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A Neimark–Sacker bifurcation in a discrete SIS model

Abstract In this paper we analyse a possiblity of occurrence of a Neimark–Sacker bifurcation in a two–dimensional SIS discrete–time model. As a discretization method, we applied the Explicit Euler Scheme. We choose a step size of discretization method as a bifurcation parameter, what is not a typical approach. We phrase conditions giving the bifurcation appearance depending on the step size. Firstly, we determine terms on the step size enabling the eigenvalues of Jacobian matrix for the endemic stationary state of system being complex and having absolute value equal to 1. Then we use the Centre Manifold Theorem in order to exclude values of step size which disable the occurrence of bifurcation. We accomplish our results with numerical simulations.

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1. Introduction One of type of bifurcation occurring in discrete systems is a Neimark–Sacker bifurcation (NSB). It can appear when the Jacobian matrix for a determined stationary state of system has two complex coupled eigenvalues with modulus equal to 1 and non–zero imaginary parts. This bifurcation is a counterpart of Hopf bifurcation appearing in continuous systems.

Let us indicate a bifurcation parameter ξ, which a given critical value ξ_c gives a desirable eigenvalue. We indicate two types of NSB: a supercritical and subcritical one. In a case of supercritical NSB for $\xi < \xi_c$ a given stationary state is a locally stable focus. After crossing ξ_c this state becomes unstable and there appears a stable isolated closed invariant curve surrounding the stationary state. In a case of subcritical NSB for $\xi < \xi_c$ we have an analogical unstable curve and a stable stationary state surrounded by this curve. Crossing ξ_c results in a disappearance of curve and loss of stability of stationary state. In a case of unstable stationary state for $\xi < \xi_c$ and in a case of stable stationary state for $\xi > \xi_c$ the invariant curve is analogically unstable and stable.

Investigating an occurrence of NSB in discrete epidemic models is not common because of complicated computations. In [7] authors investigated an appearance of NSB in a two–dimensional SIS (*susceptible–infected–susceptible*) model with respect to a step size of discretization method. Choosing this parameter as a bifurcation one is not a typical approach. The basic reproduction number is considered as a bifurcation parameter in [2], where an occurrence of NSB in a three–dimensional SIR (*susceptible–infected–recovered*) model is analysed.

In this paper our aim is to describe conditions for the NSB in a given two–dimensional discrete–time SIS model, which was introduced and analysed in our previous paper [3]. The Explicit Euler Method was chosen as a discretization method. Here we will follow an approach presented in [7]. We applied this approach in our paper [5], where we investigated an appearance of flip bifurcation.

The paper is organised as it follows. In Section 2 we introduce the analysed model and present its basic properties. Section 3 deals with determining conditions of NSB appearance. Numerical simulations are included in Section 4. In Section 5 we summarize our results. Section 6 is a list of values of coefficients of a given polynomial presented in a latter analysis. At the end one can find the references.

2. A mathematical model Let us first introduce a continuous model which we analysed in [1]. This model describes epidemic dynamics in a homogeneous population which consists of two groups: susceptible (S) and infected (I) people. An illness transmission function is based on the Mass Action Law. The considered model has the following form:

$$\begin{aligned} \dot{S} &= C - \beta S I + \gamma I - \mu S, \\ \dot{I} &= \beta S I - (\gamma + \alpha + \mu) I, \end{aligned} \tag{1}$$

where C denotes a constant inflow into the population, β is a transmission coefficient, γ reflects recovery, μ stands for a natural death rate and α is a disease-related death parameter. Each coefficient is fixed and positive.

Let us reduce a number of parameters appearing in the model. Firstly we introduce a new independent variable $\tau = \gamma t$, for which we have

$$\frac{1}{\gamma}\frac{dS}{dt} = \frac{dS}{d\tau} = S', \qquad \frac{1}{\gamma}\frac{dI}{dt} = \frac{dI}{d\tau} = I'.$$

Hence System (1) can be rewritten as

$$\begin{aligned} S' &= C - \beta S I + I - \mu S, \\ I' &= \beta S I - (1 + \alpha + \mu) I, \end{aligned}$$

where the parameters are scaled by γ and the variables S and I are now functions of the τ variable.

