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Optimal control of advection-diffusion problems for cropping systems in polluted soils*

by

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Abstract: The article studies the nutrient transfer mechanism for cropping systems in polluted soils from a mathematical and optimal control point of view. The problem under consideration is governed by an advection-diffusion PDE in a bounded domain.

The existence of a solution is obtained. We also determine the optimal amount of required nutrients at the root surface for plants where the soil is polluted by an unknown source. The characterization of the optimal control by a singular optimality system is obtained.

Keywords: nutrient uptake, polluted soils, Nye-Tinker-Barber (NTB) model, advection-diffusion system, low-regret optimal control, singular optimality system (SOS)

1. Introduction

Plant growth is strongly linked to the amount of soil nutrients absorbed from plant roots. These nutrients are produced naturally and are present in the groundsoil at various levels of concentration. They may also be provided by humans or by secondary companion crop plants. In the French West Indies (FWI), the nitrogen fixation and transfer from nitrogen fixing crops to commercial crops in mixed cropping systems have been an important subject of studies (see Jalonen et al., 2009).

In such studies, however, one has to take into account the pollution already existing in the soil. Indeed, in the FWI, banana plant farmers have been using chlordecone (CLD), an organochlorine insecticide to fight against the banana weevil *Cosmopolite Sordidus*, from 1972 to 1978, under the trademark Kepone

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(5% CLD), and from 1982 to 1993, under the trademark Curlone (5% CLD) (see Cabidoche et al., 2009). Several years after the beginning of Kepone spreading, the topsoil of some banana fields exhibited CLD content higher than 9 mg.kg^{-1} (see Cabidoche et al., 2009). Its use was definitively banned in early 1993. However, this molecule is very persistent in the natural environment, and is still being detected in soils, rivers, spring water, as well as in drinking water and food crop produce (see Clostre et al., 2014; Clostre, Letourmy and Lesueur-Jannoyer, 2015). This pesticide led to a contamination of the cropped soils.

Modelling is required to understand nutrient dynamics and transfer, and to test the scenarii for the optimization of crop nutrition in the context of polluted soils. The root nutrient uptake, as well as the solute movement in the soil, have been well explained by Tinker and Nye (2000). They describe the nutrient uptake and nutrient motion processes from the biological and chemical points of view, using partial differential equations (PDE's), known as the Nye-Tinker-Barber (NTB) system. We also refer to the work by Nye and Marriott (1969), and the work by Itoh and Barber (1983), who suggested a general framework of the model of nutrient uptake by roots, in which a term called source (or sink) is added, modelling either the increase or the decrease in solute concentration with respect to time and space.

More recently, Roose, Fowler and Darrah (2001) used the NTB model in order to reflect in a more accurate way the morphology of the root system (modelling of root growth, root hair, mycorrhizae, ...) and the spatio-temporal dynamics of the solute in the soil. We also mention the work by Ptashnyk (2010), where she studied a process of nutrient uptake by a single root branch, using the asymptotic expansion method. For the optimal control, Louison et al. (2015) studied the optimal control for the NTB model to determine the optimal amount of nutrients required for plant growth. In all these works, pollution is not taken into acount.

In this article, we focus on the modeling of plant nutrient uptake by roots in polluted soils. We introduce pollution as an unknown source function, and we consider the NTB system with pollution as a problem of incomplete data. We consider the optimal control problem and, for this purpose, we use the notion of low-regret or least-regret optimal control, introduced by Lions (1992), which is well adapted for PDE problems of incomplete data (see also Lions, 2000; Diaz and Lions, 1994). The concept of regret and least-regret were previously introduced by Savage (1972) in statistics. The low-regret control method is applied to systems, where there are controls and unknown perturbations. The method is general, and was extended to evolution problems, as well as to nonlinear problems by Nakoulima, Omrane and Vélin (2002 and 2003, for example). Recently, Jacob and Omrane (2010) used the method of Lions to a population dynamics problem of the initial missing birth rate. We extend here the method to the NTB advection-diffusion problem, where the concentration of nutrients is perturbed by pollution.

The article is organized as follows: in Section 2 we present the NTB model of nutrient uptake and we provide a proof of existence and positivity of a unique solution. In Section 3, we study the optimal control question (existence, uniqueness, ...) and adapt it to the case of polluted soils. Finally, we give a characterization of the low-regret optimal control by a singular optimality system (SOS) in Section 4. Some concluding remarks about the method and about numerical analysis are presented in the last section.

2. Statement of the problem

2.1. The setting

Let $\Omega \subset I\!R^d$, $d = 2, 3$, be the part of the soil close to the root surface, called the rhizosphere. We suppose that Ω has a regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2$. Here, Γ_1 represents the root surface (inner boundary) and Γ_2 plays the role of the rhizosphere frontier (outer boundary) with another plant root system or with the rest of the soil (see Fig. 1). During a time interval of $t \in [0, T[$, the transport and diffusion of nutrients and their uptake by roots is governed by the following Nye-Tinker-Barber (NTB) system:

$$\left\{ \begin{array}{rclll} \mathcal{A}\,c & = & g & \text{in} & Q :=]0,T[\times\Omega, \\ \operatorname{div}\mathbf{q} & = & 0 & \text{in} & Q, \\ (\mathcal{B}\,c).\mathbf{n} & = & \dfrac{Ic}{K} & \text{on} & \Sigma_1 :=]0,T[\times\Gamma_1, \\ (\mathcal{B}\,c).\mathbf{n} & = & -v & \text{on} & \Sigma_2 :=]0,T[\times\Gamma_2, \\ c(0,x) & = & c_0(x) & \text{in} & \Omega, \end{array} \right. \tag{1}$$

