leq \varepsilon\, C'_m(V(x)|\xi|^p + |\xi|^{q_s^*}). \tag{2.5}$$

Integrating (2.5) over $\mathcal{B}_R^c(0)$ we get, for all $u \in \mathbb{X}$,

$$\int\limits_{\mathcal{B}_R^c(0)} K(x)|u|^m\,dx \leq \varepsilon\, C'_m \int\limits_{\mathcal{B}_R^c(0)} (V(x)|u|^p + |u|^{q_s^*})\,dx = \varepsilon\, C'_m\left(\|u\|^p + |u|_{q_s^*}^{q_s^*}\right). \tag{2.6}$$

Now, if $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{X}$ is a sequence such that $u_n \rightharpoonup u$ in $\mathbb{X}$, then by (2.6) we have

$$\int\limits_{\mathcal{B}_R^c(0)} K(x)|u|^m\,dx \leq \varepsilon\, C''_m \qquad \text{for any } n \in \mathbb{N}. \tag{2.7}$$

Gathering (2.4) and (2.7) we get the thesis. □

The last lemma of this section is a compactness result related to the nonlinearity.

Lemma 2.4. *Assume that $(V, K) \in \mathcal{K}$ and f verifies (f_1)–(f_2) or $(\tilde{f}_1)$–(f_2). If $\{u_n\}_{n\in\mathbb{N}}$ is a sequence such that $u_n \rightharpoonup u$ in $\mathbb{X}$, then*

$$\int\limits_{\mathbb{R}^N} K(x)F(u_n)\,dx \to \int\limits_{\mathbb{R}^N} K(x)F(u)\,dx$$

and

$$\int\limits_{\mathbb{R}^N} K(x)f(u_n)u_n\,dx \to \int\limits_{\mathbb{R}^N} K(x)f(u)u\,dx.$$

Proof. We prove that

$$\int_{\mathbb{R}^N} K(x)f(u_n)u_n\,dx \to \int_{\mathbb{R}^N} K(x)f(u)u\,dx \quad \text{as } n\to\infty.$$

We note that

$$\int_{\mathbb{R}^N} K(x)f(u_n)u_n\,dx = \int_{\mathcal{B}_r(0)} K(x)f(u_n)u_n\,dx + \int_{\mathcal{B}_r^c(0)} K(x)f(u_n)u_n\,dx. \tag{2.8}$$

Concerning the first integral in (2.8) we can apply the Strauss Lemma to deduce that

$$\int_{\mathcal{B}_r(0)} K(x)f(u_n)u_n\,dx \to \int_{\mathcal{B}_r(0)} K(x)f(u)u\,dx. \tag{2.9}$$

Now we consider the second integral in (2.8).

Assume that (VK_3) is in force and let us observe that by (f_1)–(f_2), fixed $r\in(q,q_s^*)$ and given $\varepsilon>0$, we can find $\xi_0,\xi_1>0$ with $\xi_0<\xi_1$ and a constant $C>0$ such that

$$|K(x)f(\xi)\xi| \le \varepsilon\, C\left(V(x)|\xi|^p + |\xi|^{q_s^*}\right) + CK(x)\chi_{[\xi_0,\xi_1]}(|\xi|)|\xi|^r. \tag{2.10}$$

We point out that since $\{u_n\}_{n\in\mathbb{N}}\subset\mathbb{X}$ is bounded, we can infer that

$$\int_{\mathbb{R}^N} V(x)|u_n|^p\,dx \le C' \quad \text{and} \quad \int_{\mathbb{R}^N} |u_n|^{q_s^*}\,dx \le C' \quad \text{for all } n\in\mathbb{N}. \tag{2.11}$$

Now, as in the proof of Lemma 2.3 (i), we can demonstrate that

$$\int_{\mathcal{B}_r^c(0)} K(x)\,dx \le \frac{\varepsilon}{\xi_1^r} \quad \text{for all } n\in\mathbb{N},$$

which together with (2.10) and (2.11) implies that

$$\int_{\mathcal{B}_r^c(0)} K(x)f(u_n)u_n\,dx \le C\,\varepsilon\,. \tag{2.12}$$

Combining (2.9) and (2.12) we conclude the proof.

