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On the relaxation of state-constrained linear control problems via Henig dilating cones*†

by

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Abstract: We discuss a regularization of state-constrained optimal control problems via a Henig relaxation of ordering cones. Considering a state-constrained optimal control problem, the pointwise state constraint is replaced by an inequality condition involving a so-called Henig dilating cone. It is shown that this class of cones provides a reasonable solid approximation of the typically non-solid ordering cones which correspond to pointwise state constraints. Thereby, constraint qualifications, which are based on the existence of interior points, can be applied to given problems. Moreover, we characterize admissibility and solvability of the original problem by analyzing the associated relaxed problem. We also show that the optimality system for the original problem can be obtained through the limit passage in the corresponding optimality system for the relaxed problem. As an example of our approach, we derive the optimality conditions for a state constrained Neumann boundary optimal control problem and show that in this case the corresponding Lagrange multipliers are more regular than Borel measures.

Keywords: optimal control, state constrains, relaxation, Henig dilating cone, optimality conditions

1. Introduction

Over the last years, state-constrained optimal control problems have attracted increasing attention. As a rule, the state constraints are very important in various applications of the optimal control of PDEs. However, the associated numerical analysis is known to be quite complicated, since the corresponding Lagrange

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multipliers for the state constraints are in general regular Borel measures. We can refer to many contributions in this field (see, for instance, Bergounioux and Kunisch, 2002a,b,c; Casas, 1992; Casas and Mateos, 2002; Casas and Tröltzsch, 2010; Meyer, Rösch and Tröltzsch, 2006; Meyer and Tröltzsch, 2006; Hintermüller and Kunish, 2008; or Tröltzsch, 2006, and references therein). In some cases, the analysis is often simpler for problems with mixed pointwise control–state constraints, since Lagrange multipliers are more regular there. For the elliptic case with quadratic cost functional and linear equation, this has been shown in Tröltzsch (2006), however, the corresponding proofs are quite technical. In the meantime, in contrast to the control-constrained problems, many interesting questions are still open or not yet satisfactorily solved with state constraints.

Considering optimal control problems with pointwise state constraints, two problems occur: Firstly, since state constraints are very strong conditions, the admissible set of the problem is often very thin or even empty. It is a largely open question how to characterize the existence of admissible pairs for state-constrained problems properly (see, for instance, Barbu, 1993; Bonnans and Shapiro, 2000; Fursikov, 2000; Hinze, Pinnau and Ulbrich, 2009; Kogut and Leugering, 2011; Kogut and Manzo, 2013; Mel'nik, 1986; Roubíček, 1997; Tröltzsch, 2006). Secondly, in an infinite dimensional setting the convex cone (called *ordering cone*) corresponding to pointwise inequality constraints is typically nonsolid, i.e. it has an empty topological interior. If so, it is not possible to apply constraint qualifications, such as the Slater condition, which require the existence of interior points of the ordering cone. Because of this, in the majority of existing results, the state-constrained optimal control problems are studied for the dimensions $N = 2$ or $N = 3$ only. Such restriction, typically, leads to the existence and uniqueness of solutions to the corresponding boundary value problems in $C(\overline{\Omega}) \cap H^1(\Omega)$. As a result, the pointwise state constraints can be characterized by a solid ordering cone in $C(\overline{\Omega})$.

The main question that we study in this paper is to provide a regularization for a wide class of state constrained optimal control problems by replacing the ordering cone having an empty topological interior by its solid Henig approximation (see Zhuang, 1994) and show that the solvability of the original problem can be characterized by solving the corresponding Henig relaxed problems in the limit as some small parameter ε tends to zero.

As was shown in the recent publication by Khan and Sama (2013), the proposed approach is numerically viable for some state-constrained convex optimization problems. In particular, using the finite element discretization of the Henig dilating cone of positive functions, it has been shown in Khan and Sama (2013) that the above approximation scheme, called conical regularization, leads to a finite-dimensional optimization problem, which can conveniently be treated by known numerical techniques. If the standard multiplier rule can be applicable to the regularized problems and their solutions converge in some appropriate topology to the solution of the original problem, then the benefit of such approach is unquestionable if only the corresponding feasible set to the original problem is nonempty.

The second aim of this article is to apply this approximation to deriving optimality systems (the necessary optimality conditions) for the pointwise state constrained case. In particular, we show that in order to derive the optimality conditions for an optimal control problem

$$I(u,y) \to \inf, \quad L(u,y) + f = 0, \quad F(y) + F_0 \in \Lambda, \quad u \in U_{ad},$$

the key question is compactness properties of the linear operator $F : \mathbb{Y}_1 \to \mathbb{W}$ and some special continuity properties of the cost functional $I : \mathbb{U} \times \mathbb{Y} \to \overline{\mathbb{R}}$. We also present an example of a state constrained Neumann boundary optimal control problem for a linear elliptic equation in $\Omega \subset \mathbb{R}^N$, for which the optimality system takes a clearly defined structure and the corresponding Lagrange multipliers are fairly regular.

2. Statement of the control problem and main motivation

Let $\mathbb{Y}$ and $\mathbb{Z}$ be a linear normed spaces and let $\mathbb{Y}_1$ be a reflexive Banach space such that $\mathbb{Y}_1$ is continuously and densely embedded in $\mathbb{Y}$. Let $\mathbb{V}$ be a separable Banach space and $\mathbb{U} = \mathbb{V}^*$ be its dual. Let U_{ad} be a closed convex subset of $\mathbb{U}$, $\mathbb{W}$ be a reflexive Banach space, and $\Lambda \subset \mathbb{W}$ be a closed convex pointed ordering cone.

We consider the following extremal problem, which is standard in optimal control theory with PDEs: Find a pair $(u^0, y^0) \in U_{ad} \times Y_1$ such that

$$I(u,y) \to \inf \tag{1}$$

subject to the restrictions

$$L(u,y) + f = 0, \tag{2}$$

$$F(y) + F_0 \in \Lambda, \quad u \in U_{ad}, \tag{3}$$

where $L : \mathbb{U} \times \mathbb{Y}_1 \to \mathbb{Z}$ and $F : \mathbb{Y}_1 \to \mathbb{W}$ are linear mappings, $F_0 \in \mathbb{W}$ and $f \in \mathbb{Z}$ are given elements, and $I : \mathbb{U} \times \mathbb{Y} \to \overline{\mathbb{R}}$ is a convex functional, which is always assumed to be lower semicontinuous and bounded from below. We call (1)–(3) a state-constrained control problem $(SCCP)$.

Hereinafter we refer to $u \in U_{ad}$ as an admissible control, $y \in \mathbb{Y}_1$ as a state of the control object, the cone Λ as the state constraints, and $I(u,y)$ as a cost functional.

Let us note also that in the majority of applications, the state space $\mathbb{Y}_1$, endowed with its weak topology, and the control space $\mathbb{U}$, endowed with its weak-$*$ topology, are not metrizable in general. Nevertheless, due to some growth assumptions on the cost functional I, we may work with a bounded subset of $\mathbb{Y}_1 \times \mathbb{U}$ which is, due to the initial assumptions, metrizable.

It is worth to notice that the state y of our control object, as a solution of the linear problem $L(u,y) + f = 0$ under some admissible control $u \in U_{ad}$, may be not unique, in general. On the other hand, in the framework of the given statement of $SCCP$, here and in the sequel we admit the situation when there exist controls $u \in \mathbb{U}$ such that there is no $y \in \mathbb{Y}_1$ satisfying either equation $L(u,y)+f=0$, or the state constraint $F(y)+F_0 \in \Lambda$, or even both of them. Thus, we need to introduce the feasible set for the problem (1)–(3) as follows:

$$\Xi = \left\{ (u,y) \in \mathbb{U} \times \mathbb{Y}_1 \;\middle|\; \begin{array}{c} L(u,y)+f=0, \\ u \in U_{ad},\ F(y)+F_0 \in \Lambda, \\ I(u,y) < +\infty \end{array} \right\}. \tag{4}$$

We say that a pair $(u^0,y^0) \in \mathbb{U} \times \mathbb{Y}_1$ is a solution to the problem (1)–(3) if

$$(u^0,y^0) \in \Xi \quad \text{and} \quad I(u^0,y^0) = \inf_{(u,y)\in\Xi} I(u,y).$$

We also say that a sequence of pairs $\{(u_k,y_k)\}_{k=1}^{\infty} \subset \mathbb{U} \times \mathbb{Y}_1$ is τ-convergent to a pair $(u,y) \in \mathbb{U} \times \mathbb{Y}_1$ if $u_k \overset{*}{\rightharpoonup} u$ in $\mathbb{U}$ and $y_k \rightharpoonup y$ in $\mathbb{Y}_1$. In what follows, by τ-topology on $\mathbb{U} \times \mathbb{Y}_1 \to \mathbb{Z}$ we mean the topology which is induced by τ-convergence.

In what follows, we assume that the following hypotheses hold true:

(H1) (Nontriviality condition) $\Xi \neq \emptyset$.

(H2) (Coerciveness condition) For every $\lambda > 0$, the set

$$\Xi_\lambda = \{(u,y) \in \Xi \ : \ I(u,y) \leq \lambda\}$$

is bounded in $\mathbb{U} \times \mathbb{Y}_1$.

In order to explain the role of these assumptions, for the reader's convenience, we give the following result (see Fursikov, 2000; Kogut and Leugering, 2011).

Theorem 1 *Assume that*

(i) L is a linear continuous mapping from $\mathbb{U} \times \mathbb{Y}_1 \to \mathbb{Z}$ endowed with the τ-topology into $\mathbb{Z}$ endowed with the weak topology;

(ii) $F : \mathbb{Y}_1 \to \mathbb{W}$ is a linear continuous mapping with respect to the norm topology of $\mathbb{Z}$ and the weak topology of $\mathbb{Y}_1$.

Then, under hypotheses (H1)–(H2), the $SCCP$ (1)–(3) admits at least one solution.

Proof 1 Since the cost functional $I : \mathbb{U} \times \mathbb{Y} \to \overline{\mathbb{R}}$ is bounded from below and $\Xi \neq \emptyset$, it follows that there exists a sequence $\{(u_k,y_k)\}_{k\in\mathbb{N}} \subset \Xi$ such that

$$\lim_{k\to\infty} I(u_k,y_k) = \inf_{(u,y)\in\Xi} I(u,y) > -\infty.$$

Hence, $\sup_{k\in\mathbb{N}} I(u_k,y_k) \leq C$, where the constant C is independent of k. Then, in view of the coercivity condition (H2), we have $\sup_{n\in\mathbb{N}} [\|u_k\|_{\mathbb{U}} + \|y_k\|_{\mathbb{Y}_1}] \leq C$. Hence, by the Banach–Alaoglu theorem, we may suppose that (up to a subsequence)

$$(u_k,y_k) \overset{\tau}{\to} (u^0,y^0) \ \text{ in } \ \mathbb{U} \times \mathbb{Y}_1 \ \text{ and, hence, } \ y_k \rightharpoonup y^0 \ \text{ in } \ \mathbb{Y}$$

by continuity of the embedding $\mathbb{Y}_1 \hookrightarrow \mathbb{Y}$.

Since the set U_{ad} is convex and closed, it follows that U_{ad} is sequentially closed with respect to the weak-$*$ topology. Therefore, $u^0 \in U_{ad}$. In addition, for an arbitrary element $z^* \in \mathbb{Z}^*$, we have

$$\langle z^*, L(u_k, y_k)\rangle_{\mathbb{Z}^*,\mathbb{Z}} \xrightarrow[k\to\infty]{\text{by (i)}} \langle z^*, L(u^0, y^0)\rangle_{\mathbb{Z}^*,\mathbb{Z}}, \tag{5}$$

$$F(y_k) \xrightarrow[k\to\infty]{\text{by (ii)}} F(y^0) \quad \text{in} \quad \mathbb{W}. \tag{6}$$

Since the cone Λ is closed, it follows from (6) that $F(y^0) + F_0 \in \Lambda$. Thus, in order to conclude that (u^0, y^0) is an admissible pair to problem (1)–(3), it remains to show that $I(u^0, y^0) < +\infty$. However, this fact immediately follows from the lower semicontinuity property of the cost functional I with respect to the product of the weak-$*$ topology of $\mathbb{U}$ and of the weak topology for $\mathbb{Y}$. Indeed, since $(u_k, y_k) \overset{\tau}{\to} (u^0, y^0)$ in $\mathbb{U} \times \mathbb{Y}_1$, by continuity of the embedding $\mathbb{Y}_1 \hookrightarrow \mathbb{Y}$, we get

$$-\infty < I(u^0, y^0) \le \varliminf_{n\to\infty} I(u_n, y_n) = \inf_{(u,y)\in\Xi} I(u, y) \le C. \tag{7}$$

Hence, summing up the results obtained above – namely $L(u^0, y^0) + f = 0$, $u^0 \in U_{ad}$, $F(y^0) + F_0 \in \Lambda$, and $I(u^0, y^0) < +\infty$ – we conclude that $(u^0, y^0) \in \Xi$ and, because of the relation (7), the pair (u^0, y^0) is optimal for the problem (1)–(3). □

Obviously, in order to have a non-trivial problem, the set $\Xi \neq \emptyset$ should be sufficiently rich. It is not surprising that the mathematical analysis of control and especially state constrained problems is typically difficult (see, for instance, Casas, 1992; Kogut and Leugering, 2011; Roubíček, 1997), so that the verification of Hypothesis (H1) for the problem (1)–(3) remains open for many relevant situations.

Besides the problems in verifying the Hypothesis (H1), the ordering cone Λ is typically not solid in the objective space $\mathbb{W}$, i.e. $\operatorname{int}_{\|\cdot\|_{\mathbb{W}}} \Lambda = \emptyset$. This implies that we cannot apply standard constraint qualifications, such as the Slater condtion, to $SCCP$ (1)–(3). However, in the absence of such constraint qualifications, there is no guarantee that optimal pairs to the $SCCP$ (1)–(3) can be obtained via the standard multiplier rule. In other words, in such a case no optimality system is available for a computation of the solution (see Raymond, 1997, for the details). On the other hand, it makes sense to emphasize that the existence of Lagrange multipliers can be guaranteed not only by the Slater condition but also by other constraint qualification conditions, which do not require explicitly nonemptiness of the interiors of ordering cones (see, for instance, Bonnans and Shapiro, 2000).