where we have:

$$\mathcal{A} = \alpha\frac{\partial}{\partial t} + \mathbf{q}.\nabla - D\Delta \quad \text{and} \quad \mathcal{B} = D\nabla - \frac{1}{2}\mathbf{q},$$

and where $c = c(t, x)$ is the concentration of nutrient at time t in the position x, the function $g = g(t, x) \in G$, G being a closed vector subspace of $L^2(Q)$, is the source of pollution. The coefficient $\alpha = b + \theta$ is a constant, b represents the buffer power and θ is the liquid saturation. The vector function $\mathbf{q} = \mathbf{q}(t, x)$ represents the Darcy flux, and D is a positive constant, representing the diffusion coefficient. The function $h(c) = Ic/K$ is the Michaelis-Menten uptake function, which represents the inflow nutrient density at the root surface. Here, I and K are, respectively, the maximum uptake and the Michaelis-Menten constants. Finally, $v := v(t, x) \geq 0$ is the control function. It represents the addition of nutrients entering the rhizosphere via Σ_2 (from a plant service, for example), and $c_0(x)$ is the initial concentration in Ω at time $t = 0$.

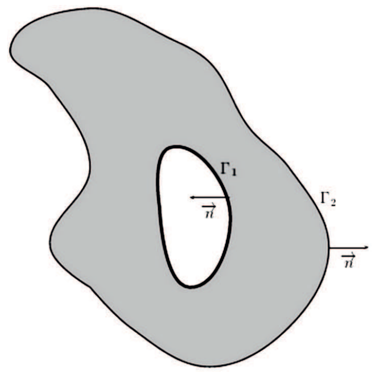

Figure 1. The rhizosphere Ω is delimited by the two boundaries, Γ_1 and Γ_2, where Γ_1 represents the root surface and where Γ_2 is the rhizosphere boundary

2.2. Existence of a solution to the NTB uptake system

It is well known that problem (1) does not have a regular solution in general. This section is devoted to the weak formulation of the problem and to the existence of a unique nonnegative solution $c(t,x)$ in the sense of distributions. We establish the weak formulation of (1) in the following lemma:

LEMMA 1 *The NTB problem (1) has the equivalent weak formulation:*
Find $c : t \in [0,T] \mapsto c(t;.) \in H^1(\Omega)$ *such that:*

$$\alpha \frac{d}{dt} \int_\Omega c(t;.)\, \psi \, dx + a(t;c,\psi) = L(t;\psi) \qquad a.e.\ t \in]0,T[, \quad \forall\, \psi \in H^1(\Omega), \quad (2)$$

with $c(0,x) = c_0(x)$, *where:*

$$a(t;c,\psi) = \frac{1}{2} \int_\Omega \mathbf{q}.\left(\psi \nabla c - c \nabla \psi\right)\, dx + D \int_\Omega \nabla c\, \nabla \psi \, dx - \int_{\Gamma_1} \frac{Ic}{K} \psi \, dx, \quad (3)$$

and where:

$$L(t;\psi) = \int_\Omega g(t;.) \psi \, dx - \int_{\Gamma_2} v\, \psi \, dx. \quad (4)$$

PROOF We multiply the first equation in (1) by $\psi \in H^1(\Omega)$ and integrate over Ω, so that we have:

$$\alpha \frac{d}{dt} \int_\Omega c(t;.)\, \psi \, dx + \int_\Omega \mathbf{q}.\psi \nabla c \, dx - D \int_\Omega \psi \, \Delta c \, dx = \int_\Omega g(t;.) \psi \, dx.$$

We begin by the advection term. Since we have: $\operatorname{div}(\mathbf{q}.c\psi) = c\psi \operatorname{div}\mathbf{q} + \mathbf{q}.(\psi\nabla c + c\nabla\psi)$, and $\operatorname{div}\mathbf{q} = 0$, we obtain:

$$\begin{aligned}\int_\Omega \mathbf{q}.\psi\nabla c\,dx &= \int_\Omega \operatorname{div}(\mathbf{q}.c\psi)\,dx - \int_\Omega \mathbf{q}.c\nabla\psi\,dx \\ &= \int_\Gamma \mathbf{q}.c\psi\,d\Gamma - \int_\Omega \mathbf{q}.c\nabla\psi\,dx\end{aligned}$$

using the divergence theorem. Adding to both sides the term $\int_\Omega \mathbf{q}.\psi\nabla c\,dx$, we finally obtain:

$$\int_\Omega \mathbf{q}.\psi\nabla c\,dx = \frac{1}{2}\int_\Gamma \mathbf{q}.c\psi\,d\Gamma + \frac{1}{2}\int_\Omega \mathbf{q}.(\psi\nabla c - c\nabla\psi)\,dx.$$

Now, for the diffusion term, we use Green's formula:

$$-D\int_\Omega \psi\,\Delta c\,dx = -D\int_\Gamma \psi\nabla c\,d\Gamma + D\int_\Omega \nabla\psi\,\nabla c\,dx.$$