Next, assume that (VK_4) holds true. Then, as in Lemma 2.3, using (2.5) and assumptions $(\tilde{f}_1)$–(f_2), there exist $C,\xi_0,\xi_1>0$ such that

$$K(x)|f(\xi)\xi| \le \varepsilon\, C'_m\left(V(x)|\xi|^p + |\xi|^{q_s^*}\right)$$

for every $\xi\in\mathbb{A}$, where $\mathbb{A}=\{\xi\in\mathbb{R} : |\xi|<\xi_0 \text{ or } |\xi|>\xi_1\}$, and $|x|>R$.

Gathering the boundedness of $\{u_n\}_{n\in\mathbb{N}}\subset\mathbb{X}$ and the estimate in (2.7) we get the desired thesis. □

3. EXISTENCE OF SOLUTIONS

To study (1.1) we look for critical points of the functional $\mathcal{I} : \mathbb{X} \to \mathbb{R}$ given by

$$\mathcal{I}(u) = \frac{1}{p}\|u\|_{V,p}^p + \frac{1}{q}\|u\|_{V,q}^q - \int_{\mathbb{R}^N} K(x)F(u)\,dx.$$

It is easy to check that $\mathcal{I} \in \mathcal{C}^1(\mathbb{X}, \mathbb{R})$ and its differential is given by

$$\begin{aligned}\langle \mathcal{I}'(u), v\rangle = &\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,dxdy \\ &+ \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sq}}\,dxdy \\ &+ \int_{\mathbb{R}^N} V(x)|u|^{p-2}uv\,dx + \int_{\mathbb{R}^N} V(x)|u|^{q-2}uv\,dx - \int_{\mathbb{R}^N} K(x)f(u)v\,dx\end{aligned}$$

for any $u, v \in \mathbb{X}$. Since we aim to prove the existence of positive solutions, we further assume that

(f_5) $f(t) = 0$ for all $t \leq 0$.

Now we prove that $\mathcal{I}$ possesses a mountain pass geometry [5].

Lemma 3.1. *The functional $\mathcal{I}$ satisfies the following conditions:*

(i) *there exist $\alpha, \rho > 0$ such that $\mathcal{I}(u) \geq \alpha$ with $\|u\| = \rho$,*
(ii) *there exists $e \in \mathbb{X}$ with $\|e\| > \rho$ such that $\mathcal{I}(e) < 0$.*

Proof. (i) Assume that (VK_3) holds true. From (f_1)–(f_2) it follows that fixed $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|F(t)| \leq \frac{\varepsilon}{p}|t|^p + \frac{C_\varepsilon}{q_s^*}|t|^{q_s^*}.$$

Thus we have

$$\begin{aligned}\mathcal{I}(u) &\geq \frac{1}{p}\|u\|_{V,p}^p + \frac{1}{q}\|u\|_{V,q}^q - \frac{\varepsilon}{p}\int_{\mathbb{R}^N} K(x)|u|^p\,dx - \frac{C_\varepsilon}{q_s^*}\int_{\mathbb{R}^N} K(x)|u|^{q_s^*}\,dx \\ &\geq \frac{1}{p}\|u\|_{V,p}^p + \frac{1}{q}\|u\|_{V,q}^q - \frac{\varepsilon}{p}\left|\frac{K}{V}\right|_\infty \int_{\mathbb{R}^N} V(x)|u|^p\,dx - \frac{C_\varepsilon}{q_s^*}|K|_\infty |u|_{q_s^*}^{q_s^*}.\end{aligned}$$

Choosing $\|u\| = \rho \in (0,1)$, taking into account that $1 < p < q$ and using the Sobolev embedding we can infer that

$$\begin{aligned}\mathcal{I}(u) &\geq C_1\left(\|u\|_{V,p}^p + \|u\|_{V,q}^q\right) - C_2\|u\|^{q_s^*} \\ &\geq C_1\left(\|u\|_{V,p}^q + \|u\|_{V,q}^q\right) - C_2\|u\|^{q_s^*} \\ &\geq \bar{C}_1\|u\|^q - C_2\|u\|^{q_s^*}.\end{aligned}$$

As a consequence, there exists $\alpha > 0$ such that $\mathcal{I}(u) \geq \alpha$ and $\|u\| = \rho$.

