

SPATIAL CONSTRUCTION OF INTERSECTION POINTS OF TWO ELLIPSES

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Abstract: A spatial construction of intersection points of two coplanar ellipses, whose major axes lie on the same line and minor axes are equal in length, is given.

Key Words: ellipse, major and minor axes of an ellipse, elliptical cone, affinity.

1. Introduction

The authors have already considered a certain problem of the intersection of two coplanar smooth conics by using the properties of two cones circumscribed on the same sphere and have presented a spatial solution in [1]. The present paper gives some simplification of the solution in the case of two coplanar ellipses whose major axes are lying on the same straight line and minor axes are equal in length. Elliptical cylinders are considered instead of cones.

The authors hope that a simpler method will be more accessible to a greater number of students. They also think that an application of the Theorem about the reducibility of two quadrics intersection line as well as the use of the affinity may be very useful.

2. Construction of Two Ellipses Intersection Points

Ellipses e_1 and e_2 are lying in the same plane. Their major axes are lying on the same straight line and partially coincide, their minor axes are equal in length.

Our construction of the required intersection points of these ellipses needs some preparation.

Suppose the ellipses are placed on the horizontal projection plane, the straight line including their major axes is parallel to the ground line x of the projection system (Fig.1).

We can choose two elliptical cylinders Φ_1 and Φ_2 passing through the ellipses e_1 and e_2 respectively so that the axes of the cylinders intersect and are parallel to the vertical projection plane. In Fig.1 the axis of Φ_2 is assumed vertical (perpendicular to the horizontal projection plane). The horizontal projection contours of the cylinders Φ_1 and Φ_2 are included in the planes ε_1 and ε_2 which pass parallel to the vertical projection plane through the ends of the minor axes of the ellipses e_1 and e_2 . The contours of the cylinders on the elevation intersect at the points which form the parallelogram 1"2"3"4".

It should be noticed that the planes ε_1 and ε_2 of the horizontal projection contours of the cylinders Φ_1 and Φ_2 are two planes simultaneously tangent to the cylinders. That means that we can apply the well known theorem on the reducibility of the intersection curve of two second-order surfaces (see [2], [3]). Therefore, the intersection curve of the cylinders Φ_1 and Φ_2 consists of two conics. Lying in the cylinders, the conics are evidently ellipses and their vertical projections are the line segments 1"3" and 2"4".

Having given the detailed explanation of Fig.1, we can now present our method to find the common points of the ellipses e_1 and e_2 resting in the particular position described above. In fact, the following two statements hold.

