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BACKWARD STOCHASTIC VARIATIONAL INEQUALITIES DRIVEN BY MULTIDIMENSIONAL FRACTIONAL BROWNIAN MOTION

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Abstract. We study the existence and uniqueness of the backward stochastic variational inequalities driven by m-dimensional fractional Brownian motion with Hurst parameters H_k $(k = 1, \dots m)$ greater than 1/2. The stochastic integral used throughout the paper is the divergence type integral.

Keywords: backward stochastic differential equation, fractional Brownian motion, backward stochastic variational inequalities, subdifferential operator.

Mathematics Subject Classification: 60H05, 60H07, 60H22.

1. INTRODUCTION

A centred Gaussian process $B^H = \{B_t^H,\ t \geq 0\}$ is called a fractional Brownian motion (fBm for short) with Hurst parameter $H \in (0, 1)$ if it has the covariance function

$$R_H(s,t) = E\left(B_s^H B_t^H\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

This process for any H is a self-similar, i.e. B_{at}^H has the same law as $a^H B_t^H$ for any $a > 0$, it has homogeneous increments. For $H = 1/2$ we obtain a standard Wiener process, but for $H \neq 1/2$, the process B^H is not a semimartingale. If $H > 1/2$, B^H has a long range dependence, that is if we put $r(n) = cov(B_1^H, B_{n+1}^H - B_n^H)$, then $\sum_{n=1}^{\infty} r(n) = +\infty$. These properties make this process a useful tool in models related to network traffic analysis, mathematical finance, physics, signal processing and many other fields. Unfortunately we cannot use the classical theory of stochastic calculus to define the fractional stochastic integral because generally B^H is not a semimartingale (it happens only if $H = 1/2$). Nevertheless an efficient stochastic calculus of B^H

has been developed. Firstly definitions of integrals of Stratonovich type with respect to fBm were introduced (see [7, 14]), but they did not satisfy the natural property $E\int_o^t f_s dB_s^H = 0$. Next, a new type of stochastic integral with respect to fBm was defined to satisfy the mentioned property. The definition introduced in [8] is the divergence operator (Skorokhod integral), defined as the adjoint of the derivative operator in the framework of the Malliavin calculus and the equivalent for $H > 1/2$ definition introduced in [9] is based on the Wick product as the limit of Riemann sums.

Pardoux and Peng were first who proved the existence and uniqueness of solution of the backward stochastic differential equations (BSDEs) with respect to Wiener process (see [19]). Since then many papers have been devoted to the study of BSDEs, mainly due to their applications. The main purpose of studying these equations was to give a probabilistic interpretation for viscosity solutions of semilinear partial differential equations.

BSDEs driven by a fBm with Hurst parameter $H > 1/2$ were first considered in [3] and with Hurst parameter $H \in (0, 1)$ in [2]. Next, Hu and Peng in [12] considered the nonlinear BSDEs with respect to a fBm with Hurst parameter $H > 1/2$. However, the existence and uniqueness of the solution of the BSDE driven by a fBm was obtained there with some restrictive assumption. Maticiuc and Nie, [15] improved their result and omitted this assumption. Moreover, they developed a theory of backward stochastic variational inequalities, that is they proved the existence and uniqueness of the solution of the reflected BSDEs driven by a fBm. The existence and uniqueness of the generalized BSDEs driven by a fBm with Hurst parameter H greater than $1/2$ was shown in [13] and in [4] the existence and uniqueness of the generalized backward stochastic variational inequalities driven by a fBm with Hurst parameter H greater than $1/2$ was proven.

The existence and uniqueness of BSDE driven by multidimensional fBm was shown with a very restrictive assumption by Miao and Yang in [17]. In [5] it was shown that this assumption is redundant.

In the current paper we show the existence and uniqueness of the backward stochastic variational inequalities (BSVI for short) driven by multidimensional fBm, i.e. we consider the reflected BSDE. In our approach we use as stochastic integral the divergence operator.