Now, we assume that (VK_4) is true. From $(\tilde{f}_1)$–(f_2) it follows that fixed $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|F(t)| \leq \frac{\varepsilon}{m}|t|^m + \frac{C_\varepsilon}{q_s^*}|t|^{q_s^*}.$$

Thus we have

$$\mathcal{I}(u) \geq \frac{1}{p}\|u\|_{V,p}^p + \frac{1}{q}\|u\|_{V,q}^q - \frac{\varepsilon}{m}\int\limits_{\mathbb{R}^N} K(x)|u|^m\,dx - \frac{C_\varepsilon}{q_s^*}\int\limits_{\mathbb{R}^N} K(x)|u|^{q_s^*}\,dx$$

which combined with (2.5) yields

$$\begin{aligned}\mathcal{I}(u) &\geq \frac{1}{p}\|u\|_{V,p}^p + \frac{1}{q}\|u\|_{V,q}^q - \frac{\varepsilon}{m}\int\limits_{\mathcal{B}_r} K(x)|u|^m\,dx - \frac{\varepsilon}{m}\int\limits_{\mathbb{R}^N} V(x)|u|^p\,dx \\ &\quad - \left(\frac{\varepsilon}{m} + \frac{C_\varepsilon}{q_s^*}|K|_\infty\right)|u|_{q_s^*}^{q_s^*} \\ &\geq C_p\|u\|_{V,p}^p + C_q\|u\|_{V,q}^q - \frac{\varepsilon}{m}|K|_\infty|u|_{L^m(\mathcal{B}_r)}^m - C|u|_{q_s^*}^{q_s^*}.\end{aligned}$$

Choosing $\|u\| = \rho \in (0,1)$, taking into account that $1 < p < q$ and using the Sobolev embedding we can infer that

$$\begin{aligned}\mathcal{I}(u) &\geq C_1\left(\|u\|_{V,p}^p + \|u\|_{V,q}^q\right) - C_2\|u\|^m - C_3\|u\|^{q_s^*} \\ &\geq C_1\left(\|u\|_{V,p}^q + \|u\|_{V,q}^q\right) - C_2\|u\|^m - C_3\|u\|^{q_s^*} \\ &\geq \bar{C}_1\|u\|^q - C_2\|u\|^m - C_3\|u\|^{q_s^*}.\end{aligned}$$

(ii) From (f_3) we have that there exist $C_1, C_2 > 0$ such that

$$F(t) \geq C_1 t^\vartheta - C_2 \quad \text{for all } t > 0.$$

Hence, for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ such that $\varphi \geq 0$ in $\mathbb{R}^N$ and $\varphi \not\equiv 0$, we have

$$\begin{aligned}\mathcal{I}(t\varphi) &\leq \frac{t^p}{p}\|\varphi\|_{V,p}^p + \frac{t^q}{q}\|\varphi\|_{V,q}^q - C_1 t^\vartheta \int\limits_{\operatorname{supp}\varphi} K(x)|\varphi|^\vartheta\,dx + C_2\int\limits_{\operatorname{supp}\varphi} K(x)\,dx \\ &\leq \frac{t^p}{p}\|\varphi\|_{V,p}^p + \frac{t^q}{q}\|\varphi\|_{V,q}^q - C_1 t^\vartheta \int\limits_{\operatorname{supp}\varphi} K(x)|\varphi|^\vartheta\,dx + C_2|K|_\infty|\operatorname{supp}\varphi|,\end{aligned}$$

for any $t > 0$. Since $\vartheta \in (q, q_s^*)$, we get $\mathcal{I}(t\varphi) \to -\infty$ as $t \to +\infty$. □

Hence, there exists a Palais–Smale sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{X}$ ([44]) such that

$$\mathcal{I}(u_n) \to c \quad \text{and} \quad \mathcal{I}'(u_n) \to 0 \text{ in } \mathbb{X}',$$

where

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{I}(\gamma(t)), \quad \Gamma = \{v \in C([0,1],\mathbb{X}) \,:\, \gamma(0) = 0 \text{ and } \gamma(1) = e\}.$$

Let us observe that thanks to (f_5) we can assume that u_n is nonnegative for all $n \in \mathbb{N}$.