Now we multiply the above equation by β. We obtain

$$\begin{aligned} x' &= C - xy + y - \mu x, \\ y' &= xy - ky, \end{aligned} \tag{2}$$

where

$$x = \beta S, \quad y = \beta I, \quad k = \alpha + \mu + 1$$

and C is scaled. Observe that $k > 1$.

In [3] we discretized System (2) using the Explicit Euler Method and obtained

$$\begin{aligned} x_{n+1} &= x_n + h\left(C - x_n y_n + y_n - \mu x_n\right), \\ y_{n+1} &= y_n + h\left(x_n y_n - k y_n\right), \end{aligned} \tag{3}$$

where $n \in \mathbb{N}$ and $h > 0$ means a step size of discretization method.

2.1. Basic properties Now let us remind some basic properties of System (3). We introduce a set

$$\Omega := \left\{(x, y) \in \mathbb{R}^2 \quad x + y \leq \frac{C}{\mu}\right\},$$

which its desirable property is its invariance. In [3] we proved a theorem:

THEOREM 2.1 *If* $(x_0, y_0) \in \Omega$ *and*

$$h < h_{\min} := \min\left(\frac{1}{k}, \frac{1}{\frac{C}{\mu} + \mu + 1}\right), \tag{4}$$

then the solutions (x_n, y_n) *of System* (3) *remain in* Ω *for every* $n \in \mathbb{N}_+$.

System (3) has two stationary states:

- disease–free: $E_d := (x_d, y_d) = \left(\frac{C}{\mu}, 0\right)$, always existing;
- endemic: $E_e := (x_e, y_e) = \left(k, \frac{C-\mu k}{k-1}\right)$, existing if $C > \mu k$.

Below we express conditions for local stability of the E_d state:

THEOREM 2.2 E_d *is a stable node (sink) for* $C < \mu k$*, non-hyperbolic for* $C = \mu k$ *and a saddle for* $C > \mu k$*.*

This theorem was proved in [3]. Now let us define parameters:

$$\begin{aligned} \delta &:= (C - \mu)^2 - 4(C - \mu k)(k - 1)^2, \quad \nu := C - \mu k, \\ h_{1,2} &:= \frac{4(k-1)}{C - \mu \pm \sqrt{\delta}}, \quad h_3 := \frac{C - \mu}{(k-1)(C - \mu k)}. \end{aligned} \tag{5}$$

For the state E_e the Jacobian matrix of System (3) reads

$$M(E_e) := \begin{pmatrix} 1 - h\frac{C-\mu}{\alpha+\mu} & -h(\alpha+\mu) \\ h\frac{C-\mu k}{\alpha+\mu} & 1 \end{pmatrix}$$

and its eigenvalues are

$$\lambda_{1,2} := 1 - h\frac{(C-\mu) \pm \sqrt{\delta}}{2(k-1)}.$$

Let us quote a theorem, which was proved in [3]:

THEOREM 2.3 *For $C > \mu k$ the state E_e exists and is*

- *a stable node (sink) for $\delta \geq 0$ and $h < h_1$ or $\delta < 0$ and $h < h_3$,*
- *non-hyperbolic if $\delta \geq 0$ and $h \in \{h_1, h_2\}$ or $\delta < 0$ and $h = h_3$,*
- *an unstable node (source) for $\delta \geq 0$ and $h > h_2$ or $\delta < 0$ and $h > h_3$,*
- *a saddle if $\delta \geq 0$ and $h_1 < h < h_2$.*

See that Theorem 2.3 is formulated without reference to the invariant set Ω. Considering assumptions from Theorem 2.1 we formulate a corollary:

COROLLARY 2.4 *Assume that the state E_e of System (3) exists and assumptions of Theorem 2.1 hold. Let $\delta < 0$. Then E_e is a sink and for every $n \in \mathbb{N}_+$ we have $(x_n, y_n) \in \Omega$.*