By summing up, we find:

$$\begin{aligned}&\alpha\frac{d}{dt}\int_\Omega c(t;.)\,\psi\,dx + \frac{1}{2}\int_\Gamma \mathbf{q}.c\psi\,d\Gamma + \frac{1}{2}\int_\Omega \mathbf{q}.(\psi\nabla c - c\nabla\psi)\,dx \\ &\quad -D\int_\Gamma \psi\nabla c\,d\Gamma + D\int_\Omega \nabla\psi\,\nabla c\,dx = \int_\Omega g(t;.)\psi\,dx.\end{aligned}$$

By separating Γ into $\Gamma_1 + \Gamma_2$ and using the boundary conditions, we get the desired result. □

Proposition 1 (Wellposedness) *Let $c_0 \in L^2(\Omega)$, $g \in L^2(Q)$ and $v \in L^2(\Sigma_2)$. We suppose that the flux $\mathbf{q}$ is uniformly bounded i.e., $\mathbf{q} \in (L^\infty(Q))^d$. Then, there exists a unique solution $c(t,x)$ to the problem (1), such that:*

$$c \in L^2(]0,T[;H^1(\Omega)) \cap \mathcal{C}(]0,T[;L^2(\Omega)).$$

Proof We consider the equivalent weak problem (2)-(4). First, we show that the bilinear form a is continuous. We have, by the Cauchy-Schwarz inequality:

$$|a(t;c,\psi)| \le (q_\infty + D)\|c\|_{H^1(\Omega)}\|\psi\|_{H^1(\Omega)} + \|c\|_{L^2(\Gamma_1)}\|\psi\|_{L^2(\Gamma_1)}$$

where $q_\infty = \|\mathbf{q}\|_{(L^\infty(Q))^d}$. Then, owing to the trace theorem on $H^1(\Omega)$, there is a constant $\beta = \beta(\Omega)$ such that

$$\|\varphi\|_{L^2(\Gamma_1)} \le \beta\|\varphi\|_{H^1(\Omega)}, \quad \forall\varphi \in H^1(\Omega).$$

Hence,

$$|a(t;c,\psi)| \le C\|c\|_{H^1(\Omega)}\|\psi\|_{H^1(\Omega)}$$

where $C = q_\infty + D + \beta^2$, which implies that a is continuous.

Now we show that a is semi-coercive. We have, indeed:

$$a(t;c,c) = D\int_\Omega |\nabla c|^2\,dx - \int_{\Gamma_1} \frac{I}{K}|c|^2\,dx \geq D\|\nabla c\|^2_{L^2(\Omega)} - \|c\|^2_{L^2(\Gamma_1)} \geq D\|\nabla c\|^2_{L^2(\Omega)} - \beta^2\|c\|^2_{H^1(\Omega)}.$$

Using the fact that $\|\nabla c\|^2_{L^2(\Omega)} = \|c\|^2_{H^1(\Omega)} - \|c\|^2_{L^2(\Omega)}$, we obtain:

$$a(t;c,c) \geq \left(D - \beta^2\right)\|c\|^2_{H^1(\Omega)} - D\|c\|^2_{L^2(\Omega)}, \qquad \forall\, c \in H^1(\Omega), \tag{5}$$

so that a is semi-coercive (when $D > \beta^2$ in our case).

Now, it is easy to see that $L(t;\psi)$ is continuous, since:

$$\begin{aligned} |L(t;\psi)| &\leq \|g\|_{L^2(Q)}\|\psi\|_{L^2(\Omega)} + \|v\|_{L^2(\Sigma_2)}\|\psi\|_{L^2(\Gamma_2)} \\ &\leq \left(\|g\|_{L^2(Q)} + \beta\|v\|_{L^2(\Sigma_2)}\right)\|\psi\|_{H^1(\Omega)} \end{aligned}$$

using the trace theorem.

Hence, the bilinear form a satisfies the conditions of J.-L. Lions's theorem, and consequently (1) has a unique weak solution $c(t,x)$ with

$$c \in L^2(]0,T[;H^1(\Omega)) \cap \mathcal{C}(]0,T[;L^2(\Omega))$$

(see Lions and Magenes, 1972, chapter 3, section 4).

Remark 1 *The diffusion process is considered as the dominant one in plant nutrition as this is shown in the literature (see, for example, Jungk and Claassen, 1997). As we have seen for the NTB model, the semi-coercivity is proven only for large diffusion $D > \beta^2$, where β depends on Ω. This condition is restrictive from the mathematical viewpoint, but it is naturally fulfilled in applications, since the advection term is proven to be negligible in front of diffusion (see also Leitner et al., 2010).*

Lemma 2 (Positivity) *Let c be the solution to the NTB system. Suppose that $c_0 \geq 0$ and $c_{|_{\Sigma_1}} \geq 0$ and that $v \geq 0$ and $g \geq 0$. Then we have:*

$$c(T,.) \geq 0, \quad \forall\, T > 0.$$

Proof As usual, we decompose the solution $c := c(t,x)$ as $c = c^+ - c^-$, where c^+ and c^- are the classical nonnegative parts of c. We will show that $c^-(T,.) = 0$ for every $T > 0$.