One of the possible ways to circumvent the restrictions arising from the condition $\operatorname{int}_{\|\cdot\|_{\mathbb{W}}} \Lambda = \emptyset$, is to introduce a special Banach space $\mathbb{W}_0$ with a continuous embedding $\mathbb{W}_0 \hookrightarrow \mathbb{W}$ and work with relative interior such that

$$\operatorname{int}_{\|\cdot\|_{\mathbb{W}_0}} \left((\Lambda - F(y^0) - F_0) \cap \mathbb{W}_0 \right) \neq \emptyset, \tag{8}$$

where y^0 is an optimal state to the original problem (1)–(3). In particular, if the feasible set Ξ contains at least one pair (u^*, y^*) different from an optimal one (u^0, y^0), then $\mathbb{W}_0$ can be constructed as the line passing through the origin of $\mathbb{W}$ and the point $F(y^*) - F(y^0)$. However, to deduce the necessary and sufficient conditions for $SCCP$ (1)–(3), it makes a sense to choose $\mathbb{W}_0$ in such a way that the set $(\Lambda - F(y^0) - F_0) \cap \mathbb{W}_0$ be as large as possible. To this end, we make use of the following result (see Lemma 2.1.1 in Fursikov, 2000).

Lemma 1 *Let Λ be a convex closed subset of a Banach space $\mathbb{W}$ such that Λ contains a countable dense subset, and let $w^0 := F(y^0) + F_0 \in \Lambda$. Then, there exists a Banach space $\mathbb{W}_0$ that is continuously embedded in $\mathbb{W}$ and dense in* $\overline{\mathrm{Lin}\,(\Lambda - w^0)} := \mathrm{cl}_{\|\cdot\|_{\mathbb{W}}}\left[\mathrm{Lin}\,(\Lambda - F(y^0) - F_0)\right]$ *such that the condition* (8) *holds true.*

To deduce an optimality system for the problem (1)–(3) under the assumption that $\mathrm{int}_{\|\cdot\|_{\mathbb{W}}}\Lambda = \emptyset$, we have to assume that for given spaces $\mathbb{Y}_1$, $\mathbb{Y}$, $\mathbb{Z}$, $\mathbb{U}$, and $\mathbb{W}$, in which the problem (1)–(3) is stated and its solution is defined, we can indicate a new collection of Banach spaces $\mathbb{Y}_0$, $\mathbb{Z}_0$, $\mathbb{U}_0$, and $\mathbb{W}_0$ such that the following conditions for mappings

$$\begin{aligned} J(u,y) &= I(u^0+u, y^0+y) - J(u^0,y^0),\\ G(u,y) &= L(u^0+u, y^0+y) - L(u^0,y^0),\\ H(y) &= F(y^0+y) - F(y^0) \end{aligned} \tag{9}$$

hold true:

(a) The mappings $G : \mathbb{U}_0 \times \mathbb{Y}_0 \to \mathbb{Z}_0$, $H : \mathbb{Y}_0 \to \mathbb{W}_0$, and $J : \mathbb{U}_0 \times \mathbb{Y}_0 \to \overline{\mathbb{R}}$ are well defined for each $(u,y) \in ((U_{ad} - u^0) \cap \mathbb{U}_0) \times \mathbb{Y}_0$, these mappings preserve the properties (linearity, continuity, and lower semicontinuity) described before, and the sets $(U_{ad} - u^0) \cap \mathbb{U}_0$ and $(\Lambda - F(y^0) - F_0) \cap \mathbb{W}_0$ contain more than one element.

(b) For each $u \in (U_{ad} - u^0) \cap \mathbb{U}_0$ the mappings $y \mapsto G(u,y)$ and $y \mapsto J(u,y)$ are continuously differentiable for each $y \in \mathcal{O}_0$, where $\mathcal{O}_0$ is a neighborhood of zero in the space $\mathbb{Y}_0$.

(c) The mapping $u \mapsto J(u,y)$ is convex and Gateaux-differentiable on $\mathbb{U}_0$ at the point $y = 0$.

(d) The mapping $u \mapsto G(u,y)$ is continuous as a mapping from $\mathbb{U}_0$ to $\mathbb{Z}_0$ and affine for any $y \in \mathcal{O}_0$.

(e) The assumption (8) is valid.

Then, the necessary optimality conditions for $SCCP$ (1)–(3) can be stated as follows (see Theorem 2.1.4 in Fursikov, 2000).

Theorem 2 *Let $(u,y) = (0,0) \in \mathbb{U}_0 \times \mathbb{Y}_0$ be a solution to the problem*

$$J(u,y) \to \inf, \tag{10}$$

$$G(u,y) = 0, \quad H(y) \in (\Lambda - F(y^0) - F_0) \cap \mathbb{W}_0, \quad u \in (U_{ad} - u^0) \cap \mathbb{U}_0. \tag{11}$$

Assume that conditions (a)–(d) hold true and $\operatorname{Im} G'_y(0,0)$ *is closed and has a finite codimension in* $\mathbb{Z}_0$*. Then there exists a pair of Lagrange multipliers* $(\lambda, p) \in (\mathbb{R}_+ \times \mathbb{Z}_0^*) \setminus \{0\}$ *such that the relations*

$$\left\langle \mathcal{L}'_y(0,0,\lambda,p), h\right\rangle_{\mathbb{Y}_0^*;\mathbb{Y}_0} = 0, \quad \forall\, h \in \mathbb{Y}_0 \ s.t.\ H(h) \in (\Lambda - F(y^0) - F_0) \cap \mathbb{W}_0, \tag{12}$$

$$\langle \mathcal{L}'_u(0,0,\lambda,p), u\rangle_{\mathbb{U}_0^*;\mathbb{U}_0} \geq 0, \quad \forall\, u \in (U_{ad} - u^0) \cap \mathbb{U}_0 \tag{13}$$

hold true, where the Lagrange function $\mathcal{L}$ *is given by*

$$\mathcal{L}(u,y,\lambda,p) := \lambda J(u,y) + \langle p, G(u,y)\rangle_{\mathbb{Z}_0^*;\mathbb{Z}_0}\,. \tag{14}$$

Moreover, if $\operatorname{Im} G'_y(0,0) = \mathbb{Z}_0$*, then we can suppose that* $\lambda = 1$ *in* (12)–(14).

It is amply clear that any implementation of the optimality conditions given by Theorem 2 may be difficult to realize. Therefore, a suitable relaxation of the original $SCCP$ seems to be at order. In particular, it seems to be reasonable to replace the convex cone Λ by a larger solid convex cone. The main idea here is to use the so-called *Henig dilating cone*.

3. Henig relaxation of $SCCP$s

The main goal of this section is to provide a regularization of the state constraints $F(y) \in \Lambda$ by replacing the ordering cone Λ by its solid Henig approximation $\Lambda_\varepsilon(B)$ (see Henig, 1982 a,b; Zhuang, 1994) and show that the solvability of the original $SCCP$ (1)–(3) can be characterized by solving the corresponding Henig relaxed problems in the limit as some small parameter ε tends to zero. As was shown in the recent publication by Khan and Sama (2013), the proposed approach is numerically viable for some state-constrained convex optimization problems.

We begin with some formal descriptions and abstract results. By analogy with the previous section, we suppose that $\mathbb{W}$ is a reflexive separable Banach space, and $\Lambda \subset \mathbb{W}$ is a closed convex pointed ordering cone.

Definition 1 *A nonempty convex subset* B *of a nontrivial ordering cone* $\Lambda \subset \mathbb{W}$ *(i.e.* $\Lambda \neq \{0_{\mathbb{W}}\}$*, where* $0_{\mathbb{W}}$ *is the zero element in* $\mathbb{W}$*) is called base of* Λ *if for each element* $z \in \Lambda\backslash\{0_{\mathbb{W}}\}$ *there is a unique representation* $z = \mu b$*, where* $\mu > 0$ *and* $b \in B$*.*

We note that, in general, bases are not unique. We denote the norm of $\mathbb{W}$ by $\|\cdot\|_{\mathbb{W}}$, and for arbitrary elements $z_1, z_2 \in \mathbb{W}$ we define

$$z_1 \leq_\Lambda z_2 \ \Leftrightarrow\ z_2 - z_1 \in \Lambda \ \text{ as well as } \ z_1 <_\Lambda z_2 \ \Leftrightarrow\ z_2 - z_1 \in \Lambda\backslash\{0_{\mathbb{W}}\}\,.$$

In order to introduce a representation for a base of Λ, let $\mathbb{W}^*$ be the topological dual space of $\mathbb{W}$, and let $\langle\cdot,\cdot\rangle_{\mathbb{W}^*,\mathbb{W}}$ be the dual pairing. Moreover, by

$$\Lambda^* := \left\{z^* \in \mathbb{W}^* \,\middle|\, \langle z^*, z\rangle_{\mathbb{W}^*,\mathbb{W}} \geq 0 \ \ \forall\, z \in \Lambda\right\}$$

and

$$\Lambda^\sharp := \big\{z^* \in \mathbb{W}^* \,\big|\, \langle z^*, z\rangle_{\mathbb{W}^*,\mathbb{W}} > 0 \;\; \forall\, z \in \Lambda\backslash\{0_{\mathbb{W}}\}\big\}$$

we define the *dual cone* and the *quasi-interior of the dual cone* of Λ, respectively. Using the definition of the dual cone, the ordering cone Λ can be characterized as follows (see Lemma 3.21 in Jahn, 2004):

$$\Lambda = \big\{z \in \mathbb{W} \,\big|\, \langle z^*, z\rangle_{\mathbb{W}^*,\mathbb{W}} \geq 0 \;\; \forall\, z^* \in \Lambda^*\big\}.$$

Due to Lemma 1.28 from Jahn (2004), we can give the following result:

Lemma 2 *Let $\Lambda \subset \mathbb{W}$ be a nontrivial ordering cone. Then the set $B := \{z \in \Lambda \,|\, \langle z^*, z\rangle_{\mathbb{W}^*,\mathbb{W}} = 1\}$ is a base of Λ for every $z^* \in \Lambda^\sharp$. Moreover, if Λ is reproducing in $\mathbb{W}$, i.e. if $\Lambda - \Lambda = \mathbb{W}$, and if B is a base of Λ, then there is an element $z^* \in \Lambda^\sharp$ satisfying $B = \{z \in \Lambda \,|\, \langle z^*, z\rangle_{\mathbb{W}^*,\mathbb{W}} = 1\}$.*

Taking into account this lemma, we assume the validity of the following Hypothesis:

(H3) (Quasi-interior non-emptiness condition) There exists an element $z^* \in \mathbb{W}$ such that $z^* \in \Lambda^\sharp$, that is, the ordering cone Λ has a closed base $B \subset \Lambda$.

Now, we are prepared to introduce the definition of a so-called *Henig dilating cone* (see Zhuang, 1994) which is based on the existence of a closed base of ordering cone Λ.

Definition 2 *Let $\Lambda \subset \mathbb{W}$ be a closed ordering cone with a closed base B. Upon choosing $\varepsilon > 0$ arbitrarily, the corresponding Henig dilating cone is defined by*

$$\Lambda_\varepsilon(B) := \mathrm{cl}_{\|\cdot\|_{\mathbb{W}}}\Big(\mathrm{cone}\,\big(B + B_\varepsilon(0_{\mathbb{W}})\big)\Big) := \mathrm{cl}_{\|\cdot\|_{\mathbb{W}}}\Big(\big\{\mu z \,\big|\, \mu \geq 0, z \in B + B_\varepsilon(0_{\mathbb{W}})\big\}\Big), \tag{15}$$

where $\frac{1}{\varepsilon}B_\varepsilon(0_{\mathbb{W}}) := \{v \in \mathbb{W} \mid \|v\|_{\mathbb{W}} \leq 1\}$ is the closed unit ball in $\mathbb{W}$ centered at the origin.

It is clear that $\Lambda_\varepsilon(B)$ depends on the particular choice of B. As follows from this definition, $\mathrm{int}\,(\Lambda)_\varepsilon(B) \neq \emptyset$ for every $\varepsilon > 0$, i.e. the Henig dilating cone is proper solid. Moreover, we have the following properties of such cones (see Schiel, 2014; Zhuang, 1994).

Proposition 1 *Let $\mathbb{W}$ be a normed space, and let $\Lambda \subset \mathbb{W}$ be a closed ordering cone with a closed base B. Upon choosing $\varepsilon \in (0, \delta)$, where*

$$\delta := \inf\big\{\|b\|_{\mathbb{W}} \,\big|\, b \in B\big\} > 0, \tag{16}$$

the following statements hold true:

(i) $\Lambda_\varepsilon(B)$ *is pointed, i.e.* $\Lambda_\varepsilon(B) \cap \big(-\Lambda_\varepsilon(B)\big) = \{0_{\mathbb{W}}\}$;

(ii) $\Lambda_\varepsilon(B) \subset \Lambda_{\varepsilon+\gamma}(B) \;\; \forall\, \gamma > 0$;

(iii) $\Lambda_\varepsilon(B) = \mathrm{cone}\,\Big(\mathrm{cl}_{\|\cdot\|_{\mathbb{W}}}\big(B + B_\varepsilon(0_{\mathbb{W}})\big)\Big)$;

(iv) $\Lambda = \bigcap_{0<\varepsilon<\delta} \Lambda_\varepsilon(B)$;

(v) *the implication*

$$\xi \in \Lambda_\varepsilon(B) \quad \Longrightarrow \quad \frac{\varepsilon}{\kappa+\varepsilon}\|\xi\|_{\mathbb{W}} + \xi \notin (-\Lambda), \tag{17}$$

$$i.\ e.\ \ \xi \not\leq_\Lambda -\frac{\varepsilon}{\kappa+\varepsilon}\|\xi\|_{\mathbb{W}}$$

holds true with $\kappa = \sup\{\|\zeta\|_{\mathbb{W}} \ : \ \zeta \in B\}$.

In the context of the constraint qualifications problem, the following result plays an important role.

PROPOSITION 2 *Let* $\Lambda \subset \mathbb{W}$ *be a closed ordering cone with a closed base* B. *Upon choosing* $\varepsilon \in (0, \delta)$ *arbitrarily, where the parameter* δ *is defined by* (16), *the inclusion*

$$\Lambda \subset \{0_{\mathbb{W}}\} \cup \mathrm{int}\big(\Lambda_\varepsilon(B)\big) \tag{18}$$

holds true.