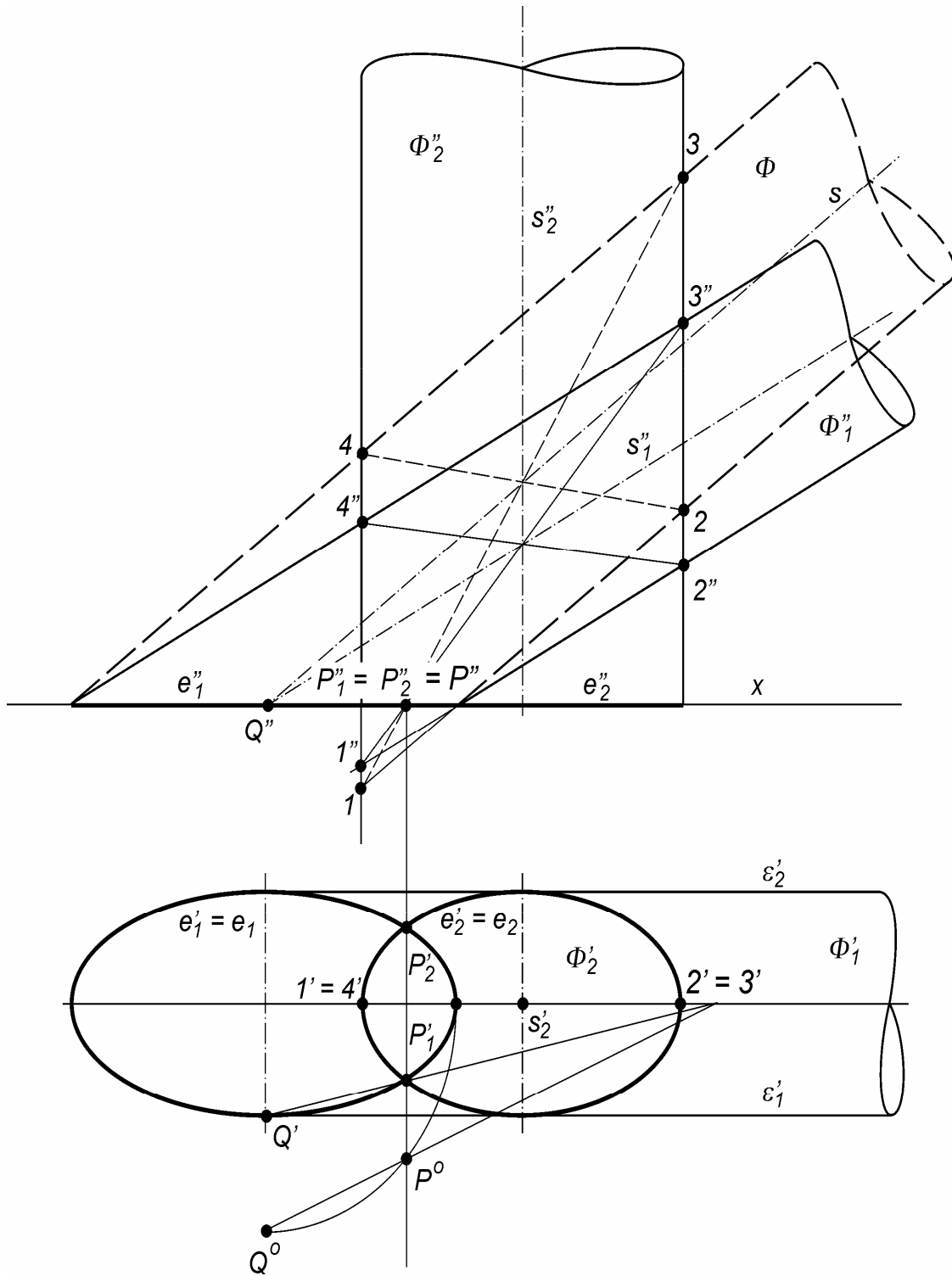


Figure 1

Statement 1 (about the construction):

One of the diagonals of the parallelogram $1''2''3''4''$ formed by the vertical projection contours of the cylinders Φ_1 and Φ_2 represented in the Fig.1 intersects the common part of the vertical projection of the ellipses e_1 and e_2 at the point P'' which is the vertical projection of the intersection points of these ellipses.

Proof: The ellipses e_1 and e_2 are cut by the horizontal projection plane from the cylinders Φ_1 and Φ_2 . In this section the points of the cylinder Φ_1 belong to the ellipse e_1 and the points of the cylinder Φ_2 belong to e_2 . Therefore, the common points of the cylinders Φ_1 and Φ_2 lying in the horizontal projection plane belong simultaneously to the two ellipses, that is they are the intersection points of these ellipses.

In Figure 1 the diagonal $1''3''$, which is the vertical projection of one part of the intersection curve of the cylinders, intersects the common part of the vertical projection of the ellipses e_1 and e_2 at the point P'' which is the vertical projection of the points common to both the ellipses e_1 and e_2 .

Statement 2 (about the uniqueness of the construction):

The vertical projection P'' of the intersection points of the ellipses e_1 and e_2 is unique and it is not dependent on the choice of the inclination angles of the axes of the cylinders Φ_1 and Φ_2 .

Proof: Instead of the cylinder Φ_1 let us consider another cylinder Φ passing through the ellipse e_1 such that its axis makes with the horizontal projection plane an angle different from the angle made by the axis of the cylinder Φ_1 . The vertical contours of the cylinders Φ_2 and Φ form the parallelogram 1234 which is related with the parallelogram $1''2''3''4''$ by an orthogonal affinity whose axis is the line x . Since the diagonals 13 and $1''3''$ of these parallelograms are corresponding by the affinity they intersect at a point lying on the affinity axis x . Therefore, the diagonals 13 and $1''3''$ intersect the common part of the vertical projection of the ellipses e_1 and e_2 at the same point P'' .

Remark: The intersection points P_1 and P_2 of the ellipses e_1 and e_2 in the Fig.1 may be found in the following way:

First transform the ellipse e_1 on a circle (a quadrant suffices) by the orthogonal affinity whose axis is the line of the major axis of the ellipse. Next refer, perpendicularly to the axis x , the point P'' to the point P^o on the quadrant. To finish the construction return, by the affinity, the point P^o at the point P_1 on the ellipse e_1 . The other intersection point P_2 is symmetric to P_1 relatively to the line containing the major axes of the ellipses e_1 and e_2 .

References

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PRZESTRZENNA KONSTRUKCJA WSPÓLNYCH PUNKTÓW DWÓCH ELIPS

Rozpad linii przenikania dwóch walców został tu wykorzystany do wyznaczenia punktów przecięcia się dwóch elips będących w określonym szczególnym położeniu – elipsy leżą w jednej płaszczyźnie, ich duże osie leżą na jednej prostej i częściowo pokrywają się, ich małe osie są równej długości.