We multiply the first equation of the NTB system (1) by c^-, and we obtain:

$$\alpha\left(\frac{\partial c}{\partial t}\right)c^- + (\mathbf{q}.\nabla c)\, c^- - D(\Delta c)c^- = gc^- \geq 0.$$

Since $(\partial_t c^+)c^- = (\nabla c^+)c^- = (\Delta c^+)c^- = 0$, we obtain:

$$-\alpha\frac{\partial c^-}{\partial t}c^- - (\mathbf{q}.\nabla c^-)c^- + D(\Delta c^-)c^- \geq 0.$$

We integrate by parts over Q. Since $c(t=0) = c_0 \geq 0$, we have:

$$-\alpha\int_0^T\int_\Omega\left(\frac{\partial c^-}{\partial t}\right)c^-\, dxdt = -\frac{\alpha}{2}\int_\Omega |c^-(T)|^2\, dx,$$

and since $div\,\mathbf{q} = 0\,\mathrm{in}\,\Omega$, we have:

$$-\int_Q(\mathbf{q}.\nabla c^-)c^-\, dxdt = -\frac{1}{2}\int_\Sigma(\mathbf{q}.c^-)c^-.\mathbf{n}\, dx\, dt.$$

Finally we have:

$$D\int_Q\left(\Delta c^-\right)c^-\, dxdt = D\int_\Sigma\Big(\nabla c^-\Big)c^-.\mathbf{n}\, dxdt - D\int_Q\left|\nabla c^-\right|^2 dxdt.$$

Summing up the three integrals, we find:

$$-\frac{\alpha}{2}\int_\Omega |c^-(T)|^2\, dx + \int_{\Sigma_1}\frac{I}{K}|c^-|^2.\mathbf{n}\, dxdt$$
$$\geq D\int_Q |\nabla c^-|^2\, dxdt + \int_{\Sigma_2} vc^-.\mathbf{n}\, dxdt \geq 0.$$

Now, since $c^-|_{\Sigma_1} = 0$, we obtain $\|c^-(T,.)\|_{L^2(\Omega)} \leq 0$. □

3. Optimal control for the NTB problem with pollution

3.1. The formulation

In this section we study the optimal control for the NTB problem (1). Nutrients enter by exudates through Γ_2. It is there that we put a control function $-v$, where $v := v(t,x)$ is a positive control function depending on t and x. It corresponds to the addition of nutrients into the soil through the frontier Γ_2. The rhizosphere contains pollution, represented by the unknown perturbation $g := g(t,x)$. The solution is then denoted by:

$$c(t,x;v,g) := c(v,g).$$

We are looking at the controls v which minimize the quadratic cost function:

$$J(v,g) = \|c(v,g) - \widetilde{c}\|^2_{L^2(\Sigma_1)} + N\|v\|^2_{L^2(\Sigma_2)}, \tag{6}$$

where $\tilde{c} \in L^2(\Sigma_1)$ is the observation, and where $N > 0$ is a positive weighting constant.

If g is known, say $g = g_0$, then $J(v, g_0) = J_{g_0}(v)$ does not depend on g, and the problem becomes a standard optimal control problem. But when we have a perturbation, expressed by $g \in G$, where G is a (closed) vector subspace of $L^2(Q)$, the problem of finding:

$$\inf_{v\in L^2(\Sigma_2)} J(v,g), \qquad \forall\, g \in G,$$

has no solution (the perturbation elements g may be of infinite number).

We then use the low-regret concept of Lions (1992), which is generally applied to systems where there are controls and unknown perturbations. The low-regret control - if it exists - is the solution of the *minmax* problem:

$$\inf_{v\in L^2(\Sigma_2)} \Big(\sup_{g\in G} \big(J(v,g) - J(0,g) - \gamma\|g\|^2_G\big) \Big). \tag{7}$$

One then looks for the control(s) not making things worse with respect to a nominal control $v = 0$ (which corresponds to the case where no control acts on the system), and yet somewhat better, up to a small real positive term $\gamma\|g\|^2_G$ with $\gamma << 1$.

In the following, we introduce the adjoint problem of system (1):

$$\left\{ \begin{array}{lcccc} \mathcal{A}^*\xi & = & 0 & \text{in} & Q, \\ div\,\mathbf{q} & = & 0 & \text{in} & Q, \\ \mathcal{B}^*\,\xi\,.\,\mathbf{n} & = & -c(v,0) - \frac{I}{K}\xi(v) & \text{on} & \Sigma_1, \\ \mathcal{B}^*\,\xi\,.\,\mathbf{n} & = & 0 & \text{on} & \Sigma_2, \\ \xi(T) & = & 0 & \text{in} & \Omega, \end{array} \right. \tag{8}$$

where

$$\mathcal{A}^* = -\alpha\frac{\partial}{\partial t} - \mathbf{q}.\nabla - D\Delta \quad \text{and} \quad \mathcal{B}^* = -D\nabla - \frac{1}{2}\mathbf{q},$$

and where $\xi := \xi(t,x;v) = \xi(v)$, and $c(v,0)$ is a concentration of nutrients when $g = 0$. For simplicity, we have taken at the initial time $\xi(t = T) = \xi_T = 0$, we also will take $c(t = 0) = c_0 = 0$.

In the case of non-homogeneous initial condition, and as it is classical, one should introduce a new function $\bar{c}(t,x) = c - c_0$ (respectively $\bar{\xi} = \xi - \xi_T$), where here we suppose that $c_0 \in L^2(\Omega)$ as in Proposition 1 (respectively $\xi_T \in L^2(\Omega)$).