Lemma 3.2. *If $\{u_n\}_{n\in\mathbb{N}}$ is a Palais–Smale sequence for $\mathcal{I}$ at the level c, then $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}$.*

Proof. Using the fact that $\{u_n\}_{n\in\mathbb{N}}$ is a Palais–Smale sequence for $\mathcal{I}$ at the level c, and assumption (f_3), we have

$$\begin{aligned}
&C(1+\|u_n\|)\\
&\geq \mathcal{I}(u_n) - \frac{1}{\vartheta}\langle \mathcal{I}'(u_n), u_n\rangle\\
&= \left(\frac{1}{p}-\frac{1}{\vartheta}\right)\|u_n\|_{V,p}^p + \left(\frac{1}{q}-\frac{1}{\vartheta}\right)\|u_n\|_{V,q}^q + \frac{1}{\vartheta}\int\limits_{\mathbb{R}^N} K(x)\left(f(u_n)u_n - \vartheta F(u_n)\right)\,dx\\
&\geq \left(\frac{1}{q}-\frac{1}{\vartheta}\right)(\|u_n\|_{V,p}^p + \|u_n\|_{V,q}^q).
\end{aligned}$$

Now, let us assume by contradiction that $\|u_n\|\to\infty$. Then we have the following cases:
(1) $\|u_n\|_{V,p}\to\infty$ and $\|u_n\|_{V,q}\to\infty$,
(2) $\|u_n\|_{V,p}\to\infty$ and $\|u_n\|_{V,q}$ is bounded,
(3) $\|u_n\|_{V,p}$ is bounded and $\|u_n\|_{V,q}\to\infty$.

In the first case, let us note that from $p<q$ and for n sufficiently large, we have that $\|u_n\|_{V,q}^{q-p}\geq 1$, that is $\|u_n\|_{V,q}^q \geq \|u_n\|_{V,q}^p$, hence

$$\begin{aligned}
C(1+\|u_n\|) &\geq \left(\frac{1}{q}-\frac{1}{\vartheta}\right)(\|u_n\|_{V,p}^p + \|u_n\|_{V,q}^p)\\
&\geq C_1(\|u_n\|_{V,p} + \|u_n\|_{V,q})^p = C_1\|u_n\|^p,
\end{aligned}$$

which gives a contradiction.

For what concerns the case (2), we can see that

$$C\left(1+\|u_n\|_{V,p}+\|u_n\|_{V,q}\right) \geq \left(\frac{1}{q}-\frac{1}{\vartheta}\right)\|u_n\|_{V,p}^p$$

implies

$$C\left(\frac{1}{\|u_n\|_{V,p}^p} + \frac{1}{\|u_n\|_{V,p}^{p-1}} + \frac{\|u_n\|_{V,q}}{\|u_n\|_{V,p}^p}\right) \geq \left(\frac{1}{q}-\frac{1}{\vartheta}\right),$$

and letting $n\to\infty$, we get $0 \geq \left(\frac{1}{q}-\frac{1}{\vartheta}\right) > 0$, which yields a contradiction.

We can proceed similarly for the case (3). Hence we have that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}$. □

Next we prove that $\mathcal{I}$ satisfies the Palais–Smale condition.

Lemma 3.3. *Suppose that $(V,K)\in\mathcal{K}$ and f satisfies (f_1) or $(\tilde{f}_1)$ and (f_2)–(f_4). Then, every sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}$ such that*

$$\mathcal{I}(u_n)\to c \quad \text{and} \quad \mathcal{I}'(u_n)\to 0 \ \text{ as } n\to\infty,$$

converges in $\mathbb{X}$ up to a subsequence.

Proof. From Lemma 3.2 we have that up to a subsequence there exists $u \in \mathbb{X}$ such that $u_n \rightharpoonup u$ in $\mathbb{X}$ and $u_n \to u$ in $L^r_{loc}(\mathbb{R}^N)$ for any $r \in [1, q_s^*)$. Now, we aim to prove the strong convergence of u_n to u in $\mathbb{X}$.

Set $t \in \{p, q\}$. Let us observe that the sequence

$$\left\{ \frac{|u_n(x) - u_n(y)|^{t-2}(u_n(x) - u_n(y))}{|x-y|^{(N+st)(1-\frac{1}{t})}} \right\}_{n \in \mathbb{N}} \quad \text{is bounded in } L^{\frac{t}{t-1}}(\mathbb{R}^{2N})$$

and

$$\frac{|u_n(x) - u_n(y)|^{t-2}(u_n(x) - u_n(y))}{|x-y|^{(N+st)(1-\frac{1}{t})}} \to \frac{|u(x) - u(y)|^{t-2}(u(x) - u(y))}{|x-y|^{(N+st)(1-\frac{1}{t})}} \text{ a.e. in } \mathbb{R}^{2N}.$$