The paper is organized as follows. In Section 2 we recall some definitions and results about a fractional stochastic integral, which will be needed throughout the paper. Section 3 contains assumptions, the definition of BSDE driven by multidimensional fBm and formulation of the main theorem of the paper. Section 4 is devoted for some a priori estimates. Finally, in Section 5 we prove the main theorem using a penalization method.

2. FRACTIONAL CALCULUS – MULTIDIMENSIONAL FBM

Assume that $B^{H_1}, B^{H_2}, \ldots, B^{H_m}$ are independent fractional Brownian motions with Hurst parameters $H_1, H_2, \ldots, H_m$ respectively, where $H_k \in \left(\frac{1}{2}, 1\right)$, $k = 1, 2, \ldots, m$.

It is well known that $EB_t^{H_k} = 0$ and

$$E\left(B_t^{H_k} \cdot B_s^{H_j}\right) = \frac{1}{2}\left(|t|^{2H_k} + |s|^{2H_k} - |t-s|^{2H_k}\right)\delta_{kj},$$

$1 \leq k, j \leq m$, where $\delta_{kj} = 1$ for $k = j$ and $\delta_{kj} = 0$ for $k \neq j$.

Now, fix a Hurst index $H_k \in (\frac{1}{2}, 1)$ and define

$$\phi_k(x) = H_k(2H_k - 1)|x|^{2H_k - 2}, \quad x \in \mathbb{R}.$$

Let $\xi, \eta : [0, T] \to \mathbb{R}$ be two continuous function. Define

$$\langle \xi, \eta \rangle_{k,t} = \int_0^t \int_0^t \phi_k(u - v)\xi(u)\eta(v)dudv, \quad 0 \leq t \leq T$$

and for $\xi = \eta$,

$$|\xi|_{k,T}^2 = \langle \xi, \xi \rangle_{k,t} = \int_0^T \int_0^T \phi_k(u - v)\xi(u)\xi(v)dudv < \infty.$$

The Malliavin derivative operator D^{H_k} of an element

$$F(\omega) = f\left(\int_0^T \xi_1(t)dB_t^{H_k}, \ldots, \int_0^T \xi_l(t)dB_t^{H_k}\right),$$

where f is a polynomial function of l variables, is defined as follows:

$$D_s^{H_k} F = \sum_{i=1}^{l} \frac{\partial f}{\partial x_i}\left(\int_0^T \xi_1(t)dB_t^{H_k}, \ldots, \int_0^T \xi_l(t)dB_t^{H_k}\right)\xi_i(s), \quad s \in [0, T].$$

Now, introduce another derivative

$$\mathbb{D}_t^{H_k} F = \int_0^T \phi_k(t - s)D_s^{H_k} F ds.$$

Theorem 2.1. *Let $F : (\Omega, \mathcal{F}, \mathcal{P}) \to \mathcal{H}$ be a stochastic process such that*

$$E\left(\|F\|_{k,T}^2 + \int_0^T \int_0^T |\mathbb{D}_s^{H_k} F_t|^2 dsdt\right) < \infty.$$

Then, the Itô-type stochastic integral denoted by $\int_0^T F_s dB_s^{H_k}$ exists in $L^2(\Omega, \mathcal{F})$. Moreover,

$$E\left(\int_0^T F_s dB_s^{H_k}\right) = 0$$

and

$$E\left(\int_0^T F_s dB_s^{H_k}\right)^2 = E\left(\|F\|_{k,T}^2 + \int_0^T\int_0^T \mathbb{D}_s^{H_k} F_t \mathbb{D}_t^{H_k} F_s ds dt\right).$$