3. An appearance of NSB

Here we focus on determining an appearance of NSB concerning the state E_e. As a bifurcation parameter we choose the step size of discretization method. Firstly, let us formulate a condition on the step size enabling the NSB. We consider complex eigenvalues with non–zero imaginary parts, so we assume that $\delta < 0$. Moreover, we require that the modulus of these eigenvalues equals 1. Hence, the next condition is $|\lambda_{1,2}| = 1$, what can be written as $h = h_3$. We formulate a proposition:

PROPOSITION 3.1 *Let δ and h_3 be defined by (5). If in System (3) the state E_e exists, then for $\delta < 0$ the NSB can happen if h varies in a small neighbourhood of h_3.*

Let us look at Corollary 2.4. For the assumptions given in this corollary we do not obtain the NSB, because in this case we have unconditional local stability of E_e. However, analysis of System (3) can be conducted without an epidemiological approach. Moreover, if E_e is a sink, then some iterations of the variables (x_n, y_n) can be not included in Ω. For this reason in this section we do not consider the assumptions from Corollary 2.4 and we assume that Theorem 2.3 holds for $\delta < 0$. Hence, we assume that the solutions (x_n, y_n) do not have to belong to Ω. Moreover, these solutions can be negative.

Now we determine sufficient conditions on h giving the NSB. The approach from [7] will be used. We denote by h^* a small perturbation of h in System (3). The system can be rephrased as

$$\begin{aligned} x_{n+1} &= x_n + (h + h^*)\,(C - x_n y_n + y_n - \mu x_n)\,, \\ y_{n+1} &= y_n + (h + h^*)\,(x_n y_n - k y_n)\,. \end{aligned} \tag{6}$$

In a further analysis we will use the notation

$$\rho := \frac{C - \mu}{\alpha + \mu}. \tag{7}$$

In order to analyse System (6) for the zero stationary state we introduce new variables

$$u_n := x_n - x_e, \quad v_n := y_n - y_e. \tag{8}$$

Substituting (8) in (6) gives

$$\begin{aligned} u_{n+1} &= u_n + (h + h^*)\,(C - u_n v_n - u_n y_e - v_n x_e + x_e y_e) \\ &\quad + (h + h^*)\,(v_n + y_e - \mu u_n - \mu x_e)\,, \\ v_{n+1} &= v_n + (h + h^*)\,(u_n v_n + u_n y_e + v_n x_e + x_e y_e - k v_n - k y_e)\,. \end{aligned} \tag{9}$$

Taking the second-order Taylor approximation of the right–hand side of the above system for the point $(0, 0)$ gives

$$\begin{aligned} u_{n+1} &= a_{11} u_n + a_{12} v_n + a_{14} u_n v_n + b_{11} h^* u_n \\ &\quad + b_{12} h^* v_n + b_{14} h^* u_n v_n + o(u_n + v_n), \\ v_{n+1} &= a_{21} u_n + a_{22} v_n + a_{24} u_n v_n + b_{21} h^* u_n \\ &\quad + b_{22} h^* v_n + b_{24} h^* u_n v_n + o(u_n + v_n), \end{aligned} \tag{10}$$

where

$$a_{11} = 1 - h(\mu + y_e) = 1 - h\frac{C - \mu}{\alpha + \mu}, \quad b_{11} = -(\mu + y_e) = -\frac{C - \mu}{\alpha + \mu}, \tag{11}$$

and

$$\begin{aligned} & a_{12} = h(1 - x_e) = -h(\alpha + \mu), \quad a_{14} = -h, \\ & b_{12} = 1 - x_e = -(\alpha + \mu), \quad b_{14} = 1, \quad a_{21} = h y_e = h\frac{C - \mu k}{\alpha + \mu}, \\ & a_{22} = 1, \quad a_{24} = h, \quad b_{21} = y_e = \frac{C - \mu k}{\alpha + \mu}, \quad b_{22} = 0, \quad b_{24} = 1 \end{aligned} \tag{12}$$

and the function $o(u_n + v_n)$ fulfills the condition $\lim\limits_{n\to\infty} \frac{o(u_n+v_n)}{u_n+v_n} = 0$. Let us express the coefficients from (11) with the use of ρ from (7) as

$$a_{11} = 1 - h\rho, \quad b_{11} = -\rho. \tag{13}$$

The Jacobian matrix of System (9) for the zero stationary state has the form

$$\begin{pmatrix} a_{11} + b_{11}h^* & a_{12} + b_{12}h^* \\ a_{21} + b_{21}h^* & a_{22} + b_{22}h^* \end{pmatrix}.$$