3.2. Existence of a solution to the adjoint problem and some properties

Define the space:

$$V = \left\{ \psi \in H^1(\Omega); \quad \psi_{|\Gamma_2} = 0 \right\}$$

equipped with the norm $\|\cdot\|_V$ (equivalent to the $\|\cdot\|_{H_0^1(\Omega)}$ norm). Then, the problem (8) has the equivalent weak formulation given by the following:

Find $\xi : t \in [0,T] \mapsto \xi(t,.;v) := \xi(v) \in V$ such that:

$$\begin{cases} \alpha \dfrac{d}{dt} \displaystyle\int_\Omega \xi(v)\,\psi\,dx + b(t;\xi,\psi) = \ell(t;\psi) \qquad a.e.\ t \in]0,T[, \quad \forall\,\psi \in V, \\ \xi(T) = 0, \end{cases} \tag{9}$$

where:

$$b(t;\xi,\psi) = \frac{1}{2}\int_\Omega \mathbf{q}\cdot(\psi\nabla\xi - \xi\nabla\psi)\ dx + D\int_\Omega \nabla\xi\cdot\nabla\psi\,dx - \int_{\Gamma_1} \frac{I}{K}\xi(v)\psi\,dx \tag{10}$$

and where:

$$\ell(t;\psi) = \int_{\Gamma_1} c(v,0)\psi\,dx. \tag{11}$$

We state an existence result in the following lemma:

Lemma 3 (Existence) *We suppose that the flux $|\mathbf{q}|$ is uniformly bounded i.e., $\mathbf{q} \in (L^\infty(Q))^d$. Then, there is a unique solution $\xi \in V$ to the problem (8) (or equivalently (9)-(11)), such that:*

$$\xi \in L^2(]0,T[;V) \cap \mathcal{C}(]0,T[;L^2(\Omega)).$$

Proof The proof is similar to the proof in Proposition 1. Indeed, the bilinear form b satisfies the condition of Lions's theorem. Consequently, (8) has a unique weak solution $\xi \in L^2(]0,T[;V) \cap \mathcal{C}(]0,T[;L^2(\Omega))$.

Proposition 2 *Let $\xi := \xi(v)$ be the solution to the adjoint problem (8). Then we have:*

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2\langle \xi(v), g\rangle_{G',G} \tag{12}$$

where $c(0,g)$ is the nutrient concentration with $v = 0$ and $c(v,0)$ the concentration with $g = 0$.

Proof From the linearity of problem (1), we have $c(v,g) = c(v,0) + c(0,g)$. We then easily get:

$$\begin{aligned} J(v,g) - J(0,g) &= \|c(v,g) - \widetilde{c}\|^2_{L^2(\Sigma_1)} + N\|v\|^2_{L^2(\Sigma_2)} - \|c(0,g) - \widetilde{c}\|^2_{L^2(\Sigma_1)} \\ &= J(v,0) - J(0,0) + 2\langle c(0,g), c(v,0)\rangle_{L^2(\Sigma_1)}. \end{aligned}$$

Now, we show that $\langle c(v,0), c(0,g)\rangle_{L^2(\Sigma_1)} = \langle \xi(v), g\rangle_{G',G}$:

We multiply the first equation in (8) by $c(0,g)$, and we integrate by parts over Q.

Since $\xi(t=T) = c(t=0) = 0$, we have:

$$-\alpha \int_Q \frac{\partial \xi}{\partial t}(v) c(0,g)\, dxdt = \alpha \int_Q \frac{\partial c}{\partial t}(0,g)\xi(v)\, dxdt + 0. \tag{13}$$

For the advection term we have:

$$\begin{aligned} -\int_Q \Big(\mathbf{q}.\nabla\xi(v)\Big)c(0,g)\, dxdt &= -\int_Q div\Big(\mathbf{q}\,\xi(v)\,c(0,g)\Big)\, dxdt \\ &\quad + \int_Q \xi(v)\Big(\mathbf{q}.\nabla c(0,g)\Big)\, dxdt \\ &= -\int_\Sigma \mathbf{q}\,\xi(v)\,c(0,g).\mathbf{n}\, d\Sigma \\ &\quad + \int_Q \xi(v)\Big(\mathbf{q}.\nabla c(0,g)\Big)\, dxdt. \end{aligned}$$

And, Green's formula for the Laplacian gives:

$$\begin{aligned} & -\int_Q \Big(D\Delta\xi(v)\Big)c(0,g)\, dxdt \\ = & -\int_\Sigma \Big(D\nabla\xi(v)\Big)c(0,g).\mathbf{n}\, d\sigma dt + \int_\Sigma \Big(D\nabla c(0,g)\Big)\xi(v).\mathbf{n}\, d\sigma dt \\ & -\int_Q \Big(D\Delta c(0,g)\Big)\xi(v)\, dxdt. \end{aligned}$$