PROOF Let $z \in \Lambda \setminus \{0_{\mathbb{W}}\}$ be chosen arbitrarily. By the definition of a base, there is a unique representation $z = \lambda b$ with $\lambda > 0$ and $b \in B$. Obviously,

$$z \in \mathrm{int}\big(\{\lambda b\} + B_{\lambda\varepsilon}(0_{\mathbb{W}})\big) = \mathrm{int}\big(B_{\lambda\varepsilon}(\lambda b)\big)$$

holds true. Assume for a moment that

$$B_{\lambda\varepsilon}(\lambda b) \subseteq \mathrm{cone}\big(\{b\} + B_\varepsilon(0_{\mathbb{W}})\big). \tag{19}$$

Then we obtain

$$z \in \mathrm{int}\Big(\mathrm{cone}\big(\{b\} + B_\varepsilon(0_{\mathbb{W}})\big)\Big) \subseteq \mathrm{int}\Big(\mathrm{cone}\big(B + B_\varepsilon(0_{\mathbb{W}})\big)\Big) = \mathrm{int}\big(\Lambda_\varepsilon(B)\big),$$

which completes the proof. In order to show (19), let $x \in B_{\lambda\varepsilon}(\lambda b)$ be chosen arbitrarily, i.e.

$$\|x - \lambda b\|_{\mathbb{W}} \le \lambda\varepsilon.$$

Then

$$\Big\|\frac{x}{\lambda} - b\Big\|_{\mathbb{W}} = \frac{1}{\lambda}\|x - \lambda b\|_{\mathbb{W}} \le \frac{\lambda\varepsilon}{\lambda} = \varepsilon$$

yields

$$x \in \big\{\mu y \,\big|\, \|y - b\|_{\mathbb{W}} \le \varepsilon, \mu \ge 0\big\} = \mathrm{cone}\big(\{b\} + B_\varepsilon(0_{\mathbb{W}})\big).$$

As a result, (19) is satisfied. □

REMARK 1 *The following property, deriving from Proposition 2, turns out to be rather useful: in order to prove the inclusion* $z \in \mathrm{int}_{\|\cdot\|_{\mathbb{W}}}\Lambda_\varepsilon(B)$, *it is sufficient to check whether* $z \in \Lambda \setminus \{0_{\mathbb{W}}\}$.

The following result shows that Henig dilating cones $\Lambda_\varepsilon(B)$ possess good approximation properties.

Proposition 3 *Let Λ be a closed ordering cone in a normed space $\mathbb{W}$, and let B be an arbitrary closed base of Λ. Let parameter δ be defined as in* (16)*, and let $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,\delta)$ be a monotonically decreasing sequence such that $\lim_{k\to\infty} \varepsilon_k = 0$. Then, the sequence of cones $\{\Lambda_{\varepsilon_k}(B)\}_{k\in\mathbb{N}}$ converges to Λ in Kuratowski sense with respect to the norm topology of $\mathbb{W}$ as k tends to infinity, that is*

$$K-\liminf_{k\to\infty} \Lambda_{\varepsilon_k}(B) = \Lambda = K-\limsup_{k\to\infty} \Lambda_{\varepsilon_k}(B),$$

where

$$\begin{aligned} K-\liminf_{k\to\infty} \Lambda_{\varepsilon_k}(B) &:= \big\{z \in \mathbb{W} \,\big|\, \textit{for all neighborhoods } N \textit{ of } z \textit{ there is a} \\ &\qquad k_0 \in \mathbb{N} \textit{ such that } N \cap \Lambda_{\varepsilon_k}(B) \neq \emptyset \;\; \forall\, k \geq k_0\big\}, \\ K-\limsup_{k\to\infty} \Lambda_{\varepsilon_k}(B) &:= \big\{z \in \mathbb{W} \,\big|\, \textit{for all neighborhoods } N \textit{ of } z \textit{ and every } k_0 \in \mathbb{N} \\ &\qquad \textit{there is a } k \geq k_0 \textit{ such that } N \cap \Lambda_{\varepsilon_k}(B) \neq \emptyset\big\}. \end{aligned}$$

Proof Let $z \in \Lambda$ be chosen arbitrarily. Then $N \cap \Lambda \neq \emptyset$ holds true for every neighborhood N of z, and due to the inclusions $\Lambda \subset \Lambda_{\varepsilon_k}$ $\forall\, k \in \mathbb{N}$, we see that $N \cap \Lambda_{\varepsilon_k} \neq \emptyset$ for all $k \in \mathbb{N}$. Hence,

$$\Lambda \subseteq K-\liminf_{k\to\infty} \Lambda_{\varepsilon_k}(B). \tag{20}$$

Taking into account the inclusion (20) and the fact that

$$K-\liminf_{k\to\infty} \Lambda_{\varepsilon_k}(B) \subseteq K-\limsup_{k\to\infty} \Lambda_{\varepsilon_k}(B),$$

we get

$$\Lambda \subseteq K-\liminf_{k\to\infty} \Lambda_{\varepsilon_k}(B) \subseteq K-\limsup_{k\to\infty} \Lambda_{\varepsilon_k}(B). \tag{21}$$

To show that the sequence $\{\Lambda_{\varepsilon_k}(B)\}_{k\in\mathbb{N}}$ converges to Λ in Kuratowski sense, it remains to show that

$$K-\limsup_{k\to\infty} \Lambda_{\varepsilon_k}(B) \subseteq \Lambda. \tag{22}$$

However, the inclusion (22) is equivalent to

$$(\mathbb{W}\backslash\Lambda) \subseteq \left(\mathbb{W}\backslash K-\limsup_{k\to\infty} \Lambda_{\varepsilon_k}(B)\right). \tag{23}$$

Let $\bar{z} \in \mathbb{W}\backslash\Lambda$ be an arbitrary element. Since Λ is closed, there is an open neighborhood $\bar{N}$ of $\bar{z}$ with respect to the norm topology of $\mathbb{W}$ such that $\bar{N} \cap \Lambda =$

$\emptyset$. By Proposition 1 (see item (iv)), there is a sufficiently big index $k_0 \in \mathbb{N}$ such that

$$\bar{N} \cap \Lambda_{\varepsilon_k}(B) = \emptyset \ \ \forall\, k \geq k_0.$$

This implies

$$\bar{z} \in \mathbb{W} \setminus \limsup_{k\to\infty} \Lambda_{\varepsilon_k}(B).$$

Combining (21), (22), and (23), we arrive at the relation

$$\Lambda \subseteq K\!-\liminf_{k\to\infty} \Lambda_{\varepsilon_k}(B) \subseteq K\!-\limsup_{k\to\infty} \Lambda_{\varepsilon_k}(B) \subseteq \Lambda.$$

Thus, $\Lambda = K\!-\lim_{k\to\infty} \Lambda_{\varepsilon_k}(B)$ and the proof is complete. □

Taking these results into account, we associate with $SCCP$ (1)–(3) the following Henig relaxed problem

$$I(u,y) \to \inf \tag{24}$$

subject to the constraints

$$\left.\begin{array}{c} L(u,y) + f = 0, \\ F(y) + F_0 \in \Lambda_\varepsilon(B), \quad u \in U_{ad}, \end{array}\right\} \tag{25}$$

or, in a more compact form, this problem can be stated as follows

$$\inf_{(u,y)\in \Xi_\varepsilon} I(u,y), \quad \forall\, \varepsilon \in (0,\delta), \tag{26}$$

where

$$\delta = \inf\left\{ \|\xi\|_{\mathbb{W}} \ : \ \xi \in B \right\}, \tag{27}$$

the base B takes the form $B = \{z \in \Lambda \,|\, \langle z^*, z\rangle_{\mathbb{W}^*,\mathbb{W}} = 1\}$ for a given element $z^* \in \Lambda^\sharp$, and we define the set of admissible solutions $\Xi_\varepsilon \subset \mathbb{U} \times \mathbb{Y}_1$ as follows: $(u,y) \in \Xi_\varepsilon$ if and only if $u \in U_{ad}$, $y \in \mathbb{Y}_1$, $I(u,y) < +\infty$, the pair (u,y) is related by linear operator equation $L(u,y) + f = 0$, and $F(y) + F_0 \in \Lambda_\varepsilon(B)$.

Remark 2 *Since, by Proposition 2, the inclusion $\Xi \subseteq \Xi_\varepsilon$ holds true for all $\varepsilon > 0$, it is reasonable to call the $SCCP_\epsilon$ (26) a relaxation of $SCCP$ (1)–(3). Moreover, as this obviously follows from Proposition 3, the convergence $\Xi_\varepsilon \overset{\varepsilon\to 0}{\to} \Xi$ in Kuratowski sense holds true with respect to the τ-topology on $\mathbb{U} \times \mathbb{Y}_1$.*

We are now in a position to show that using the relaxation approach we can reduce the main suppositions of Theorem 1. In particular, we can characterize Hypothesis (H_1) by the nontriviality property of the corresponding Henig relaxed problems.

Theorem 3 *Let $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\delta)$ be a monotonically decreasing sequence converging to 0 as $k\to\infty$. Assume that conditions (i)–(ii) of Theorem 1 and Hypothesis (H3) are valid. Then, the Hypothesis (H1) implies that the Henig relaxed problem* (26) *has a nonempty feasible set Ξ_ε for all $\varepsilon=\varepsilon_k$, $k\in\mathbb{N}$. And vice versa, if there exists a sequence $\left\{(u^k,y^k)\right\}_{k\in\mathbb{N}}$ such that*

$$(u^k,y^k)\in\Xi_{\varepsilon_k} \ \text{ for all } \ k\in\mathbb{N}, \quad \text{ and } \ \left\{(u^k,y^k)\right\}_{k\in\mathbb{N}} \text{ is bounded in } \mathbb{U}\times\mathbb{Y}_1, \tag{28}$$

then each of τ-cluster pairs of this sequence is admissible with respect to the original SCCP (1)–(3).

Proof Since the implication $\left(\Xi\neq\emptyset\right)\Longrightarrow\left(\Xi_\varepsilon\neq\emptyset \text{ for all } \ \varepsilon>0\right)$ is obvious by Proposition 3 and Remark 2, we concentrate on the proof of the inverse statement — non-triviality of the Henig relaxed problems $\inf_{(u,y)\in\Xi_{\varepsilon_k}} I(u,y)$ for all $k\in\mathbb{N}$, together with the extra property (28), imply the existence of at least one pair (u,y) such that $(u,y)\in\Xi$.

Let $\left\{(u^k,y^k)\in\Xi_{\varepsilon_k}\right\}_{k\in\mathbb{N}}\subset\mathbb{U}\times\mathbb{Y}_1$ be an arbitrary bounded sequence. Then, by compactness arguments, we can assert the existence of a subsequence of $\left\{(u^k,y^k)\right\}_{k\in\mathbb{N}}$ (still denoted by the same index) and a pair $(u^*,y^*)\in\mathbb{U}\times\mathbb{Y}_1$ such that

$$(u^k,y^k)\xrightarrow{\ \tau\ }(u^*,y^*) \ \text{ in } \mathbb{U}\times\mathbb{Y}_1 \text{ as } k\to\infty.$$

Closely following the proof of Theorem 1, it can be shown that the limit pair (u^*,y^*) is such that $u^*\in U_\partial$, $I(u^*,y^*)<+\infty$, and $L(u^*,y^*)+f=0$. It remains to establish that

$$F(y^*)+F_0\in\Lambda. \tag{29}$$

By contraposition, let us assume that $\xi^*:=F(y^*)+F_0\in\mathbb{W}\setminus\Lambda$. Since the cone Λ is closed, it follows that there is a neighborhood $\mathcal{N}(\xi^*)$ of ξ^* in $\mathbb{W}$ such that $\mathcal{N}(\xi^*)\cap\Lambda=\emptyset$. Using the fact that

$$\Lambda\subset(\Lambda)_{\varepsilon_k}(B)\subseteq(\Lambda)_{\varepsilon_l}(B), \quad \forall\, k\geq l,$$

by Proposition 3 and definition of the Kuratowski limit, it is easy to conclude the existence of an index $k_0\in\mathbb{N}$ such that

$$\mathcal{N}(\xi^*)\cap(\Lambda)_{\varepsilon_k}(B)=\emptyset, \quad \forall\, k\geq k_0. \tag{30}$$

However, due to the compactness property of the mapping $F:\mathbb{Y}\to\mathbb{W}$ (see assumption (ii) of Theorem 1), there is an index $k_1\in\mathbb{N}$ satisfying

$$\xi^k\in\mathcal{N}(\xi^*), \quad \forall\, k\geq k_1. \tag{31}$$

Combining (30) and (31), we finally obtain

$$\xi^k=F(y^k)+F_0\in\mathbb{W}\setminus(\Lambda)_{\varepsilon_k}(B), \quad \forall\, k\geq\max\{k_0,k_1\}.$$

This, however, is a contradiction to

$$F(y^k) + F_0 \in \Lambda, \quad \forall\, k \in \mathbb{N}.$$

Thus, $F(y^*) + F_0 \in \Lambda$. Hence, the pair (u^*, y^*) is admissible for $SCCP$ (1)–(3). □

Our next intention is to show that some optimal solutions for the original $SCCP$ (1)–(3) can be attained by solving the corresponding Henig relaxed problems (24)–(25). To begin with, it is worth to notice that as an obvious consequence of Theorems 1 and 3, we have the following quite significant property of Henig relaxed problems:

COROLLARY 1 *Assume that conditions (i)–(ii) of Theorem 1 and Hypotheses (H1) and (H3) are valid. Assume also that, instead of (H2), the following enhanced Hypothesis is satisfied*

(H2) (Enhanced coerciveness condition) For every* $\lambda > 0$*, the set*

$$\Xi_{\delta,\lambda} = \{(u, y) \in \Xi_\delta \;:\; I(u, y) \le \lambda\}$$

is bounded in $\mathbb{U} \times \mathbb{Y}_1$*, where*

$$\Xi_\delta := \left\{ (u, y) \in \mathbb{U} \times \mathbb{Y}_1 \;\middle|\; \begin{array}{c} L(u, y) + f = 0, \\ u \in U_{ad},\; F(y) + F_0 \in \Lambda_\delta(B), \\ I(u, y) < +\infty \end{array} \right\}.$$

Then the Henig relaxed problem (26) *is solvable for each* $\varepsilon \in (0, \delta)$*.*

As for the asymptotic properties of solutions to the relaxed problems (24)–(25), we have:

THEOREM 4 *Let* $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0, \delta)$ *be a monotonically decreasing numerical sequence such that* $\varepsilon_k \to 0$ *as* $k \to \infty$*, where* $\delta > 0$ *is defined by* (27)*. Assume that conditions (i)–(ii) of Theorem 1 and Hypotheses (H1),(H2*), and (H3) are valid. Let* $\left\{(u^{k,0}, y^{k,0}) \in \Xi_{\varepsilon_k}\right\}_{k\in\mathbb{N}}$ *be an arbitrary sequence of optimal solutions to the Henig relaxed problems* (24)–(25)*. Then, this sequence is relatively compact with respect to the* τ*-convergence in* $\mathbb{U} \times \mathbb{Y}_1$ *and each of its* τ*-cluster pairs* (u^0, y^0) *is such that*

$$(u^0, y^0) \in \Xi, \quad \text{and} \quad \inf_{(u,y)\in\Xi} I(u, y) = I(u^0, y^0) = \lim_{k\to\infty} I(u^{k,0}, y^{k,0}) = I(u^0, y^0). \tag{32}$$