This matrix with the use of notation from (12) and (13) can be written as

$$\begin{pmatrix} 1 - (h + h^*)\frac{C-\mu}{\alpha+\mu} & -(h + h^*)(\alpha + \mu) \\ (h + h^*)\frac{C-\mu k}{\alpha+\mu} & 1 \end{pmatrix}. \tag{14}$$

The characteristic polynomial of the matrix reads $\lambda^2 + P(h^*)\lambda + Q(h^*)$, where

$$P(h^*) = (h + h^*)\rho - 2, \quad Q(h^*) = 1 - (h + h^*)\rho + (h + h^*)^2(C - \mu k).$$

A condition for the appearance of NSB is $\lambda_{1,2}^m \neq 1$ for $h^* = 0$ and $m \in \{1, 2, 3, 4\}$ [7], what corresponds to

$$P(0) \notin \{-2, 0, 1, 2\}. \tag{15}$$

Observe that $P(0) = h\rho - 2$. From the definition of the parameters we get that the condition $P(0) \neq -2$ always holds. Analysing the remaining possibilities from (15) we state that (15) can be expressed as

$$h \notin \left\{ \frac{\alpha + \mu}{C - \mu}, \frac{2(\alpha + \mu)}{C - \mu}, \frac{4(\alpha + \mu)}{C - \mu} \right\}. \tag{16}$$

In a NSB we assume that eigenvalues of a determined Jacobian matrix have non–zero imaginary part, hence the eigenvalues of the matrix (14) for $h^* = 0$ are

$$\lambda_{1,2} = 1 - \frac{h\rho}{2} \pm i\frac{h}{2}\sqrt{4(C - \mu k) - \rho^2}, \tag{17}$$

where

$$4(C - \mu k) - \rho^2 > 0. \tag{18}$$

Further we will use notations

$$\nu := C - \mu k, \quad \alpha := 1 - \frac{h\rho}{2}, \quad \beta := \frac{h}{2}\sqrt{4\nu - \rho^2}. \tag{19}$$

Obviously from the definition of ν and from (18) we have $4\nu - \rho^2 > 0$. What is more, Eq. (17) can be written as $\lambda_{1,2} = \alpha \pm \beta i$.

We define an invertible matrix

$$\bar{T} := \begin{pmatrix} 0 & 1 \\ \beta & \alpha \end{pmatrix}$$

and a transformation

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \bar{T} \begin{pmatrix} U_n \\ V_n \end{pmatrix}.$$

Considering this transformation in (9) provides a system

$$\begin{aligned} U_{n+1} &= \alpha U_n - \beta V_n + \bar{F}(U_n, V_n), \\ V_{n+1} &= \beta U_n + \alpha V_n + \bar{G}(U_n, V_n), \end{aligned}$$

where

$$\begin{aligned} \bar{F}(U_n, V_n) &= \frac{h(1+\alpha)}{\beta} V_n(\beta U_n + \alpha V_n) + o\Big(\big(|U_n| + |V_n|\big)^2\Big), \\ \bar{G}(U_n, V_n) &= -hV_n(\beta U_n + \alpha V_n) + o\Big(\big(|U_n| + |V_n|\big)^2\Big). \end{aligned}$$

Let us define

$$\begin{aligned} F_1 &:= \frac{\partial^2 F}{\partial U_n \partial V_n}\Big|_{(0,0)} = h(1+\alpha), \quad F_2 = \frac{\partial^2 F}{\partial V_n^2}\Big|_{(0,0)} = \frac{2h\alpha(1+\alpha)}{\beta}, \\ G_1 &:= \frac{\partial^2 G}{\partial U_n^2}\Big|_{(0,0)} = 2\beta^2, \quad G_2 := \frac{\partial^2 G}{\partial U_n \partial V_n}\Big|_{(0,0)} = -h\beta, \\ G_3 &:= \frac{\partial^2 G}{\partial V_n^2}\Big|_{(0,0)} = -2h\alpha. \end{aligned} \tag{20}$$