Theorem 2.2 (multidimensional fractional Itô formula)**.** *Let $\sigma_{ik} \in L^2([0,T])$, $i = 1,\dots d$, $k = 1,\dots,m$, be deterministic functions. Suppose that $\langle \sigma_{ik}, \sigma_{jk}\rangle_{k,t}$ are continuously differentiable as functions of $t \in [0,T]$. Set $X_t = (X_t^1,\dots,X_t^d)$ where*

$$X_t^i = X_0^i + \int_0^t b_i(s)ds + \sum_{k=1}^m \int_0^t \sigma_{ik}(s)dB_s^{H_k}, \quad t \in [0,T], \quad i = 1,\dots,d,$$

$X_0 = (X_0^1,\dots,X_0^d)$ is a constant vector and b_i are deterministic continuous functions with $\int_0^T |b_i(s)|ds < \infty$, $i = 1,\dots,d$. Let F be continuously differentiable with respect to t and twice continuously differentiable with respect to x. Then

$$\begin{aligned}
F(t,X_t) &= F(0,X_0) + \int_0^t \frac{\partial F}{\partial s}(s,X_s)ds + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(s,X_s)dX_s^i \\
&\quad + \sum_{i,j=1}^d \int_0^t \frac{\partial^2 F}{\partial x_i x_j}(s,X_s)\sum_{k=1}^m \sigma_{ik}(s)\mathbb{D}_s^{H_k}(X_s^j)ds \\
&= F(0,X_0) + \int_0^t \frac{\partial F}{\partial s}(s,X_s)ds + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(s,X_s)b_i(s)ds \\
&\quad + \sum_{i=1}^d\sum_{k=1}^m \int_0^t \frac{\partial F}{\partial x_i}(s,X_s)\sigma_{ik}(s)dB_s^{H_k} \\
&\quad + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 F}{\partial x_i x_j}(s,X_s)\sum_{k=1}^m \frac{d}{ds}\left(\langle \sigma_{ik},\sigma_{jk}\rangle_{k,s}\right)ds, \quad t \in [0,T].
\end{aligned}$$

Theorem 2.3 (fractional Itô chain rule)**.** *Let $T \in (0,\infty)$ and let $b_1(s)$, $b_2(s)$, $\sigma_{1k}(s)$, $\sigma_{2k}(s)$ be in $\mathbb{D}_{1,2}$ and*

$$E\left(\int_0^T (|b_i(s)| + |\sigma_{ij}(s)|)\, ds\right) < \infty, \quad i = 1,2,\ j = 1,\dots,m.$$

Assume that $\mathbb{D}_t^{H_j}\sigma_{ij}(s)$ are continuously differentiable with respect to $(s,t) \in [0,T]\times[0,T]$ for almost all $\omega \in \Omega$. Suppose that

$$E\int_0^T\int_0^T |\mathbb{D}_t^{H_j}\sigma_{ij}(s)|^2 ds dt < \infty, \quad i = 1,2,\ j = 1,\dots,m.$$

Denote

$$F(t) = F(0) + \int_0^t b_1(s)ds + \sum_{k=1}^m \int_0^t \sigma_{1k}(s)dB_s^{H_k}, \quad t \in [0,T]$$

and

$$G(t) = G(0) + \int_0^t b_2(s)ds + \sum_{k=1}^m \int_0^t \sigma_{2k}(s)dB_s^{H_k}, \quad t \in [0,T].$$

Then

$$\begin{aligned} F(t)G(t) = F(0)G(0) &+ \int_0^t F(s)dG(s) + \int_0^t G(s)dF(s) \\ &+ \sum_{k=1}^m \left(\int_0^t \mathbb{D}_s^{H_k} F(s)\sigma_{2k}(s)ds + \int_0^t \mathbb{D}_s^{H_k} G(s)\sigma_{1k}(s)ds \right) \\ = F(0)G(0) &+ \int_0^t F(s)b_2(s)ds + \sum_{k=1}^m \int_0^t F(s)\sigma_{2k}(s)dB_s^{H_k} \\ &+ \int_0^t G(s)b_1(s)ds + \sum_{k=1}^m \int_0^t G(s)\sigma_{1k}(s)dB_s^{H_k} \\ &+ \sum_{k=1}^m \left(\int_0^t \mathbb{D}_s^{H_k} F(s)\sigma_{2k}(s)ds + \int_0^t \mathbb{D}_s^{H_k} G(s)\sigma_{1k}(s)ds \right). \end{aligned}$$

The above theorems can be found in [9, 10, 12, 15, 17] and for a deeper discussion we refer the reader to [10, 18].