Summing up these terms, we find that:

$$\begin{aligned} 0 &= \int_Q (\mathcal{A}^*\xi(v))\, c(0,g)\, dxdt \\ &= \int_Q \xi(v)\,(\mathcal{A}\, c(0,g))\ dxdt - \int_\Sigma \mathbf{q}\,\xi(v)\,c(0,g).\mathbf{n}\, d\Sigma \\ &\quad - \int_\Sigma \Big(D\nabla\xi(v)\Big)c(0,g).\mathbf{n}\, d\sigma dt + \int_\Sigma \Big(D\nabla c(0,g)\Big)\xi(v).\mathbf{n}\, d\sigma dt \\ &= \int_Q \xi(v)\,(\mathcal{A}\, c(0,g))\ dxdt - \int_\Sigma \Big(D\nabla\xi(v) + \frac{1}{2}\mathbf{q}\xi(v)\Big)c(0,g).\mathbf{n}\, d\sigma dt \\ &\quad + \int_\Sigma \Big(D\nabla c(0,g) - \frac{1}{2}\mathbf{q}c(0,g)\Big)\xi(v).\mathbf{n}\, d\sigma dt. \end{aligned}$$

Now, using the boundary conditions in (1) and (8), we obtain:

$$
\begin{aligned}
0 &= \int_Q \xi(v)\, g\, dxdt - \int_{\Sigma_1} \Big(c(v,0) + \frac{I}{K}\xi(v) \Big) c(0,g).\mathbf{n}\, d\sigma_1 dt \\
&\quad + \int_{\Sigma_1} \Big(\frac{I}{K} c(0,g) \Big) \xi(v).\mathbf{n}\, d\sigma_1 dt \\
&= \int_Q \xi(v)\, g\, dxdt - \int_{\Sigma_1} c(v,0) c(0,g).\mathbf{n}\, d\sigma_1 dt.
\end{aligned}
$$

That is:

$$\langle c(0,g), c(v,0) \rangle_{L^2(\Sigma_1)} = \langle \xi(v), g \rangle_{G',G}. \qquad \square$$

3.3. Existence of the low-regret optimal control

Here, we prove the existence of the low-regret optimal control. We prepare this by the following lemma:

LEMMA 4 *The problem (7) is equivalent to the classical optimal control problem:*

$$\inf_{v \in L^2(\Sigma_2)} \mathcal{J}^\gamma(v) \tag{14}$$

where

$$\mathcal{J}^\gamma(v) = J(v,0) - J(0,0) + \frac{1}{\gamma} \|\xi(v)\|^2_{G'} \tag{15}$$

and where G' is the dual of G.

PROOF Indeed, we have from the above:

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2\langle \xi(v), g \rangle_{G',G} - \gamma \|g\|^2_G.$$

Hence,

$$
\begin{aligned}
&\sup_{g \in G} \Big(J(v,g) - J(0,g) - \gamma \|g\|^2_G \Big) \\
&\qquad = J(v,0) - J(0,0) + \sup_{g \in G} \Big(2\langle \xi(v), g \rangle_{G',G} - \gamma \|g\|^2_G \Big).
\end{aligned}
$$

But, using the conjugate property we have:

$$\sup_{g \in G} \Big(2\langle \xi(v), g \rangle_{G',G} - \gamma \|g\|^2_G \Big) = \frac{1}{\gamma} \|\xi(v)\|^2_{G'}.$$

Hence, the low-regret control - if it exists - satisfies the classical optimal control problem:

$$\inf_{v\in L^2(\Sigma_2)}\left(\sup_{g\in G}\left(J(v,g)-J(0,g)-\gamma\|g\|_G^2\right)\right)=\inf_{v\in L^2(\Sigma_2)}\mathcal{J}^\gamma(v)$$

where $\mathcal{J}^\gamma(v)$ is given by (15). □

Proposition 3 *The minimization problem (14)-(15) for the NTB system with pollution admits a unique solution u_γ called the low-regret optimal control.*

Proof The cost function $\mathcal{J}^\gamma(v)$ satisfies $\mathcal{J}^\gamma(v)\geq -J(0,0)$, for any $v\in L^2(\Sigma_2)$. Therefore, there exists $k_\gamma=\inf\limits_{v\in L^2(\Sigma_2)}\mathcal{J}^\gamma(v)$. We consider a minimizing sequence $\{v_n(\gamma)\}=\{v_n\}$. Then, it converges to k_γ (which is independent of n). We obtain $-J(0,0)\leq \mathcal{J}^\gamma(v_n)\leq k_\gamma+1$, so that:

$$\|c(v_n,0)-\tilde{c}\|^2_{L^2(\Sigma_1)}+N\|v_n\|^2_{L^2(\Sigma_2)}-\|\tilde{c}\|^2_{L^2(\Sigma_1)}+\frac{1}{\gamma}\|\xi(v)\|^2_{G'}\leq k_\gamma+1.$$

In particular, we have:

$$\|v_n\|_{L^2(\Sigma_2)}\leq\sqrt{\frac{k_\gamma+1+\|\tilde{c}\|^2_{L^2(\Sigma_1)}}{N}}=C_\gamma.$$

Hence, there is a subsequence that we can still denote by $\{v_n\}$, which converges weakly to a low-regret control $u_\gamma\in L^2(\Sigma_2)$.

Moreover, the sequence $c(v_n,0)$ converges to $c_\gamma=c(u_\gamma,0)$ for the weak topology of $L^2(\Sigma_1)$. On the other hand, $c(v_n,0)$ satisfy the state equation (1) and we have:

$$\langle\mathcal{A}c(v_n,0),\varphi\rangle=\langle c(v_n,0),\mathcal{A}^*\varphi\rangle,$$

which converges to $\langle c(u_\gamma,0),\mathcal{A}^*\varphi\rangle=\langle\mathcal{A}c(u_\gamma,0),\varphi\rangle=0$, for every $\varphi\in\mathcal{D}(Q)$.