According to Theorem 3.5.2 from [6], a condition for the appearance of NSB is

$$\Gamma := \frac{\bar{B}\bar{D} - \bar{A}\bar{C}}{64(1-\alpha)} - \frac{1}{2}||\xi_1||^2 - ||\xi_2||^2 \neq 0, \tag{21}$$

for

$$\begin{aligned} \bar{A} &= 2\beta^2(1+\alpha-4\alpha^2) + (1-\alpha)\Big(4\alpha(1-\alpha^2) + (1-2\alpha)(2\alpha^2-1)\Big), \\ \bar{B} &= \beta\Big(4\alpha(1-\alpha^2) + (1-2\alpha)(2\alpha^2-1)\Big) - 2\beta(1-\alpha)(1+\alpha-4\alpha^2), \\ \bar{C} &= F_2(-F_2 + 2G_2) - (G_1 + G_3)(G_1 - G_3 - 2F_1), \\ \bar{D} &= F_2(G_1 - G_3 - 2F_1) - (G_1 + G_3)(F_2 + 2G_2), \\ \xi_1 &= \frac{1}{4}\Big(F_2 + i(G_1 + G_3)\Big), \\ \xi_2 &= \frac{1}{8}\Big(-F_1 + 2G_2 + i(G_1 - G_3 + 2F_1)\Big). \end{aligned} \tag{22}$$

Using (19), (20) and (22), the condition (21) reads

$$\Gamma(h) = -\frac{4}{4\nu - \rho^2} + \sum_{j=1}^{7} a_j h^j \neq 0, \tag{23}$$

where the values of coefficients a_j are presented in Section 6. Observe that $\Gamma(h)$ is a seventh degree polynomial. Because of the E_e existence, we have $\nu > 0$. The definition of the variables yields $\Gamma(0) < 0$. Hence, we can state that for a sufficiently small h we have $\Gamma(h) < 0$. Determining the sign of Γ depending on h is complicated and was omitted in this paper.

On the grounds of Theorem 3.2. from [7] based on the Central Manifold Theorem (Theorem 3.4.2.) from [6] we formulate a lemma:

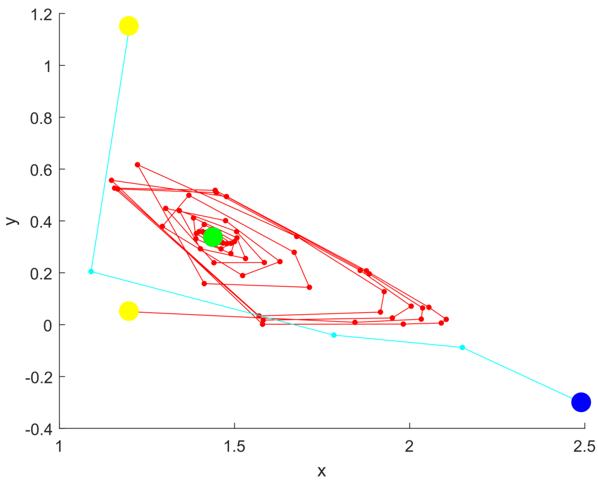
Lemma 3.2 *In System (6) for the existing state E_e the NSB appears for h^* varying in a small neighbourhood of 0 if (16) and $\Gamma(h) \neq 0$. If $\Gamma(h) < 0$, then an attracting closed invariant curve bifurcates from E_e for $h^* > 0$. For $\Gamma(h) > 0$ a repelling closed invariant curve bifurcates from E_e for $h^* < 0$.*

We omit a proof of Lemma 3.2. In a context of System (6) and Lemma 3.2 we formulate a proposition:

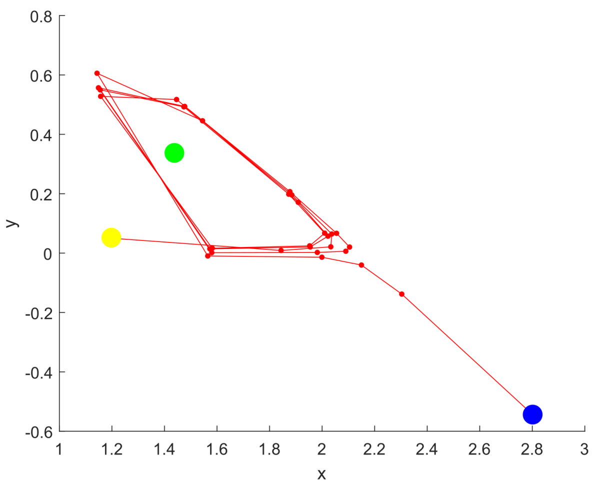
Proposition 3.3 *We assume that $\delta < 0$ and $h = h_3$. In System (6) there appears a NSB for the existing state E_e for h^* varying in a small neighbourhood of 0 if (16) and $\Gamma(h) \neq 0$. For a sufficient small h an attracting closed invariant curves bifurcates from E_e for $h^* > 0$.*

4. Numerical simulations In this section we include numerical simulations reflecting the NSB occurrence for the state E_e. Figure 1 shows consecutive iterations of System (6). As values of parameters we chose $C = 0.45$, $\mu = 0.21$ and $\alpha = 0.23$. For this values we get $\delta < 0$ so that E_e is a sink. Choosing appropriate values enabling an NSB appearance is generally difficult. The initial conditions is $(x_0, y_0) = (1.2; 0.05)$, for Fig. 1(a) we also consider $(x_0^*, y_0^*) = (1.2; 1.15)$. In the case of Fig. 1(a) we assumed $h = 3.42437$ and for (x_0, y_0) and (x_0^*, y_0^*) accordingly 2000 and 5 initial iterations are presented. For Fig. 1(b) we chose $h = 3.42438$ and depicted 30 initial iterations. For $h = 3.42437$ we had the local stability of E_e and for $h = 3.42438$ this state loses the stability. We conclude that the critical value of the bifurcation parameter belongs to an interval $D := (3.42437; 3.42438)$.

Figure 1 suggests existence of an unstable closed invariant curve, which disappears while h crosses the critical value. Moreover, we obtain a stationary state loosing local stability. Hence, we state that in System (6) for the given values of the parameters there is a subcritical NSB. Observe that in Proposition 3.3 we assume that h is sufficiently small. We treat the value of h from D as big enough so that Proposition 3.3 is not applied.



(a)



(b)

Figure 1: Points being iterations of System (6) for accordingly (a) $h = 3.42437$ and (b) $h = 3.42438$. The initial conditions are depicted with yellow dots. The consecutive iterations are marked with red or azure points and linked with red and azure lines, respectively. The stationary state is depicted with a green dot, the last iteration is marked with a blue dot. For the case (a) because of the E_e local stability the coordinates of the last iteration are virtually equal to those for E_e. Hence, the point reflecting the last iteration is not visible.

5. Summary In this paper we analysed the possibility of NSB appearance in the two–dimensional discrete–time SIS model. As the bifurcation parameter the step size of discretization method was chosen. We indicated the values of step size excluding the NSB occurrence. We also conducted the numerical simulations illustrating the NSB. The parameters were chosen so that they reflect dynamics of epidemic. The simulations suggest that for the given values of parameters there appears the subcritical NSB.

In our previous paper [4] we analysed a discrete–time system where a non–standard discretization was used. This system has the form

$$\begin{aligned} x_{n+1} &= \frac{x_n(1-h\mu)+hC+hy_n}{1+hy_n}, \\ y_{n+1} &= y_n(1+h(x_n-k)). \end{aligned} \tag{24}$$

For this system reaching conditions for the NBS appearance is hard, that is why we investigated this issue only by conducting numerical simulations. They suggested a supercritical NSB appearance. However, in order to prove that for both Systems (3) and (24) there are different types of NSB, mathematical analysis of System (24) should be conducted.