PROOF Let $\left\{(u^{k,0}, y^{k,0}) \in \Xi_{\varepsilon_k}\right\}_{k\in\mathbb{N}}$ be a given sequence of optimal solutions to the Henig relaxed problems. Following Hypothesis (H1), we can fix an arbitrary pair $(\widehat{u}, \widehat{y})$ such that $(\widehat{u}, \widehat{y}) \in \Xi$. Then, $(\widehat{u}, \widehat{y})$ is admissible for each of the Henig relaxed problems (24)–(25), and, hence,

$$\inf_{(u,y)\in\Xi_{\varepsilon_k}} I(u, y) = I(u^{k,0}, y^{k,0}) \le I(\widehat{u}, \widehat{y}) < +\infty, \quad \forall\, k \in \mathbb{N}. \tag{33}$$

By Proposition 2, the inclusion $\Xi \subseteq \Xi_{\varepsilon_k}$ holds true for all $\varepsilon_k > 0$, and the sequence $\{\Xi_{\varepsilon_k}\}_{k\in\mathbb{N}}$ is monotone in the following sense (because of the property (ii) of Proposition 1)

$$\Xi_\delta \supseteq \Xi_{\varepsilon_1} \supseteq \Xi_{\varepsilon_2} \supseteq \cdots \supseteq \Xi_{\varepsilon_k} \supseteq \cdots \supseteq \Xi \neq \emptyset. \tag{34}$$

Hence,

$$\inf_{(u,y)\in\Xi_\delta} I(u,y) \le \cdots \le \inf_{(u,y)\in\Xi_{\varepsilon_k}} I(u,y) \le \cdots \le \inf_{(u,y)\in\Xi} I(u,y) \le I(\widehat{u},\widehat{y}).$$

As a result, (33) and Hypothesis (H2*) imply the boundedness of the sequence of optimal pairs $\left\{(u^{k,0},y^{k,0}) \in \Xi_{\varepsilon_k}\right\}_{k\in\mathbb{N}}$ in $\mathbb{U}\times\mathbb{Y}_1$. Hence, this sequence is relatively τ-compact, and for each of its τ-cluster pairs (u^0,y^0) the inclusion $(u^0,y^0)\in\Xi$ is a direct consequence of Theorem 3. It remains to show that the limit pair (u^0,y^0) is a solution to $SCCP$ (1)–(3). Indeed, by Theorem 1, the original $SCCP$ has a nonempty set of solutions. Let (u^*,y^*) be one of them. Then, the following inequality is obvious

$$I(u^*,y^*) \le I(u^0,y^0). \tag{35}$$

On the other hand, by Proposition 1 (see property (iv)), we have $(u^*,y^*)\in\Xi_{\varepsilon_k}$ for every $k\in\mathbb{N}$. Since $\left\{(u^{k,0},y^{k,0})\right\}_{i\in\mathbb{N}}$ are the solutions to the corresponding relaxed problems (26), it follows that

$$\inf_{(u,y)\in\Xi_{\varepsilon_k}} I(u,y) = I(u^{k,0},y^{k,0}) \le I(u^*,y^*), \quad \forall\, k\in\mathbb{N}. \tag{36}$$

As a result, taking into account the relations (35) and (36), and the lower semicontinuity property of the cost functional I with respect to the τ-convergence, we have

$$\begin{aligned}\inf_{(u,y)\in\Xi} I(u,y) = I(u^*,y^*) &\overset{\text{by (36)}}{\ge} \limsup_{k\to\infty} I(u^{k,0},y^{k,0}) \\ &\ge \liminf_{i\to\infty} I(u^{k_i,0},y^{k_i,0}) \ge I(u^0,y^0) \overset{\text{by (35)}}{\ge} I(u^*,y^*).\end{aligned}$$

Thus,

$$\inf_{(u,y)\in\Xi} I(u,y) = \lim_{k\to\infty} I(u^{k,0},y^{k,0}) = I(u^0,y^0),$$

and we arrive at the desired property $(32)_2$. The proof is complete. □

4. Optimality conditions for the Henig relaxed problem and their asymptotic analysis

As was mentioned at the beginning of Section 3, the main benefit of the relaxed optimal control problems (24)–(25) comes from the fact that the Henig dilating cone $(\Lambda)_\varepsilon(B)$ has a nonempty topological interior. Hence, there is an opportunity to apply the Slater-type constraint qualification in order to characterize the optimal solutions for the $SCCP_\epsilon$ (24)–(25) at each ε-level by the standard multiplier rule, see Itô and Kunisch (2008). With that in mind, we make use of the following well-know result, proved by Bonnans and Casas (2004, 2008).

THEOREM 5 *Let U and Z be two Banach spaces, U being separable, and $K \subset U$ and $C \subset Z$ be two convex subsets, C having a nonempty interior. Given $z_0 \in \mathrm{int}_{\|\cdot\|_Z} C$ and $\sigma > 0$, let $C_\sigma = (1-\sigma)z_0 + \sigma C$ and $u^0 \in K$ be a solution of the problem*

$$J(u) \to \inf, \quad u \in K \quad \text{and} \quad G(u) \in C_\sigma, \tag{37}$$

where $J : U \to \mathbb{R}$ and $G : U \to Z$ are of class C^1. Then there exist a real number $\lambda \geq 0$ and an element $\mu \in Z^*$ such that

$$\lambda + \|\mu\|_{Z^*} > 0, \tag{38}$$

$$\left\langle \mu, z - G(u^0) \right\rangle_{Z^*;Z} \leq 0, \quad \forall\, z \in C_\sigma, \tag{39}$$

$$\left\langle \lambda J'(u^0) + \left[DG(u^0)\right]^* \mu, u - u^0 \right\rangle_{U^*;U} \geq 0, \quad \forall\, u \in K. \tag{40}$$

Moreover, λ can be taken equal to one in the following cases:

1. *If the following Slater-type condition is satisfied:*

$$\exists \widehat{u} \in K \quad \text{such that} \quad G(u^0) + \left[DG(u^0)\right](\widehat{u} - u^0) \in \mathrm{int}_{\|\cdot\|_Z} C_\sigma. \tag{41}$$

2. *There exists an interval $[\sigma_1, \sigma_2]$ such that the extremal problem (37) has a solution for each σ of this interval.*

We begin with some specifications of the main assumptions of Section 2.

(j) Let $\mathbb{Y}$ and $\mathbb{Z}$ be linear normed spaces, let $\mathbb{Y}_1$, $\mathbb{U}$, and $\mathbb{W}$ be reflexive Banach spaces, let $\mathbb{Y}_1$ be continuously and densely embedded in $\mathbb{Y}$, let U_{ad} be a closed convex subset of $\mathbb{U}$, and let $\Lambda \subset \mathbb{W}$ be a closed convex pointed cone with an empty topological interior.

(jj) Let $L(u,y) := L_1(u) + L_2(y)$ be a linear mapping such that L_1 continuously acts from $(\mathbb{U}, w_{\mathbb{U}})$ to $(\mathbb{Z}, w_{\mathbb{Z}})$, and L_2 is a continuous linear operator from $(\mathbb{Y}_1, w_{\mathbb{Y}_1})$ to $(\mathbb{Z}, w_{\mathbb{Z}})$. Moreover, there exists a subspace $\mathbb{S}$ of $\mathbb{Y}_1$, possibly empty, such that $y \mapsto L_2(y)$ is an isomorphism from the quotient space $\mathbb{Y}_1/\mathbb{S}$ onto $\mathbb{Z}$.

(jjj) Let $F : \mathbb{Y}_1 \to \mathbb{W}$ be a compact operator, that is, F is a linear bounded operator such that the image under F of any bounded subset of $\mathbb{Y}_1$ is a relatively compact subset of $\mathbb{W}$.

(jv) Let τ-topology be the weak topology on $\mathbb{U} \times \mathbb{Y}_1$ induced by the product of weak topology $w_{\mathbb{U}}$ of $\mathbb{U}$ and weak topology $w_{\mathbb{Y}_1}$ of $\mathbb{Y}_1$. Let $I : \mathbb{U} \times \mathbb{Y} \to \overline{\mathbb{R}}$ be a convex τ-lower semicontinuous and bounded from below cost functional such that $u \mapsto I(u,y)$ and $y \mapsto I(u,y)$ are continuously differentiable on

$\mathbb{U}\times\mathbb{Y}$. In particular, for each $(\widehat{u},\widehat{y})\in\mathbb{U}\times\mathbb{Y}$, there exist linear continuous functionals $I'_u(\widehat{u},\widehat{y})\in\mathbb{U}^*$ and $I'_y(\widehat{u},\widehat{y})\in\mathbb{Y}^*$, satisfying the equalities

$$[D_u I(\widehat{u},\widehat{y})]\,(v)=\langle I'_u(\widehat{u},\widehat{y}),v\rangle_{\mathbb{U}^*;\mathbb{U}}\,,\quad [D_y I(\widehat{u},\widehat{y})]\,(h)=\left\langle I'_y(\widehat{u},\widehat{y}),h\right\rangle_{\mathbb{Y}^*;\mathbb{Y}} \tag{42}$$

for all $v\in\mathbb{U}$ and $h\in\mathbb{Y}$.

Before we pass to the implementation of Theorem 5 for the case of $SCCP_\epsilon$ (24)–(25), we give the following Frechèt differentiability property of the relation $L(u,y)+f=0$.

Lemma 3 *Assume that conditions (j)–(jj) are satisfied. Let $\Phi:\mathbb{U}\to\mathbb{Y}_1/\mathbb{S}$ be the mapping defined by the rule: $\Phi(u)=y$ if and only if the pair (u,y) is related by the operator equation $L(u,y)+f=0$. Then $\Phi:\mathbb{U}\to\mathbb{Y}_1/\mathbb{S}$ is infinitely Frechèt differentiable. Moreover, for every $u,v\in\mathbb{U}$, an element $z:=[D\Phi(u)]\,(v)$ is the unique solution in $\mathbb{Y}_1/\mathbb{S}$ of the following operator equation*

$$L_2(z)+L_1(v)=0\quad in\ \ \mathbb{Z}. \tag{43}$$

Proof Let us define the mapping $G:\mathbb{U}\times\mathbb{Y}_1/\mathbb{S}\to\mathbb{Z}$ as follows: $G(u,y):=L(u,y)+f$. Then, by assumption (jj), this mapping is of the class C^∞. Moreover, since $G'_y(u,y):=L_2(y)$ is an isomorphism from $\mathbb{Y}_1/\mathbb{S}$ onto $\mathbb{Z}$, and $G(u,y)=0$ for each pair, which is related by the original equation $L(u,y)+f=0$, it follows from the Implicit Function Theorem that the mapping Φ is of the class C^∞ and its Frechèt differential is given by the formula

$$\left[G'_y(u,y)\right](z)=-\left[G'_u(u,y)\right](v).$$

To conclude the proof, it remains to note that

$$\left[G'_y(u,y)\right](z)=L_2(z)\quad\text{ and }\quad [G'_u(u,y)]\,(v)=L_1(v).$$

□

We are now in a position to apply Theorem 5 to the Henig relaxed problem (24)–(25). To begin with, we assume that Hypotheses (H1),(H2*), and (H3) are satisfied. Let B be a closed base of the cone Λ. Then, we construct the Henig dilating cone $\Lambda_\varepsilon(B)$ as in (15). Further, we set

$$\begin{gathered}U=\mathbb{U},\ Z=\mathbb{W},\ K=U_{ad},\ C=\Lambda_\varepsilon(B),\\ J(u)=I(u,\Phi(u))\ \text{ and }\ G(u)=F(\Phi(u))+F_0,\quad\forall\,u\in U_{ad},\\ \text{and letting }\sigma=1,\text{ we have }C_\sigma:=\Lambda_\varepsilon(B),\end{gathered} \tag{44}$$

where the mapping $\Phi:\mathbb{U}\to\mathbb{Y}_1/\mathbb{S}$ is defined in Lemma 3, that is, $L(u,\Phi(u))+f=0$ for each $u\in U_{ad}$. Let $(u^0_\varepsilon,y^0_\varepsilon)\in\Xi_\varepsilon$ be an optimal pair to the problem (24)–(25). It is clear now that, by linearity of the operator $F:\mathbb{Y}_1\to\mathbb{W}$, we have

$$\left[DG(u^0)\right](v)=\left[F\circ D\Phi(u^0_\varepsilon)\right](v),\quad\forall\,v\in\mathbb{U}. \tag{45}$$

Since $y^0_\varepsilon = \Phi(u^0_\varepsilon)$, by Lemma 3, the element

$$z := \left[D\Phi(u^0_\varepsilon)\right](u - u^0_\varepsilon) \tag{46}$$

belongs to $\mathbb{Y}_1/\mathbb{S}$ and satisfies the equation

$$L_2(z) + L_1(u - u^0_\varepsilon) = 0 \quad \text{in} \quad \mathbb{Z} \quad \forall\, u \in U_{ad}.$$

Hence, for each test element $p \in \mathbb{Z}^*$, we have the identity

$$\langle L_2^*(p), z\rangle_{\mathbb{Y}_1^*;\mathbb{Y}_1} = -\left\langle L_1^*(p), u - u^0_\varepsilon\right\rangle_{\mathbb{U}^*;\mathbb{U}}, \quad \forall\, u \in U_{ad}. \tag{47}$$

Let $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{W}^*$ be given elements. We define $p \in \mathbb{Z}^*$ as the solution of the following variational problem

$$\begin{aligned} &\text{Find} \;\; p \in \mathbb{Z}^* \;\; \text{such that} \\ \langle L_2^*(p), \varphi\rangle_{\mathbb{Y}_1^*;\mathbb{Y}_1} &= \lambda \left\langle I'_y(u^0_\varepsilon, y^0_\varepsilon), E(\varphi)\right\rangle_{\mathbb{Y}^*;\mathbb{Y}} + \langle F^*(\mu), \varphi\rangle_{\mathbb{Y}_1^*;\mathbb{Y}_1} \\ &= \left\langle \lambda E^* I'_y(u^0_\varepsilon, y^0_\varepsilon) + F^*(\mu), \varphi\right\rangle_{\mathbb{Y}_1^*;\mathbb{Y}_1}, \quad \forall\, \varphi \in \mathbb{Y}_1, \end{aligned} \tag{48}$$

where $E : \mathbb{Y}_1 \to \mathbb{Y}$ stands for the embedding operator. Note that assumption (j) implies the linearity and boundedness of this operator.