We use the same arguments for the boundary conditions, and then the state-control pair function satisfies (1).

The uniqueness of the low-regret control u_γ is obvious and comes from the strict convexity of the cost function $\mathcal{J}^\gamma$. □

4. Characterization of the low-regret control for the NTB system

In this section we give a characterization of the low-regret optimal control by an optimality system.

THEOREM 1 *The low-regret control u_γ, solution to (14)-(15), is characterized by the unique quadruplet $\{c_\gamma, \rho_\gamma, \xi_\gamma, p_\gamma\}$ solution of the optimality system:*

$$\left\{\begin{array}{lll}
\mathcal{A}c_\gamma = 0, & \mathcal{A}^*\xi_\gamma = 0, & \textit{in } Q,\\
\mathcal{B}c_\gamma.\mathbf{n} = \frac{I}{K}c_\gamma, & \mathcal{B}^*\xi_\gamma.\mathbf{n} = -c_\gamma - \frac{I}{K}\xi_\gamma, & \textit{on } \Sigma_1,\\
\mathcal{B}c_\gamma.\mathbf{n} = -u_\gamma, & \mathcal{B}^*\xi_\gamma.\mathbf{n} = 0, & \textit{on } \Sigma_2,\\
c_\gamma(0) = 0, & \xi_\gamma(T) = 0, & \textit{in } \Omega,\\
\textit{and} & & \\
\mathcal{A}\rho_\gamma = \frac{1}{\gamma}\xi_\gamma, & \mathcal{A}^*p_\gamma = 0 & \textit{in } Q,\\
\mathcal{B}\rho_\gamma.\mathbf{n} = \frac{I}{K}\rho_\gamma, & \mathcal{B}^*p_\gamma.\mathbf{n} = c_\gamma - \tilde{c} + \rho_\gamma - \frac{I}{K}p_\gamma & \textit{on } \Sigma_1,\\
\mathcal{B}\rho_\gamma.\mathbf{n} = 0, & \mathcal{B}^*p_\gamma.\mathbf{n} = 0 & \textit{on } \Sigma_2,\\
\rho_\gamma(0) = 0, & p_\gamma(T) = 0 & \textit{in } \Omega,
\end{array}\right.$$

with $div\,\mathbf{q} = 0$ and with the adjoint equation:

$$p_\gamma + Nu_\gamma = 0 \quad in \quad L^2(\Sigma_2), \tag{16}$$

where here $c_\gamma = c(u_\gamma, 0)$, $\xi_\gamma = \xi(u_\gamma, 0)$, $\rho_\gamma = \rho(u_\gamma, 0)$ and $p_\gamma = p(u_\gamma, 0)$.

PROOF Indeed, the low-regret control u_γ satisfies the Euler-Lagrange formula:

$$\lim_{\lambda\to 0}\left(\frac{\mathcal{J}^\gamma(u_\gamma + \lambda w) - \mathcal{J}^\gamma(u_\gamma)}{\lambda}\right) = 0, \qquad \forall\, w \in L^2(\Sigma_2).$$

Then we have:

$$\begin{aligned}
\mathcal{J}^\gamma(u_\gamma + \lambda w) - \mathcal{J}^\gamma(u_\gamma) &= J(u_\gamma + \lambda w, 0) + \frac{1}{\gamma}\|\xi(u_\gamma + \lambda w)\|^2_{G'}\\
&\quad - J(u_\gamma, 0) - \frac{1}{\gamma}\|\xi(u_\gamma)\|^2_{G'}\\
&= 2\lambda\langle c(u_\gamma, 0) - \tilde{c}, c(w, 0)\rangle_{L^2(\Sigma_1)} + \lambda^2\|c(w, 0)\|^2_{L^2(\Sigma_1)}\\
&\quad + 2\lambda N\langle u_\gamma, w\rangle_{L^2(\Sigma_2)} + \lambda^2 N\|w\|^2_{L^2(\Sigma_2)}\\
&\quad + \frac{\lambda^2}{\gamma}\|\xi(w)\|^2_{G'} + 2\frac{\lambda}{\gamma}\langle\xi(u_\gamma), \xi(w)\rangle_{G'}.
\end{aligned}$$

When $\lambda \to 0$, we obtain:

$$\langle c(u_\gamma, 0) - \tilde{c}, c(w, 0)\rangle_{L^2(\Sigma_1)} + N\langle u_\gamma, w\rangle_{L^2(\Sigma_2)} + \langle\frac{1}{\gamma}\xi(u_\gamma), \xi(w)\rangle_{G'} = 0. \tag{17}$$

We develop the term $\langle\frac{1}{\gamma}\xi(u_\gamma), \xi(w)\rangle_{G'}$ in the following manner. We define