6. Appendix - forms of the coefficients from Eq. (23)

$$\begin{aligned}
a_1 &= \frac{1}{4\nu-\rho^2}\left(15\rho-8\frac{\nu}{\rho}\right),\\
a_2 &= -\frac{3}{8}+26\frac{u}{4\nu-\rho^2}+\frac{1}{64}\rho^2-\frac{1}{16}\sqrt{4\nu-\rho^2}-\frac{1}{16}\nu-\frac{103}{4}\frac{\rho^2}{4\nu-\rho^2},\\
a_3 &= \frac{1}{8}\left(\frac{377}{2}\frac{\rho^3}{4\nu-\rho^2}-\frac{3}{4}\rho^2+\frac{25}{4}\rho-252\frac{\nu\rho}{4\nu-\rho^2}+3\nu-11\frac{\nu}{\rho}\right),\\
a_4 &= \frac{1}{8}\left(-\frac{151}{32}\rho^2+\frac{5}{2}\rho-\frac{3}{2}\nu^2+\frac{155}{8}\nu-\frac{47}{4}\rho\nu+\frac{7\nu^2}{\rho}-\frac{3}{32}\rho^4+\frac{3}{4}\nu\rho^2\right)\\
&\quad+\frac{\rho^2}{8(4\nu^2-\rho)}\left(-\frac{193}{2}\rho^2+151\nu\right),\\
a_5 &= \frac{1}{8}\left(\frac{9}{32}\rho^5-\frac{9}{4}\rho^4-\frac{5}{2}\nu\rho^3-\frac{1}{8}\rho^3+\frac{43}{4}\nu\rho^2+\frac{13}{2}\nu^2\rho-\frac{15}{2}\nu\rho-7\nu^2\right)\\
&\quad+\frac{221}{64}\frac{\rho^5}{4\nu-\rho^2}-\frac{6\nu\rho^3}{4\nu-\rho^2}-\frac{\nu^3}{2\rho},\\
a_6 &= \frac{1}{8}\left(-\frac{9}{32}\rho^6+\frac{1}{2}\rho^5+\frac{9}{8}\rho^4-\frac{3}{4}\nu\rho^2-\frac{3}{2}\nu^2\rho-\frac{13}{8}\nu\rho^3+\frac{43}{16}\nu\rho^2+7\nu^3\right)\\
&\quad+\frac{\rho^2}{32(4\nu-\rho^2)}\left(31\nu-\frac{33}{2}\rho^2\right)-\nu^2\rho^2,\\
a_7 &= \frac{\rho^2}{8}\left(-\frac{5}{8}\nu\rho^3-\frac{1}{4}\rho^3-\frac{1}{4}\nu\rho^2+\frac{1}{16}\rho^2+\frac{1}{2}\nu\rho+\nu^2\right)+\frac{1}{4}\rho\nu^2(\rho^2-\nu)\\
&\quad+\frac{\rho^5}{16(4\nu-\rho^2)}\left(\frac{\rho^2}{2}-\nu\right).
\end{aligned}$$

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Bifurkacja Neimarka–Sackera w dyskretnym modelu *SIS*

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Streszczenie W artykule zbadano możliwość wystąpienia bifurkacji Neimerka–Sackera (*BNS*) w dyskretnym dwuwymiarowym modelu *SIS*. W celu dyskretyzacji modelu ciągłego zastosowano jawny schemat Eulera. Jako parametr bifurkacyjny wybrano długość kroku dyskretyzacji, co nie jest standardowym podejściem. Sformułowaliśmy warunki wystąpienia bifurkacji w zależności od długości kroku. Najpierw sprawdzono, dla jakich warunków wartości własne macierzy Jacobiego dla endemicznego stanu stacjonarnego są zespolone oraz ich moduł wynosi 1. Następnie zastosowano twierdzenie o rozmaitości centralnej w celu wykluczenia tych wartości kroku dyskretyzacji, dla których *BNS* nie występuje. Rozważania teoretyczne są uzupełnione symulacjami numerycznymi.

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