As a result, inequality (40) can be represented as follows

$$\begin{aligned} 0 &\le \left\langle \lambda J'(u^0_\varepsilon) + \left[DG(u^0_\varepsilon)\right]^* \mu, u - u^0_\varepsilon \right\rangle_{U^*;U} = \{\text{by (42) and (45)}\} \\ &= \left\langle \lambda I'_y(u^0_\varepsilon, y^0_\varepsilon), E\left(\left[D\Phi(u^0_\varepsilon)\right](u - u^0_\varepsilon)\right)\right\rangle_{\mathbb{Y}_1^*;\mathbb{Y}_1} + \left\langle \lambda I'_u(u^0_\varepsilon, y^0_\varepsilon), u - u^0_\varepsilon\right\rangle_{\mathbb{U}^*;\mathbb{U}} \\ &\quad + \left\langle \mu, \left[F \circ D\Phi(u^0_\varepsilon)\right](u - u^0_\varepsilon)\right\rangle_{\mathbb{W}^*;\mathbb{W}} = \{\text{by (46)}\} \\ &= \left\langle \lambda E^* I'_y(u^0_\varepsilon, y^0_\varepsilon), z\right\rangle_{\mathbb{Y}_1^*;\mathbb{Y}_1} + \left\langle \lambda I'_u(u^0_\varepsilon, y^0_\varepsilon), u - u^0_\varepsilon\right\rangle_{\mathbb{U}^*;\mathbb{U}} + \langle F^*(\mu), z\rangle_{\mathbb{W}^*;\mathbb{W}} = \{\text{by (48)}\} \\ &= \left\langle \lambda I'_u(u^0_\varepsilon, y^0_\varepsilon), u - u^0_\varepsilon\right\rangle_{\mathbb{U}^*;\mathbb{U}} + \langle L_2^*(p), z\rangle_{\mathbb{Y}_1^*;\mathbb{Y}_1} = \{\text{by (47)}\} \\ &= \left\langle \lambda I'_u(u^0_\varepsilon, y^0_\varepsilon), u - u^0_\varepsilon\right\rangle_{\mathbb{U}^*;\mathbb{U}} - \left\langle L_1^*(p), u - u^0_\varepsilon\right\rangle_{\mathbb{U}^*;\mathbb{U}}, \quad \forall\, u \in U_{ad}. \end{aligned}$$

Thus, summing up the transformations given above, we finally arrive at the following specification of Theorem 5 for the Henig relaxed problems (24)–(25).

Theorem 6 *Let B be a closed base of the cone Λ. For given parameter $\varepsilon \in (0, \delta)$, let $(u^0_\varepsilon, y^0_\varepsilon) \in \Xi_\varepsilon$ be a solution to the problem (24)–(25). Then assumptions (j)–(jv) imply the existence of a real number $\lambda_\varepsilon \ge 0$ and of elements $p_\varepsilon \in \mathbb{Z}^*$ and $\mu_\varepsilon \in \mathbb{W}^*$, satisfying the following optimality system (the so-called multiplier rule)*

$$\lambda_\varepsilon + \|\mu_\varepsilon\|_{Z^*} > 0, \tag{49}$$

$$\left\langle \mu_\varepsilon, w_\varepsilon - F(y^0_\varepsilon) - F_0\right\rangle_{\mathbb{W}^*;\mathbb{W}} \le 0, \quad \forall\, w_\varepsilon \in \Lambda_\varepsilon(B), \tag{50}$$

$$L_1(u^0_\varepsilon) + L_2(y^0_\varepsilon) + f = 0 \quad \textit{in} \;\; \mathbb{Z}, \tag{51}$$

$$L_2^*(p_\varepsilon) = \lambda_\varepsilon E^* I'_y(u^0_\varepsilon, y^0_\varepsilon) + F^*(\mu_\varepsilon) \quad \textit{in} \;\; \mathbb{Y}_1^*, \tag{52}$$

$$\left\langle \lambda_\varepsilon I'_u(u^0_\varepsilon, y^0_\varepsilon) - L_1^*(p_\varepsilon), u - u^0_\varepsilon\right\rangle_{\mathbb{U}^*;\mathbb{U}} \ge 0, \quad \forall\, u \in U_{ad}. \tag{53}$$

Moreover, λ_ε can be chosen equal to one if the Slater-type condition is satisfied:

$$\begin{array}{c}\exists\,(\widehat{u},\widehat{y})\in U_{ad}\times\mathbb{Y}_1\quad \textit{such that}\\ L(\widehat{u},\widehat{y})+f=0\qquad \textit{and}\qquad F(\widehat{y})+F_0\in \operatorname{int}_{\|\cdot\|_W}\Lambda_\varepsilon(B).\end{array}\tag{54}$$

Proof It is enough to apply Theorem 5 with notation (44) and take in Theorem 5 $y^0_\varepsilon := G(u^0_\varepsilon) = F(\Phi(u^0_\varepsilon)) + F_0$ and p_ε as the solution of (48). As a result, the relations (49)–(53) follow immediately from (38)–(40). It remains to show that the Slater-type condition (41) can be represented in the form of (54).

Indeed, let $(\widehat{u},\widehat{y})\in U_{ad}\times\mathbb{Y}_1$ be a pair with properties (41). In view of the assumption (jj), we have

$$\begin{array}{l} y^0_\varepsilon + s := \Phi(u^0_\varepsilon) = L_2^{-1}\big(-f - L_1(u^0_\varepsilon)\big)\quad \forall\, s\in\mathbb{S}\\ \text{implies } z := \big[D\Phi(u^0_\varepsilon)\big]\,(\widehat{u}-u^0_\varepsilon) = -L_2^{-1}L_1(\widehat{u}-u^0_\varepsilon).\end{array}$$

Hence, the condition (41) takes the form

$$\begin{array}{c}\exists\,(\widehat{u},\widehat{y})\in U_{ad}\times\mathbb{Y}_1\\ \text{such that } F(z)+F_0\in\operatorname{int}_{\|\cdot\|_Z}\Lambda_\varepsilon(B)\ \text{ and }\ L_2(z)+L_1(\widehat{u}-u^0_\varepsilon)=0.\end{array}\tag{55}$$

Since

$$0 = L_2(z)+L_1(\widehat{u})-L_1(u^0_\varepsilon) = L_2(z)-L_2(\widehat{y})-f-L_1(u^0_\varepsilon) = L_2(z-\widehat{y})-f-L_1(u^0_\varepsilon),$$

it follows that

$$L_2(z-\widehat{y}) = f + L_1(u^0_\varepsilon) = -L_2(y^0_\varepsilon).\tag{56}$$

As a result, (54) follows immediately from (55), (56), and assumption (jj). □

In what follows, we assume that the following hypothesis holds true:

(H1*) (Strict nontriviality condition) The feasible set Ξ to the original problem (1)–(3) contains at least two different pairs $(\widehat{u},\widehat{y})$ and $(\widetilde{u},\widetilde{y})$ such that $F(\widehat{y})+F_0\neq 0_{\mathbb{W}}$ and $F(\widetilde{y})+F_0\neq 0_{\mathbb{W}}$.

Corollary 2 *Assume that Hypotheses (H1*), (H2*), (H3), and assumptions of Theorem 6 are satisfied. Then the Slater condition* (54) *is satisfied, and, hence, optimality conditions to the problem* (24)–(25) *can be represented as follows: if $(u^0_\varepsilon, y^0_\varepsilon)\in\Xi_\varepsilon$ is a solution to the Henig relaxed SCCP* (24)–(25)*, then there exist elements $p_\varepsilon\in\mathbb{Z}^*$ and $\mu_\varepsilon\in\mathbb{W}^*$ such that*

$$\big\langle \mu_\varepsilon, w_\varepsilon - F(y^0_\varepsilon) - F_0\big\rangle_{\mathbb{W}^*;\mathbb{W}} \le 0,\quad \forall\, w_\varepsilon\in\Lambda_\varepsilon(B),\tag{57}$$

$$L_1(u^0_\varepsilon)+L_2(y^0_\varepsilon)+f=0\quad \textit{in}\ \ \mathbb{Z},\tag{58}$$

$$L_2^*(p_\varepsilon) = E^* I'_y(u^0_\varepsilon,y^0_\varepsilon) + F^*(\mu_\varepsilon)\quad \textit{in}\ \ \mathbb{Y}_1^*,\tag{59}$$

$$\big\langle I'_u(u^0_\varepsilon,y^0_\varepsilon) - L_1^*(p_\varepsilon), u-u^0_\varepsilon\big\rangle_{\mathbb{U}^*;\mathbb{U}} \ge 0,\quad \forall\, u\in U_{ad}.\tag{60}$$

PROOF Let (u^0, y^0) be a τ-cluster pair of the sequence $\left\{(u^0_\varepsilon, y^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ of optimal solutions to the Henig relaxed problems (24)–(25). Since all conditions of Theorem 4 are fulfilled, it follows that $(u^0, y^0) \in \Xi$. Then, by Hypothesis (H1*), we may suppose that there exists another admissible pair $(\widehat{u}, \widehat{y})$ such that $F(\widetilde{y}) + F_0 \neq 0_{\mathbb{W}}$. Let us show that the second pair $(\widehat{u}, \widehat{y})$ satisfies the Slater condition (54). Indeed, the fulfillment of the relation $L(\widehat{u}, \widehat{y}) + f = 0$ follows immediately from the admissibility condition $(\widehat{u}, \widehat{y}) \in \Xi$. To show that $F(\widehat{y}) + F_0 \in \mathrm{int}_{\|\cdot\|_W} \Lambda_\varepsilon(B)$, it is enough to observe that $F(\widehat{y}) + F_0 \in \Lambda \setminus \{0_{\mathbb{W}}\}$ and, hence, by inclusion

$$\Lambda \subset \{0_{\mathbb{W}}\} \cup \mathrm{int}_{\|\cdot\|_{\mathbb{W}}} \Lambda_\varepsilon(B),$$

it follows from Remark 1 that $F(\widehat{y}) + F_0 \in \mathrm{int}_{\|\cdot\|_{\mathbb{W}}} \Lambda_\varepsilon(B)$. □

The next step of our analysis in this section is to provide an asymptotic analysis of optimality conditions in the form (57)–(60) as positive parameter ε tends to zero. To this end, we need to introduce a couple of additional assumptions to the collection (j)–(jv) that we postulated before. Namely, hereinafter in this section we adopt the following extra continuity properties:

(v) $\left\{(u_\varepsilon, y_\varepsilon) \rightharpoonup (u, y) \text{ weakly in } \mathbb{U} \times \mathbb{Y}\right\} \Longrightarrow I'_y(u_\varepsilon, y_\varepsilon) \rightharpoonup I'_y(u, y) \text{ in } \mathbb{Y}^*$;

(vj) $\left\{(u_\varepsilon, y_\varepsilon) \xrightarrow{\tau} (u, y) \text{ in } \mathbb{U} \times \mathbb{Y}_1\right\} \Longrightarrow I'_u(u_\varepsilon, y_\varepsilon) \to I'_u(u, y) \text{ strongly in } \mathbb{U}^*$;

(vjj) $L_1^* : \mathbb{Z}^* \to \mathbb{U}^*$ is a compact operator;

(vjjj) there exists a sequence of Lagrange multipliers $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfying relations (57)–(59) such that $\sup_{\varepsilon>0} \|\mu_\varepsilon\|_{\mathbb{W}^*} < +\infty$ (we note that, potentially, the adjoint states p_ε and the multipliers μ_ε are not unique).

Before we pass to the main result of this section, we make use of the following notion:

DEFINITION 3 *We say that an optimal pair $(u^0, y^0) \in \Xi$ to the original SCCP (1)–(3) is attainable if there exist a monotonically decreasing numerical sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0, \delta)$ such that $\varepsilon_k \to 0$ as $k \to \infty$, and a sequence of optimal pairs $\left\{(u^{k,0}, y^{k,0}) \in \Xi_{\varepsilon_k}\right\}_{k\in\mathbb{N}}$ to the Henig relaxed problems* (24)–(25)*, satisfying $(u^{k,0}, y^{k,0}) \xrightarrow{\tau} (u^0, y^0)$ in $\mathbb{U} \times \mathbb{Y}_1$ as $k \to \infty$.*

It is clear that the set of attainable solutions to the original $SCCP$ is always nonempty, provided all assumptions of Theorem 4 are satisfied. Moreover, if, in addition, the cost functional $I(u, y)$ is strictly convex, then the $SCCP$ (1)–(3) has a unique solution, which is assuredly attainable by Theorem 4.

THEOREM 7 *Assume that Hypotheses (H1*), (H2*), (H3) and conditions (j)–(vjjj) are satisfied. Let $(u^0, y^0) \in \Xi$ be an attainable solution to the original*

SCCP (1)–(3). Then, there exist elements $p \in \mathbb{Z}^*$ and $\mu \in \mathbb{W}^*$ such that

$$\left\langle \mu, w - F(y^0) - F_0 \right\rangle_{\mathbb{W}^*;\mathbb{W}} \le 0, \quad \forall\, w \in \Lambda, \tag{61}$$

$$L_1(u^0) + L_2(y^0) + f = 0 \quad in \ \ \mathbb{Z}, \tag{62}$$

$$L_2^*(p) = E^* I_y'(u^0, y^0) + F^*(\mu) \quad in \ \ \mathbb{Y}_1^*, \tag{63}$$

$$\left\langle I_u'(u^0, y^0) - L_1^*(p), u - u^0 \right\rangle_{\mathbb{U}^*;\mathbb{U}} \ge 0, \quad \forall\, u \in U_{ad}. \tag{64}$$

Proof As it follows from Definition 3, for a given pair $(u^0, y^0) \in \Xi$, we can indicate a monotonically decreasing sequence of indices $\{\varepsilon\} \subset (0, \delta)$ and a sequence of optimal pairs $\left\{(u^0_\varepsilon, y^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ to the Henig relaxed problems (24)–(25) such that

$$u^0_\varepsilon \rightharpoonup u^0 \ \text{ in } \ \mathbb{U} \ \text{ as } \ \varepsilon \to 0, \tag{65}$$

$$y^0_\varepsilon \rightharpoonup y^0 \ \text{ in } \ \mathbb{Y}_1 \ \text{ as } \ \varepsilon \to 0. \tag{66}$$

Hence, by the continuity of the embedding $\mathbb{Y}_1 \hookrightarrow \mathbb{Y}$, we have

$$y^0_\varepsilon \rightharpoonup y^0 \ \text{ in } \ \mathbb{Y} \ \text{ as } \ \varepsilon \to 0. \tag{67}$$

By Corollary 2, there exists a sequence of Lagrange multipliers $\{(p_\varepsilon, \mu_\varepsilon)\}_{\varepsilon>0} \subset \mathbb{Z}^* \times \mathbb{W}^*$ such that at each ε-level the tuple $\left(p_\varepsilon, \mu_\varepsilon, u^0_\varepsilon, y^0_\varepsilon\right) \in \mathbb{Z}^* \times \mathbb{W}^* \times \mathbb{U} \times \mathbb{Y}_1$ satisfies the optimality system (57)–(60). In view of condition (vjjj), we may suppose that

$$\|\mu_\varepsilon\|_{\mathbb{W}^*} \le C \quad \forall\, \varepsilon > 0 \quad \text{for some } C > 0.$$