Hence, up to a subsequence, we may assume that for any $h \in L^t(\mathbb{R}^{2N})$ it holds

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{t-2}(u_n(x) - u_n(y))}{|x-y|^{(N+st)(1-\frac{1}{t})}} \, h(x,y) \, dxdy \\ &\to \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{t-2}(u(x) - u(y))}{|x-y|^{(N+st)(1-\frac{1}{t})}} \, h(x,y) \, dxdy. \end{aligned} \tag{3.1}$$

Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ and set

$$h(x,y) := \frac{\phi(x) - \phi(y)}{|x-y|^{\frac{N+st}{t}}}. \tag{3.2}$$

Then, $h \in L^t(\mathbb{R}^{2N})$, and using (3.2) in (3.1) we obtain that

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{t-2}(u_n(x) - u_n(y))(\phi(x) - \phi(y))}{|x-y|^{N+st}} \, dxdy \\ &\to \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{t-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x-y|^{N+st}} \, dxdy. \end{aligned}$$

Now, taking into account that

$$\int_{\mathbb{R}^N} V(x)|u_n|^{t-2}u_n\varphi \, dx \to \int_{\mathbb{R}^N} V(x)|u|^{t-2}u\varphi \, dx$$

and using Lemma 2.4 and $\langle \mathcal{I}'(u_n), \phi \rangle = o_n(1)$, we can infer that $\langle \mathcal{I}'(u), \phi \rangle = 0$ for any $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$. Since $\mathcal{C}_c^\infty(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$, we deduce that u is a critical point of $\mathcal{I}$. In particular, $\langle \mathcal{I}'(u), u \rangle = 0$.

Now, combining $\langle \mathcal{I}'(u_n), u_n \rangle = o_n(1)$ with $\langle \mathcal{I}'(u), u \rangle = 0$, we get

$$\|u_n\|^p_{V,p} + \|u_n\|^q_{V,q} - \int_{\mathbb{R}^N} K(x) f(u_n) \, u_n \, dx = o_n(1)$$

and

$$\|u\|_{V,p}^p + \|u\|_{V,q}^q - \int\limits_{\mathbb{R}^N} K(x) f(u)\, u\, dx = 0.$$

Using Lemma 2.4 we infer

$$\|u_n\|_{V,p}^p + \|u_n\|_{V,q}^q = \|u\|_{V,p}^p + \|u\|_{V,q}^q + o_n(1). \tag{3.3}$$

In the light of the Brezis–Lieb Lemma [22] we know that

$$\|u_n - u\|_{V,p}^p = \|u_n\|_{V,p}^p - \|u\|_{V,p}^p + o_n(1)$$

and

$$\|u_n - u\|_{V,q}^q = \|u_n\|_{V,q}^q - \|u\|_{V,q}^q + o_n(1),$$

which together with (3.3) yields

$$\|u_n - u\|_{V,p}^p + \|u_n - u\|_{V,q}^q = o_n(1).$$

Hence $\|u_n - u\| = o_n(1)$. This ends the proof of lemma. □

We conclude this section giving the proof of the main result.

Proof of Theorem 1.3. In view of Lemma 3.1, Lemma 3.2 and Lemma 3.3 we can apply the mountain pass theorem [5] to deduce that there exists $u \in \mathbb{X}$ such that $\mathcal{I}(u) = c$ and $\mathcal{I}'(u) = 0$.

Now, let $u^- := \min\{u, 0\}$. Recalling that for any $x, y \in \mathbb{R}$ and $t > 1$ it holds

$$|x-y|^{t-2}(x-y)(x^- - y^-) \geq |x^- - y^-|^t,$$

and using $\langle \mathcal{I}'(u), u^- \rangle = 0$ and the fact that $f(t) = 0$ for $t \leq 0$, we have

$$\|u^-\|_{V,p}^p + \|u^-\|_{V,q}^q \leq \langle \mathcal{I}'(u), u^- \rangle = 0$$

which implies that $u^- = 0$, that is $u \geq 0$ in $\mathbb{R}^N$ and $u \not\equiv 0$.

Now, let us suppose that $f(t) = 0$ for $t \geq 0$. Then, using the same arguments we can prove the existence of $u \in \mathbb{X}$ such that $\mathcal{I}(u) = c$ and $\mathcal{I}'(u) = 0$. Moreover, $u \leq 0$ for all $x \in \mathbb{R}^N$ and $u \not\equiv 0$. □

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