$\rho_\gamma = \rho(u_\gamma, 0)$, solution of the problem:

$$\left\{\begin{array}{llll} \mathcal{A}\rho_\gamma & = & \frac{1}{\gamma}\xi_\gamma & \text{in} \quad Q, \\ \mathcal{B}\rho_\gamma.\mathbf{n} & = & \frac{I}{K}\rho_\gamma & \text{on} \quad \Sigma_1, \\ \mathcal{B}\rho_\gamma.\mathbf{n} & = & 0 & \text{on} \quad \Sigma_2, \\ \rho_\gamma(0) & = & 0 & \text{in} \quad \Omega, \end{array}\right. \tag{18}$$

where $\xi_\gamma = \xi(u_\gamma, 0)$. Then, we multiply the first equation in (18) by $\xi(w)$ and integrate by parts over Q. We obtain:

$$\begin{array}{rcl} \langle A\rho_\gamma, \xi(w)\rangle_{L^2(Q)} & = & \displaystyle\int_Q \Big(\alpha\frac{\partial}{\partial t}\rho_\gamma + \mathbf{q}.\nabla\rho_\gamma - D\Delta\rho_\gamma\Big)\xi(w)\,dxdt \\ & = & \displaystyle -\int_Q \Big(\alpha\frac{\partial}{\partial t}\xi(w) + \mathbf{q}.\nabla\xi(w) - D\Delta\xi(w)\Big)\rho_\gamma\,dxdt \\ & & \displaystyle -\int_{\Sigma_1}\Big(D\nabla\rho_\gamma - \frac{1}{2}\mathbf{q}\rho_\gamma\Big)\xi(w).\mathbf{n}\,d\Sigma_1 \\ & & \displaystyle +\int_{\Sigma_1}\Big(D\nabla\xi(w) + \frac{1}{2}\mathbf{q}\xi(w)\Big)\rho_\gamma.\mathbf{n}\,d\Sigma_1. \end{array}$$

We use the boundary conditions in (8) and (18) and we obtain:

$$\langle A\rho_\gamma, \xi(w)\rangle_{L^2(Q)} = \langle c(w,0), \rho_\gamma\rangle_{L^2(\Sigma_1)}.$$

Finally, $\langle \frac{1}{\gamma}\xi(u_\gamma), \xi(w)\rangle_{G'} = \langle c(w,0), \rho_\gamma\rangle_{L^2(\Sigma_1)}$, so that (17) reduces to:

$$\langle c_\gamma - \tilde{c} + \rho_\gamma, c(w,0)\rangle_{L^2(\Sigma_1)} + \langle Nu_\gamma, w\rangle_{L^2(\Sigma_2)} = 0. \tag{19}$$

Then, we introduce the adjoint state $p(u_\gamma, 0) = p_\gamma$ solution, to:

$$\left\{\begin{array}{llll} \mathcal{A}^* p_\gamma & = & 0 & \text{in} \quad Q, \\ \mathcal{B}^* p_\gamma.\mathbf{n} & = & c_\gamma - \tilde{c} + \rho_\gamma - \frac{I}{K}p_\gamma & \text{on} \quad \Sigma_1, \\ \mathcal{B}^* p_\gamma.\mathbf{n} & = & 0 & \text{on} \quad \Sigma_2, \\ p_\gamma(T) & = & 0 & \text{in} \quad \Omega. \end{array}\right.$$

We then have, by using Green's formula:

$$\begin{array}{rcl} 0 & = & \langle \mathcal{A}^* p_\gamma, c(w,0)\rangle_{L^2(Q)} \\ & = & \displaystyle -\int_Q \Big(\alpha\frac{\partial}{\partial t}p_\gamma + \mathbf{q}.\nabla p_\gamma + D\Delta p_\gamma\Big)c(w,0)\,dxdt \\ & = & \displaystyle \int_Q \Big(\alpha\frac{\partial}{\partial t}c(w,0) + \mathbf{q}.\nabla c(w,0) - D\Delta c(w,0)\Big)p_\gamma\,d\sigma dt \\ & & \displaystyle +\int_{\Sigma_1}(\tilde{c} - c_\gamma + \rho_\gamma)\,c(w,0).\mathbf{n}\,d\Sigma_1 - \int_{\Sigma_2} wp_\gamma.\mathbf{n}\,d\Sigma_2 \\ & = & \langle c_\gamma - \tilde{c} + \rho_\gamma, c(w,0)\rangle_{L^2(\Sigma_1)} - \langle p_\gamma, w\rangle_{L^2(\Sigma_2)}. \end{array}$$

Hence,

$$\langle c_\gamma - \tilde{c} + \rho_\gamma, c(w,0)\rangle_{L^2(\Sigma_1)} = \langle p_\gamma, w\rangle_{L^2(\Sigma_2)}. \tag{20}$$

Summing (19) and (20), we finally obtain the adjoint state equality:

$$\langle p_\gamma + N u_\gamma, w\rangle_{L^2(\Sigma_2)} = 0, \qquad \forall\, w \in L^2(\Sigma_2). \qquad \square$$

5. Conclusion

We have studied the question of optimal control for an advection-diffusion PDE of Nye-Tinker-Barber (NTB) type, which describes the nutrient transfer mechanism at the root surface of a plant, using functional analysis. In the literature, the NTB model has been mostly seen from a numerical point of view (see, for example, Leitner et al., 2010, and the references therein).

We have also studied the pollution problem, which is an important one, since pollution affects the soil in many cultures around the world. We used the low-regret control method of Lions and we obtained a low-regret optimal magnitude of nutrient uptake that we characterized by a singular optimality system.

A next step of considering numerical analysis to the NTB system with pollution could be taken, consisting in studying the mechanisms of nutrient uptake in soil-plant ecosystems, and including the soil quality. Indeed, following Tinker and Nye (2000), the soil is supposed to be quasi-homogeneous in the NTB model, but this assumption is not completely true in the presence of pollution. Numerical simulations for the low-regret optimal control should give a better indication concerning this point.

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