Hence, by reflexivity of the Banach space $\mathbb{W}$, there exists an element $\mu \in \mathbb{W}^*$ such that, up to a subsequence,

$$\mu_\varepsilon \rightharpoonup \mu \ \text{ in } \ \mathbb{W}^* \ \text{ as } \ \varepsilon \to 0. \tag{68}$$

In addition, since $y \mapsto L_2(y)$ is a Banach space isomorphism from $\mathbb{Y}_1/\mathbb{S}$ onto $\mathbb{Z}$ (see (jj)), it follows that the inverse operator $(L_2^*)^{-1} : (\mathbb{Y}_1/\mathbb{S})^* \to \mathbb{Z}^*$ is bounded. Therefore, the sequence of adjoint states

$$\left\{p_\varepsilon := (L_2^*)^{-1}\left[E^* I_y'(u^0_\varepsilon, y^0_\varepsilon) + F^*(\mu_\varepsilon)\right]\right\}_{\varepsilon>0}$$

is uniquely defined by the corresponding triplets $\left\{(u^0_\varepsilon, y^0_\varepsilon, \mu_\varepsilon)\right\}_{\varepsilon>0} \subset \mathbb{U} \times \mathbb{Y}_1 \times \mathbb{W}^*$, and, moreover, there exists a constant $C > 0$ such that

$$\|p_\varepsilon\|_{\mathbb{Z}^*} \le \|(L_2^*)^{-1}\| \left[\|E\| \sup_{\varepsilon>0} |I_y'(u_\varepsilon, y_\varepsilon)| + \|F^*\| \sup_{\varepsilon>o} \|\mu_\varepsilon\|_{\mathbb{W}^*}\right] \overset{\text{by (v) and (68)}}{\le} C. \tag{69}$$

Thus, within a subsequence, we can suppose that there exists an element $p \in \mathbb{Z}^*$ such that

$$p_\varepsilon \rightharpoonup p \ \text{ in } \ \mathbb{Z}^* \ \text{ as } \ \varepsilon \to 0. \tag{70}$$

We are now in a position to pass to the limit in relations (57)–(60) as $\varepsilon \to 0$. Let $\{w_\varepsilon \in \Lambda_\varepsilon(B)\}_{\varepsilon>0}$ be any strongly convergent sequence in $\mathbb{W}$ and let $w \in \mathbb{W}$ be its limit. Then, property (jjj) implies that $w_\varepsilon - F(y^0_\varepsilon) \to w - F(y^0)$ in $\mathbb{W}$ as $\varepsilon \to 0$ and, therefore,

$$0 \geq \lim_{\varepsilon\to 0} \left\langle \mu_\varepsilon, w_\varepsilon - F(y^0_\varepsilon) - F_0 \right\rangle_{\mathbb{W}^*;\mathbb{W}} \overset{\text{by (68)}}{=} \left\langle \mu, w - F(y^0) - F_0 \right\rangle_{\mathbb{W}^*;\mathbb{W}}$$

as the product of weakly and strongly convergent sequences. Moreover, by Proposition 3, we have $w \in \Lambda = K\!-\lim_{\varepsilon\to 0} \Lambda_\varepsilon(B)$. Taking into account the definition of the convergence $\Lambda_\varepsilon(B) \overset{\varepsilon\to 0}{\to} \Lambda$ in Kuratowski sense, we see that the choice of strongly convergent sequence $w_\varepsilon \overset{\varepsilon\to 0}{\to} w$ in $\mathbb{W}$, satisfying $w_\varepsilon \in \Lambda_\varepsilon(B)$ for each $\varepsilon > 0$, can be specified for each element w of Λ. Hence, relation (57) leads us in the limit as $\varepsilon \to 0$ to the inequality (61).

The passage to the limit in (58) can be done in a similar manner as in (5). As a result, we arrive at the relation (62).

In order to pass to the limit in the adjoint state equation (59), it is enough to note that

$$\begin{aligned}
\langle L_2^*(p), \varphi\rangle_{\mathbb{Y}_1^*,\,\mathbb{Y}_1} &= \langle p, L_2(\varphi)\rangle_{\mathbb{Z}^*,\,\mathbb{Z}} \overset{\text{by (70)}}{=} \lim_{\varepsilon\to 0} \langle p_\varepsilon, L_2(\varphi)\rangle_{\mathbb{Z}^*,\,\mathbb{Z}} \\
&= \lim_{\varepsilon\to 0} \left\langle E^* I'_y(u^0_\varepsilon, y^0_\varepsilon) + F^*(\mu_\varepsilon), \varphi \right\rangle_{\mathbb{Y}_1^*,\,\mathbb{Y}_1} \\
&= \lim_{\varepsilon\to 0} \left\langle I'_y(u^0_\varepsilon, y^0_\varepsilon), \varphi \right\rangle_{\mathbb{Y}^*,\,\mathbb{Y}} + \lim_{\varepsilon\to 0} \langle \mu_\varepsilon, F(\varphi)\rangle_{\mathbb{W}^*,\,\mathbb{W}} \\
&\overset{\text{by (v) and (68)}}{=} \left[\left\langle I'_y(u^0, y^0), \varphi \right\rangle_{\mathbb{Y}^*,\,\mathbb{Y}} + \langle \mu, F(\varphi)\rangle_{\mathbb{W}^*,\,\mathbb{W}} \right] \\
&= \left\langle E^* I'_y(u^0, y^0) + F^*(\mu), \varphi \right\rangle_{\mathbb{Y}_1^*,\,\mathbb{Y}_1}, \quad \forall\, \varphi \in \mathbb{Y}_1.
\end{aligned}$$

It remains to realize the limit passage in inequality (60). With that in mind, we make use of assumptions (vj)–(vjj). Following these properties, we see that the τ-convergence (65)–(66) implies the strong convergence

$$I'_u(u^0_\varepsilon, y^0_\varepsilon) \to I'_u(u^0, y^0) \quad \text{and} \quad L_1^*(p_\varepsilon) \to L_1^*(p) \quad \text{in} \quad \mathbb{U}^* \quad \text{as } \varepsilon \to 0.$$

As a result, the right hand side of inequality (60) is a product of weakly and strongly convergent sequences, and, therefore, in the limit we immediately arrive at inequality (60). The proof is complete. □

Remark 3 *As this obviously follows from the proof of Theorem 7, instead of properties (vj)–(vjj), we can assume the following ones.*

*(vj)** $\left\{(u_\varepsilon, y_\varepsilon) \overset{\tau}{\to} (u, y) \text{ in } \mathbb{U}\times\mathbb{Y}_1\right\} \Longrightarrow I'_u(u_\varepsilon, y_\varepsilon) \rightharpoonup I'_u(u, y) \text{ in } \mathbb{U}^*$;

*(vjj)** *The cost functional* $I : \mathbb{U}\times\mathbb{Y} \to \overline{\mathbb{R}}$ *possesses the so-called* $(\mathfrak{N})$*-property:*

$$\left\{ \begin{array}{c} (u_k, y_k) \overset{\tau}{\to} (u, y) \quad in \quad \mathbb{U}\times\mathbb{Y}_1 \\ and \quad \lim_{k\to\infty} I(u_k, y_k) = I\,(u, y) \end{array} \right\} \Longrightarrow \; u_k \to u \quad strongly\ in \quad \mathbb{U}.$$

5. Optimality conditions for state constrained Neumann boundary optimal control problem

Let Ω be a bounded open connected subset of $\mathbb{R}^N$ with $N > 2$. We assume that the boundary $\partial\Omega$ is a Lipschitz continuous $(N-1)$-dimensional manifold, such that Ω is locally situated on one side of $\partial\Omega$. Let $y_d \in L^2(\Omega)$, $h \in L^2(\Omega)$, $\zeta^{max} \in H^1(\partial\Omega)$, and $w_d \in L^2(\partial\Omega)$ be given distributions. Let $K := L^2_+(\partial\Omega)$ be the natural ordering cone of positive elements in $L^2(\partial\Omega)$, i.e.

$$K = \left\{ v \in L^2(\partial\Omega) \mid v \geq 0 \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega \right\}.$$

The optimal control problem that we consider in this section is to minimize the discrepancy between y_d and the solutions of the following state-constrained boundary value problem

$$-\Delta y = h \quad \text{in } \Omega, \tag{71}$$

$$\frac{\partial y(s)}{\partial \nu} = u \quad \text{on } \partial\Omega, \tag{72}$$

$$0 \leq y \leq \zeta^{max}(s) \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega, \tag{73}$$

by choosing an appropriate function $u \in U_{ad}$ as control, where ν is the outward normal to the boundary $\partial\Omega$, and we define the class of admissible controls U_{ad} as follows:

$$U_{ad} = \left\{ u \in L^2(\partial\Omega) \;\middle|\; \xi_1 \leq u(x) \leq \xi_2 \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega \right\}. \tag{74}$$

It is clear that U_{ad} is a nonempty convex closed subset of $L^2(\partial\Omega)$. Hereinafter, we assume that for given $\xi_1, \xi_2 \in L^2(\partial\Omega)$, $h \in L^2(\Omega)$, and $\zeta^{max} \in H^1(\partial\Omega)$,

$$\xi_2 - \xi_1 \in K, \quad \zeta^{max} \in K, \quad \text{and} \quad U_{ad} \cap D_h \neq \emptyset, \tag{75}$$

where the set D_h is defined as follows

$$D_h = \left\{ g \in L^2(\partial\Omega) \;\middle|\; \int_\Omega h\,dx + \int_{\partial\Omega} g\,d\mathcal{H}^{N-1} = 0 \right\}.$$

More precisely, we are concerned with the following optimal control problem

$$\begin{gathered} \text{Minimize } \left\{ I(u,y) = \int_\Omega |y - y_d|^2\,dx + \int_{\partial\Omega} |u - w_d|^2\,d\mathcal{H}^{N-1} \right\} \\ \text{subject to the constraints (71)–(75).} \end{gathered} \tag{76}$$

To begin with, let us show the solvability of the Neumann problem (71)–(72) for each admissible control $u \in U_{ad} \cap D_h$. To this end, we introduce the following interpolation space $H^{3/2}_\Delta(\Omega)$, which is closely related to the Laplace operator

$$H^{3/2}_\Delta(\Omega) := \left\{ y \in H^{3/2}(\Omega) \mid \Delta y \in L^2(\Omega) \right\}. \tag{77}$$

The norm in $H^{3/2}_\Delta(\Omega)$ is defined in the standard way as the graph-norm

$$\|y\|^2_{H^{3/2}_\Delta(\Omega)} = \|y\|^2_{H^{3/2}(\Omega)} + \|\Delta y\|^2_{L^2(\Omega)}$$
$$= \|y\|^2_{L^2(\Omega)} + \|\nabla y\|^2_{L^2(\Omega)^N} + \int_\Omega\int_\Omega \frac{|\nabla y(x) - \nabla y(z)|^2}{|x-z|^{N+1}}\,dxdz + \|\Delta y\|^2_{L^2(\Omega)}.$$

The following results are well known (for the details and complete proofs we refer to Lions and Magenes, 1968).

Lemma 4 *Let*

$$\gamma^0_{\partial\Omega}(y) := y|_{\partial\Omega} \quad \text{and} \quad \gamma^1_{\partial\Omega}(y) := \left.\frac{\partial y}{\partial \nu}\right|_{\partial\Omega}, \quad \forall\, y \in C^\infty(\overline{\Omega}).$$

Then the trace operator $\Gamma(y) = \left(\gamma^0_{\partial\Omega}(y), \gamma^1_{\partial\Omega}(y)\right)$ *can be extended to a continuous surjective operator* $\Gamma : H^{3/2}_\Delta(\Omega) \to H^1(\partial\Omega) \times L^2(\partial\Omega)$.

Lemma 5 *For any* $h \in L^2(\Omega)$ *and* $u \in U_{ad} \cap D_h$ *there exists a solution* $y \in H^{3/2}_\Delta(\Omega)$ *to the Neumann problem (71)–(72), and this solution is unique in the quotient space* $H^{3/2}_\Delta(\Omega)/\mathbb{R}$, *i.e. it is unique up to an additive constant. Moreover, each solution* z *to this problem satisfies the estimate*

$$\|z\|^2_{H^{3/2}_\Delta(\Omega)} \le C\left(\|h\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\Omega)} + \|z\|^2_{L^2(\Omega)}\right). \tag{78}$$

Lemma 6 *For each* $h \in L^2(\Omega)$ *and* $v \in H^1(\partial\Omega)$ *there exists a unique solution* $y \in H^{3/2}_\Delta(\Omega)$ *to the Dirichlet boundary value problem*

$$-\Delta y = h \quad \text{in } \Omega, \quad y = v \quad \text{on } \partial\Omega \tag{79}$$

such that

$$\|y\|^2_{H^{3/2}_\Delta(\Omega)} \le C\left(\|h\|^2_{L^2(\Omega)} + \|v\|^2_{H^1(\partial\Omega)}\right). \tag{80}$$

We define the feasible set for the problem (76) as follows:

$$\Xi = \left\{ (u,y) \in L^2(\partial\Omega) \times H^{3/2}_\Delta(\Omega) \;\middle|\; \begin{array}{c} -\Delta y = h, \quad \gamma^1_{\partial\Omega}(y) = u, \\ \gamma^0_{\partial\Omega}(y) \in K, \quad \zeta^{max} - \gamma^0_{\partial\Omega}(y) \in K \\ u \in U_{ad} \cap D_h, \quad I(u,y) < +\infty. \end{array} \right\}. \tag{81}$$

As this follows from Lemma 5, for each admissible control $u \in U_{ad} \cap D_h$, the Neumann boundary value problem (71)–(72) is solvable. However, because of the state constraints (73) and non-uniqueness of weak solutions, the fulfilment of the non-triviality condition (H1) is not evident. In view of this, we give the following observation:

Proposition 4 *Assume that there exists a positive value $\alpha > 0$ such that*

$$\zeta^{max}(s) \geq \alpha > 0 \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega. \tag{82}$$

For given $h \in L^2(\Omega)$ and $v \equiv 0$, let $z \in H^{3/2}_{\Delta}(\Omega)$ be a unique solution to the Dirichlet boundary value problem (79). *Let $\xi_1, \xi_2 \in L^2(\partial\Omega)$ be such that*

$$-\xi_1 + \gamma^1_{\partial\Omega}(z) \in K \quad \text{and} \quad \xi_2 - \gamma^1_{\partial\Omega}(z) \in K. \tag{83}$$

Then there exists an interval $\mathcal{I} = [(\widehat{u}, \widehat{y}); (\widetilde{u}, \widetilde{y})]$ in $L^2(\partial\Omega) \times H^{3/2}_{\Delta}(\Omega)$ such that $\mathcal{I} \in \Xi$, that is, the strict nontriviality condition (H1) for SCCP* (76) *is fulfilled.*

Proof We define the pairs $(\widehat{u}, \widehat{y})$ and $(\widetilde{u}, \widetilde{y})$ as follows:

$$\widehat{y} := z, \quad \widetilde{y} := z + \alpha, \quad \widehat{u} := \gamma^1_{\partial\Omega}(z), \quad \widetilde{u} := \widehat{u}.$$

By the initial assumptions and properties of the trace operator $\gamma^1_{\partial\Omega}$ (see Lemma 4), we have $\gamma^1_{\partial\Omega}(z) \in L^2(\partial\Omega)$ and

$$\int_\Omega (\nabla z, \nabla \varphi)\, dx - \int_{\partial\Omega} \gamma^1_{\partial\Omega}(z) \varphi \, d\mathcal{H}^{N-1} = \int_\Omega h\varphi \, dx, \quad \forall \varphi \in C^\infty(\mathbb{R}^N).$$

Hence, $\int_{\partial\Omega} \gamma^1_{\partial\Omega}(z)\, d\mathcal{H}^{N-1} = \int_\Omega h\, dx$, and in view of (83), it follows that $\widehat{u}$ is an admissible control, i.e. $\widehat{u} \in U_{ad} \cap D_h$. Moreover, since the pairs $(\widehat{u}, \widehat{y})$ and $(\widetilde{u}, \widetilde{y})$ are related by the Neumann problem (71)–(72) and

$$\widehat{y}(s) = 0 \quad \text{and} \quad \widetilde{y}(s) = \alpha \leq \zeta^{max} \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega,$$

it follows that these pairs satisfy the state constraints (73) and, therefore, $(\widetilde{u}, \widetilde{y}) \in \Xi$ and $(\widehat{u}, \widehat{y}) \in \Xi$. To conclude the proof, it remains to use the convexity of the set Ξ. As a result, we finally have

$$\mathcal{I} = [(\widehat{u}, \widehat{y}); (\widetilde{u}, \widetilde{y})] := \text{conv}\, \{(\widehat{u}, \widehat{y}); (\widetilde{u}, \widetilde{y})\} \subset \Xi.$$

□

Our next intention is to show that the optimal control problem (76) is solvable. To this end, we make use of Theorem 1 and show that this problem satisfies all assumptions formulated in Theorem 1.

Theorem 8 *If $U_{ad} \cap D_h \neq \emptyset$ and Hypothesis (H1) is fulfilled, then SCCP* (76) *has a unique solution $(u^0, y^0) \in L^2(\partial\Omega) \times H^{3/2}_{\Delta}(\Omega)$.*

Remark 4 *As follows from Proposition 4, the fulfillment of Hypothesis (H1) can be omitted, provided a special choice of distributions $\xi_1, \xi_2 \in L^2(\partial\Omega)$ and $\zeta^{max} \in H^1(\partial\Omega)$.*

Proof To begin with, we set

$$\begin{gathered}
\mathbb{Y}_1 = H^{3/2}_{\Delta}(\Omega), \quad \mathbb{Y} = L^2(\Omega), \quad \mathbb{W} = L^2(\partial\Omega)\times L^2(\partial\Omega), \quad \mathbb{U} = L^2(\partial\Omega), \\
\mathbb{Z} = \left\{ (h_1,h_2) \in L^2(\Omega)\times L^2(\partial\Omega) \;\Big|\; \int_\Omega h_1\,dx + \int_{\partial\Omega} h_2\,d\mathcal{H}^{N-1} = 0 \right\}, \\
f = (h,0), \quad L(u,y) = (-\Delta y, \gamma^1_{\partial\Omega}(y) - u), \\
F(y) = (\gamma^0_{\partial\Omega}(y), -\gamma^0_{\partial\Omega}(y)), \quad F_0 = (0,\zeta^{max}), \quad \Lambda = L^2_+(\partial\Omega)\times L^2_+(\partial\Omega).
\end{gathered} \tag{84}$$

The validity of the embedding $\mathbb{Y}_1 \hookrightarrow \mathbb{Y}$ and its continuity and density follow from the definition of the space $H^{3/2}_{\Delta}(\Omega)$. All spaces, indicated before, are obviously normed, and $\mathbb{Y}_1$, $\mathbb{U}$, and $\mathbb{W}$ are reflexive Banach spaces. As for the continuity of linear operators $L : \mathbb{U}\times\mathbb{Y}_1 \to \mathbb{Z}$ and $F : \mathbb{Y}_1 \to \mathbb{W}$, indicated in items (i)–(ii) of Theorem 1, they follow from (77) and Lemma 4. Moreover, by Lemma 5, for fixed $h \in L^2(\Omega)$ and $u \in U_{ad}\cap D_h$ there exists a unique solution $y \in H^{3/2}_{\Delta}(\Omega)/\mathbb{R}$ to the Neumann problem (71)–(72) and each representative of the class of equivalence to y satisfies the estimate (78). Hence, $y \mapsto L_2(y) := (-\Delta y, \gamma^1_{\partial\Omega}(y))$ is an isomorphism from the quotient space $H^{3/2}_{\Delta}(\Omega)/\mathbb{R}$ onto $\mathbb{Z}$. We also note that the strict convexity and lower semicontinuity of the cost functional $I : \mathbb{U}\times\mathbb{Y} \to \overline{\mathbb{R}}$ immediately follow from (76).

Thus, in order to show that all assumptions of Theorem 1 hold, it remains to check the coerciveness condition (H2). To this end, we fix an arbitrary admissible pair $(u,y) \in \Xi$ and make use of estimate (78). We obtain

$$\begin{aligned}
\|(u,y\|^2_{\mathbb{U}\times\mathbb{Y}_1} &:= \|u\|^2_{L^2(\partial\Omega)} + \|y\|^2_{H^{3/2}(\Omega)} + \|\Delta y\|^2_{L^2(\Omega)} \\
&\overset{\text{by (78)}}{\leq} (C+1)\left(\|h\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\Omega)} + \|y\|^2_{L^2(\Omega)}\right) \\
&\leq (C+1)\left[\|h\|^2_{L^2(\Omega)} + 2\|w_d\|^2_{L^2(\partial\Omega)} + 2\|y_d\|^2_{L^2(\Omega)} + 2I(u,y)\right] \\
&\leq C_1 I(u,y) + C_2,
\end{aligned} \tag{85}$$

where the constants C_1 and C_2 are independent of (u,y).

Hence, the set $\Xi_\lambda = \{(u,y)\in\Xi \;:\; I(u,y)\leq\lambda\}$ is bounded in $L^2(\partial\Omega)\times H^{3/2}_{\Delta}(\Omega)$ for each $\lambda > 0$. To conclude the proof, it remains to apply Theorem 1. □

Now, our intention is to provide the Henig regularization of $SCCP$ (76). To begin with, we note that the cone $\Lambda = L^2_+(\partial\Omega)\times L^2_+(\partial\Omega)$ is reproducing in $\mathbb{W}$ and $\Lambda^\sharp \neq \emptyset$. Hence (see Lemma 2), the set

$$B := \left\{ \eta = (\eta_1,\eta_2) \in \Lambda \;\Big|\; \int_{\partial\Omega} (\eta_1+\eta_2)\,d\mathcal{H}^{N-1} = 1 \right\} \tag{86}$$

is a closed base of Λ. Let $\delta > 0$ be defined as in (16). Then, for each $\varepsilon \in (0,\delta)$, the corresponding Henig dilating cone $\Lambda_\varepsilon(B)$ takes the form (15). In particular, the inclusion $\zeta = (\zeta_1, \zeta_2) \in \Lambda_\varepsilon(B)$ implies the existence of elements $\eta_1, \eta_2 \in K$ and $v_{\varepsilon,1}, v_{\varepsilon,2} \in L^2(\partial\Omega)$ such that

$$\int_{\partial\Omega} (\eta_1 + \eta_2)\, d\mathcal{H}^{N-1} = 1, \quad \int_{\partial\Omega} v_{\varepsilon,1}^2\, d\mathcal{H}^{N-1} \leq \varepsilon^2, \quad \int_{\partial\Omega} v_{\varepsilon,2}^2\, d\mathcal{H}^{N-1} \leq \varepsilon^2,$$
$$\text{and} \quad \zeta_i = \mu(\eta_i + v_{\varepsilon,i}),\ i = 1,2 \quad \text{for some } \mu \geq 0.$$

As a result, we can associate with $SCCP$ (76) the following Henig relaxed problem

$$\inf_{(u,y)\in\Xi_\varepsilon} I(u,y), \quad \forall\, \varepsilon \in (0,\delta), \tag{87}$$

where the feasible set $\Xi_\varepsilon \subset L^2(\partial\Omega) \times H^{3/2}_\Delta(\Omega)$ can be represented as follows:

$$\Xi_\varepsilon = \left\{ (u,y) \in L^2(\partial\Omega) \times H^{3/2}_\Delta(\Omega) \ \middle|\ \begin{array}{c} -\Delta y = h, \quad \gamma^1_{\partial\Omega}(y) = u, \quad u \in U_{ad} \cap D_h, \\ F(y) + F_0 \in \Lambda_\varepsilon(B), \quad I(u,y) < +\infty. \end{array} \right\}. \tag{88}$$

Remark 5 *Since B, given by* (86)*, is a closed base of the cone Λ, it follows from estimate* (85) *that the enhanced coerciveness condition (H2*) holds true. Hence, the Henig relaxed problem* (87) *has a unique solution $(u^0_\varepsilon, y^0_\varepsilon) \in L^2(\partial\Omega) \times H^{3/2}_\Delta(\Omega)$ for each $\varepsilon \in (0,\delta)$, provided $U_{ad} \cap D_h \neq \emptyset$ and Hypothesis (H1) is fulfilled (see Corollary 1 and Theorem 8). Moreover, in this case it is easy to conclude from the uniqueness of optimal pair for the original problem* (76) *and Theorem 4 that the sequence $\left\{(u^0_\varepsilon, y^0_\varepsilon)\right\}_{\varepsilon>0}$ weakly converges in $L^2(\partial\Omega) \times H^{3/2}_\Delta(\Omega)$ to the solution (u^0, y^0) of $SCCP$* (76)*. Thus, the unique optimal pair (u^0, y^0) to the original $SCCP$* (76) *is attainable in the sense of Definition 3.*

Since the main goal of this section is to derive optimality conditions to $SCCP$ (76), we begin with the optimality system for the Henig relaxed problem (87).

Lemma 7 *Assume that distributions $\xi_1, \xi_2 \in L^2(\partial\Omega)$ and $\zeta^{max} \in H^1(\partial\Omega)$ satisfy* (82)*–*(83)*. Let $(u^0_\varepsilon, y^0_\varepsilon) \in \Xi_\varepsilon$ be an optimal pair to the Henig relaxed $SCCP$* (87)*. Then there exist elements $p_\varepsilon \in H^1(\Omega)$ and $\mu_{1,\varepsilon}, \mu_{2,\varepsilon} \in L^2(\partial\Omega)$ such that*

$$\left.\begin{array}{l} \displaystyle\int_{\partial\Omega} \mu_{1,\varepsilon} \left(w_{1,\varepsilon} - \gamma^0_{\partial\Omega}(y^0_\varepsilon)\right)\, d\mathcal{H}^{N-1} \leq 0, \\ \displaystyle\int_{\partial\Omega} \mu_{2,\varepsilon} \left(w_{2,\varepsilon} + \gamma^0_{\partial\Omega}(y^0_\varepsilon) - \zeta^{max}\right)\, d\mathcal{H}^{N-1} \leq 0, \end{array}\right. \quad \forall\, w_\varepsilon = (w_{1,\varepsilon}, w_{2,\varepsilon}) \in \Lambda_\varepsilon(B), \tag{89}$$

$$-\Delta y^0_\varepsilon = h \quad \textit{in } \Omega, \qquad \frac{\partial y^0_\varepsilon}{\partial \nu} = u^0_\varepsilon \quad \textit{on } \partial\Omega, \tag{90}$$

$$-\Delta p_\varepsilon = 2(y^0_\varepsilon - y_d) \quad \textit{in } \Omega, \qquad \frac{\partial p_\varepsilon}{\partial \nu} = \mu_{1,\varepsilon} - \mu_{2,\varepsilon} \quad \textit{on } \partial\Omega, \tag{91}$$

$$\int_{\partial\Omega} \left[2(u^0_\varepsilon - w_d) + \gamma^0_{\partial\Omega}(p_\varepsilon)\right] \left(u - u^0_\varepsilon\right)\, d\mathcal{H}^{N-1} \geq 0, \quad \forall\, u \in U_{ad}. \tag{92}$$

Proof As this follows from Proposition 4, Theorem 8, and Remark 5, the Hypotheses (H1*), (H2*), and (H3) hold true for the problem (76). Moreover, by Lemma 4 and compactness of the inclusion $H^1(\partial\Omega) \subset L^2(\partial\Omega)$, the operator $F : H^{3/2}_\Delta(\Omega) \to L^2(\partial\Omega) \times L^2(\partial\Omega)$ is compact. We also remark that the cost functional (76) is continuously differentiable on $\mathbb{U} \times \mathbb{Y} := L^2(\partial\Omega) \times L^2(\Omega)$. In particular, for all $g \in L^2(\Omega)$ and $v \in L^2(\partial\Omega)$, we have

$$[D_u I(\widehat{u}, \widehat{y})]\,(v) = 2\int_{\partial\Omega} (\widehat{u} - w_d) v\, d\mathcal{H}^{N-1}, \quad [D_y I(\widehat{u}, \widehat{y})]\,(g) = 2\int_{\Omega} (\widehat{y} - y_d) g\, dx. \tag{93}$$

Thus, all assumptions of Theorem 6 are satisfied. Hence, by Corollary 2, for each $\varepsilon \in (0, \delta)$, there exist elements $p_\varepsilon \in \mathbb{Z}^*$ and $\mu_\varepsilon \in \mathbb{W}^*$ such that the optimality system for Henig relaxed problem (87) takes the form of relations (57)–(60). In order to specify these relations for the case of spaces (84), we need to introduce some preliminaries.

To begin with, we identify the spaces $\mathbb{Y} = L^2(\Omega)$, $\mathbb{W} = L^2(\partial\Omega) \times L^2(\partial\Omega)$, and $L^2(\Omega) \times L^2(\partial\Omega)$ with their duals. In order to describe the structure of the space $\mathbb{Z}^*$, we consider the following subspace of $L^2(\Omega) \times L^2(\partial\Omega)$:

$$\mathcal{N} = \left\{(\lambda, \lambda) \in L^2(\Omega) \times L^2(\partial\Omega) \quad \forall\, \lambda \in \mathbb{R}\right\}.$$

Since, for any pairs $(\eta_1, \eta_2) \in L^2(\Omega) \times L^2(\partial\Omega)$ and $(h_1, h_2) \in \mathbb{Z}$, we have

$$\begin{aligned}
&\int_\Omega (\eta_1 + \lambda) h_1\, dx + \int_{\partial\Omega} (\eta_2 + \lambda) h_2\, d\mathcal{H}^{N-1} \\
&\qquad = \int_\Omega \eta_1 h_1\, dx + \int_{\partial\Omega} \eta_2 h_2\, d\mathcal{H}^{N-1} + \lambda \left(\int_\Omega h_1\, dx + \int_{\partial\Omega} h_2\, d\mathcal{H}^{N-1}\right) \\
&\qquad = \int_\Omega \eta_1 h_1\, dx + \int_{\partial\Omega} \eta_2 h_2\, d\mathcal{H}^{N-1},
\end{aligned}$$

it follows that $\mathbb{Z}^*$ has a structure of the quotient space $\left[L^2(\Omega) \times L^2(\partial\Omega)\right]/\mathcal{N}$, and the value of functional $\dot{\eta} \in \mathbb{Z}^*$ at element $h = (h_1, h_2) \in \mathbb{Z}$ is as follows

$$\langle \dot{\eta}, h\rangle_{\mathbb{Z}^*;\mathbb{Z}} = \int_\Omega \eta_1 h_1\, dx + \int_{\partial\Omega} \eta_2 h_2\, d\mathcal{H}^{N-1}, \tag{94}$$

where $(\eta_1, \eta_2) \in L^2(\Omega) \times L^2(\partial\Omega)$ is an arbitrary representative of the equivalence class $\dot{\eta}$.

Let $p_\varepsilon \in \mathbb{Z}^*$ be an element indicated in (59). Then, $p_\varepsilon = \{(p_{1,\varepsilon} + \lambda, p_{2,\varepsilon} + \lambda) \mid \lambda \in \mathbb{R}\}$ for some $p_{1,\varepsilon} \in L^2(\Omega)$ and $p_{2,\varepsilon} \in L^2(\partial\Omega)$. Since the adjoint state equation (59) implies relation

$$\langle p_\varepsilon, L_2(\varphi)\rangle_{\mathbb{Z}^*;\mathbb{Z}} = \left(I'_y(u^0_\varepsilon, y^0_\varepsilon), E(\varphi)\right)_{L^2(\Omega)} + (\mu_\varepsilon, F(\varphi))_{\mathbb{W}}, \quad \forall\, \varphi \in H^{3/2}_\Delta(\Omega), \tag{95}$$

and $E(y) = y$ for all $y \in H^{3/2}_\Delta(\Omega)$, it follows from (94), (93), and (84) that

$$\langle p_\varepsilon, L_2(\varphi)\rangle_{\mathbb{Z}^*;\,\mathbb{Z}} = \int_\Omega (-\Delta p_{1,\varepsilon})\varphi\,dx + \int_{\partial\Omega} \gamma^1_{\partial\Omega}(p_{1,\varepsilon})\gamma^0_{\partial\Omega}(\varphi)\,d\mathcal{H}^{N-1} + \int_{\partial\Omega} \left(p_{2,\varepsilon} - \gamma^0_{\partial\Omega}(p_{1,\varepsilon})\right)\gamma^1_{\partial\Omega}(\varphi)\,d\mathcal{H}^{N-1},$$

$$\left(I'_y(u^0_\varepsilon, y^0_\varepsilon), E(\varphi)\right)_{L^2(\Omega)} = 2\int_\Omega (y^0_\varepsilon - y_d)\varphi\,dx,$$

$$(\mu_\varepsilon, F(\varphi))_{\mathbb{W}} = \int_{\partial\Omega} (\mu_{1,\varepsilon} - \mu_{2,\varepsilon})\,\gamma^0_{\partial\Omega}(\varphi)\,d\mathcal{H}^{N-1}.$$

Combining these relations with (95), we arrive at the identity

$$\int_{\partial\Omega} \left(p_{2,\varepsilon} - \gamma^0_{\partial\Omega}(p_{1,\varepsilon})\right)\gamma^1_{\partial\Omega}(\varphi)\,d\mathcal{H}^{N-1} = \int_\Omega \left[\Delta p_{1,\varepsilon} + 2(y^0_\varepsilon - y_d)\right]\varphi\,dx + \int_{\partial\Omega} \left(-\gamma^1_{\partial\Omega}(p_{1,\varepsilon}) + \mu_{1,\varepsilon} - \mu_{2,\varepsilon}\right)\gamma^0_{\partial\Omega}(\varphi)\,d\mathcal{H}^{N-1}, \quad \forall\,\varphi \in H^{3/2}_\Delta(\Omega).$$

After localization, this implies that $p_{2,\varepsilon} = \gamma^0_{\partial\Omega}(p_{1,\varepsilon})$ and $p_{1,\varepsilon}$ is a solution of the Neumann boundary value problem (91). Since $(y^0_\varepsilon - y_d) \in L^2(\Omega)$ and $\mu_{1,\varepsilon}, \mu_{2,\varepsilon} \in L^2(\partial\Omega)$, it follows from the well known regularity results for elliptic equations (see, for instance, Gröger, 1989; Zanger, 2000) that $p_{1,\varepsilon} \in H^{1+\delta}(\Omega)$ for some $\delta > 0$.

As for the inequalities (89), they are a direct consequence of relation (57) and the fact that

$$\begin{aligned} 0 &\ge \left\langle \mu_\varepsilon, w_\varepsilon - F(y^0_\varepsilon) - F_0\right\rangle_{\mathbb{W}^*;\mathbb{W}} \\ &= \int_{\partial\Omega} \mu_{1,\varepsilon}\left(w_{1,\varepsilon} - \gamma^0_{\partial\Omega}(y^0_\varepsilon)\right)\,d\mathcal{H}^{N-1} \\ &+ \int_{\partial\Omega} \mu_{2,\varepsilon}\left(w_{2,\varepsilon} + \gamma^0_{\partial\Omega}(y^0_\varepsilon) - \zeta^{max}\right)\,d\mathcal{H}^{N-1}. \end{aligned} \tag{96}$$

Since (96) ought be valid for all $w_\varepsilon = (w_{1,\varepsilon}, w_{2,\varepsilon}) \in \Lambda_\varepsilon(B)$, including elements like $(w_{1,\varepsilon}, \zeta^{max} - \gamma^0_{\partial\Omega}(y^0_\varepsilon))$ and $(\gamma^0_{\partial\Omega}(y^0_\varepsilon), w_{2,\varepsilon})$, it follows that relation (96) can be split up into two separate parts.

It remains to derive the inequality (92). Following our previous reasoning, the adjoint state p_ε represents a class of equivalence in $\left[L^2(\Omega) \times L^2(\partial\Omega)\right]/\mathcal{N}$. Hence, it can be defined up to an element of subspace $\mathcal{N}$. Taking into account the property $p_{2,\varepsilon} = \gamma^0_{\partial\Omega}(p_{1,\varepsilon})$, established before, we have

$$p_\varepsilon = (p_{1,\varepsilon} + \lambda, \gamma^0_{\partial\Omega}(p_{1,\varepsilon}) + \lambda) \in L^2(\Omega) \times L^2(\partial\Omega) \quad \text{for all} \quad \lambda \in \mathbb{R}.$$

Since

$$\left\langle L^*_1(p_\varepsilon), u - u^0_\varepsilon\right\rangle_{\mathbb{U}^*;\mathbb{U}} = \left\langle p_\varepsilon, L_1(u - u^0_\varepsilon)\right\rangle_{\mathbb{Z}^*;\mathbb{Z}} = \int_{\partial\Omega} \left[\gamma^0_{\partial\Omega}(p_{1,\varepsilon}) + \lambda\right]\left(-u + u^0_\varepsilon\right)\,d\mathcal{H}^{N-1},$$

it follows from (60) and (93) that

$$\int_{\partial\Omega} \left[2(u_\varepsilon^0 - w_d) + \gamma_{\partial\Omega}^0(p_{1,\varepsilon}) + \lambda\right]\left(u - u_\varepsilon^0\right)\, d\mathcal{H}^{N-1} \geq 0, \quad \forall\, u \in U_{ad}. \quad (97)$$

Using the fact that $\gamma_{\partial\Omega}^0(p_{1,\varepsilon}) + \lambda = \gamma_{\partial\Omega}^0(p_{1,\varepsilon} + \lambda)$, $p_{2,\varepsilon} + \lambda = \gamma_{\partial\Omega}^0(p_{1,\varepsilon} + \lambda)$, and the distribution $p_{1,\varepsilon} + \lambda$ satisfies the relations (91) for all $\lambda \in \mathbb{R}$, it means that parameter λ can be specified in (97) by the condition that this inequality has to be valid for all $u \in U_{ad}$. Thus, the inequality (97) implies (92). The proof is complete. □

The final step of our analysis is to pass to the limit in relations (89)–(92) as $\varepsilon \to 0$. With that in mind, we note that by Sobolev Embedding Theorem the weak convergence $(u_\varepsilon, y_\varepsilon) \rightharpoonup (u, y)$ in $L^2(\partial\Omega) \times H_\Delta^{3/2}(\Omega)$ implies the strong convergence $y_\varepsilon \to y$ in $L^2(\Omega)$ and, therefore,

$$I_y'(u_\varepsilon, y_\varepsilon) = 2(y_\varepsilon - y_d) \to 2(y - y_d) = I_y'(u, y) \;\text{ in } L^2(\Omega).$$

Hence, the assumption (v) holds true. However, the weak convergence $(u_\varepsilon, y_\varepsilon) \rightharpoonup (u, y)$ in $L^2(\partial\Omega) \times H_\Delta^{3/2}(\Omega)$ and the properties of operator $L_1 : L^2(\partial\Omega) \to L^2(\Omega) \times L^2(\partial\Omega)$ do not guarantee the fulfilment of assumptions (vj)–(vjjj). At the same time, following Theorem 4, we have the convergence (see (32))

$$\lim_{\varepsilon\to 0} I(u_\varepsilon^0, y_\varepsilon^0) = I(u^0, y^0). \quad (98)$$

Taking into account the strong convergence $y_\varepsilon^0 \to y^0$ in $L^2(\Omega)$, we can conclude from (98) that $\|u_\varepsilon^0\|_{L^2(\partial\Omega)} \to \|u^0\|_{L^2(\partial\Omega)}$. Hence, $u_\varepsilon^0 \to u^0$ in $L^2(\partial\Omega)$. Thus, we arrive at the fulfilment of the alternative assumptions (vj)*–(vjj)* (see Remark 3). As a result, Theorem 7 and the arguments, given in the proof of Lemma 7, lead us to the following conclusion.

THEOREM 9 *Assume that $U_{ad} \cap D_h \neq \emptyset$ and either Hypothesis (H1*) is satisfied or the distributions $\xi_1, \xi_2 \in L^2(\partial\Omega)$ and $\zeta^{max} \in H^1(\partial\Omega)$ are chosen in accordance with the rules* (82)–(83). *Let $(u^0, y^0) \in \Xi$ be a solution to the SCCP* (76). *Assume also that there exists a sequence of Lagrange multipliers $\{\mu_{1,\varepsilon}, \mu_{2,\varepsilon}\}_{\varepsilon>0}$ satisfying relations* (89)–(91), *such that* $\sup_{\varepsilon>0}\left[\|\mu_{1,\varepsilon}\|_{L^2(\partial\Omega)} + \|\mu_{2,\varepsilon}\|_{L^2(\partial\Omega)}\right] <$

$+\infty$. *Then, there exist elements* $p \in H^1(\Omega)$ *and* $\mu_1, \mu_2 \in L^2(\partial\Omega)$ *such that*

$$\int_{\partial\Omega} \mu_1 \left(w - \gamma^0_{\partial\Omega}(y^0)\right) \, d\mathcal{H}^{N-1} \le 0, \quad \forall\, w \in K, \tag{99}$$

$$\int_{\partial\Omega} \mu_2 \left(w + \gamma^0_{\partial\Omega}(y^0) - \zeta^{max}\right) \, d\mathcal{H}^{N-1} \le 0, \quad \forall\, w \in K, \tag{100}$$

$$-\Delta y^0 = h \quad in\ \Omega, \quad \frac{\partial y^0}{\partial \nu} = u^0 \quad on\ \partial\Omega, \tag{101}$$

$$-\Delta p = 2(y^0 - y_d) \quad in\ \Omega, \quad \frac{\partial p}{\partial \nu} = \mu_1 - \mu_2 \quad on\ \partial\Omega, \tag{102}$$

$$\int_{\partial\Omega} \left[2(u^0 - w_d) + \gamma^0_{\partial\Omega}(p)\right] \left(u - u^0\right) \, d\mathcal{H}^{N-1} \ge 0, \quad \forall\, u \in U_{ad}, \tag{103}$$

$$u^0 \in U_{ad}, \quad 0 \le y^0 \le \zeta^{max}(s) \quad \mathcal{H}^{N-1}\text{-}a.e.\ on\ \partial\Omega. \tag{104}$$

Remark 6 *One of the most restrictive assumptions of Theorem 9 is the fulfilment of the following condition*

$$\sup_{\varepsilon>0} \left[\|\mu_{1,\varepsilon}\|_{L^2(\partial\Omega)} + \|\mu_{2,\varepsilon}\|_{L^2(\partial\Omega)}\right] < +\infty.$$

Undoubtedly, the direct verification of this property is a non-trivial matter, in general. However, it is easy to indicate the following particular case, where the boundedness of the sequence $\left\{\|\mu_{1,\varepsilon}\|_{L^2(\partial\Omega)} + \|\mu_{2,\varepsilon}\|_{L^2(\partial\Omega)}\right\}_{\varepsilon>0}$ *becomes evident. Indeed, let us assume for a moment that the following conditions*

$$0_{L^2(\partial\Omega)} \in \operatorname{cor}\left(\Lambda_\varepsilon(B) - \gamma^0_{\partial\Omega}(y^0_\varepsilon)\right),$$
$$0_{L^2(\partial\Omega)} \in \operatorname{cor}\left(\Lambda_\varepsilon(B) + \gamma^0_{\partial\Omega}(y^0_\varepsilon) - \zeta^{max}\right)$$

hold true for each $\varepsilon > 0$. *Here,* $\operatorname{cor}(S)$ *stands for the algebraic interior of* $S \subset L^2(\partial\Omega)$, *i.e.*

$$\operatorname{cor}(S) = \left\{\widehat{x} \in S \mid \forall\, x \in L^2(\partial\Omega)\ \exists\, \widehat{\lambda} > 0\ s.\ t.\ \widehat{x} + \lambda x \in S\ \forall\, \lambda \in [0, \widehat{\lambda}]\right\}$$

(for the details we refer to Jahn, 2004). As a result, we immediately deduce from (89) *that*

$$\mu_{1,\varepsilon} = \mu_{2,\varepsilon} = 0_{L^2(\partial\Omega)} \quad \forall\, \varepsilon > 0.$$

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