In solving backward stochastic differential equation with respect to a fractional Brownian motion the major problem is the absence of a martingale representation type theorem. Considering such equations an important role plays the quasi conditional expectation, which was introduced in [11]. For a while, in what follows to simplify the notations we will drop the superscripts k in the Hurst index H and in a function ϕ. For any natural n define the set $\mathcal{H}^{\otimes n}$ of all real symmetric Borel functions f of n variable such that

$$\|f\|^2_{\mathcal{H}^{\otimes n},T} = \int_{[0,T]^{2n}} \prod_{i=1}^n \phi(s_i - r_i) f(s_1, \ldots, s_n) f(r_1, \ldots, r_n) ds_1 \ldots ds_n dr_1 \ldots dr_n < \infty.$$

Then we can define the iterated integral in the sense of Itô-Skorokhod (see [9])

$$I_n^H(f) = \int\limits_{[0,T]^n} f(t_1,\ldots,t_n)dB_{t_1}^H \ldots dB_{t_n}^H$$
$$= n! \int\limits_{0\le t_1<\ldots<t_n\le T} f(t_1,\ldots,t_n)dB_{t_1}^H \cdots dB_{t_n}^H.$$

For $n=0$ and f_0 being a constant we put $I_0(f_0)=f_0$ and $\|f\|^2_{\mathcal{H}^{\otimes 0},T} = f_0^2$.

Theorem 2.4. *Let $F \in L^2(\Omega, \mathcal{F}, P)$. Then there exists a sequence $f_n \in \mathcal{H}^{\otimes n}$, $n \ge 0$, such that F has the following chaos expansion:*

$$F(\omega) = \sum_{n=0}^{\infty} I_n^H(f_n).$$

Moreover,

$$E|F|^2 = \sum_{n=0}^{\infty} n! \|f_n\|_{\mathcal{H}^{\otimes n},T} < \infty.$$

Definition 2.5. The quasi-conditional expectation of some random variable of the form

$$F(\omega) = \sum_{n=0}^{\infty} I_n^H(f_n)$$

relative to a fractional Brownian motion B^H and the filtration $\{\mathcal{F}_t\}_{t\ge 0}$ generated by B^H is defined by

$$\hat{E}^H(F|\mathcal{F}_t) = \sum_{n=0}^{\infty} I_n^H\left(f_n I_{[0,t]}^{\otimes n}\right), \quad t \in [0,T]$$

if the series converges in $L^2(\Omega, \mathcal{F}, P)$. Here $I_{[0,t]}^{\otimes n}(t_1, t_2, \ldots, t_n) = I_{[0,t]}(t_1)\cdots I_{[0,t]}(t_n)$.

Remark 2.6. Note that $\hat{E}\left(\hat{E}(F|\mathcal{F}_t)|F_s\right) = \hat{E}(F|\mathcal{F}_s)$ for $0 \le s \le t \le T$.

Now, following [17], for any $k = 1, \ldots, m$ take $f_n \in \mathcal{H}_k^{\otimes n}$ and denote by $\hat{L}^2(\Omega, \mathcal{F}, P)$ the set of $F \in L^2(\Omega, \mathcal{F}, P)$ such that F has the following chaos expansion:

$$F(\omega) = \sum_{k=1}^{m} \sum_{n=0}^{\infty} I_n^{H_k}(f_n).$$

The quasi-conditional expectation of random variable $F \in \hat{L}^2(\Omega, \mathcal{F}, P)$ is defined as in the definition above.

Proposition 2.7. *Let $F : (\Omega, \mathcal{F}, \mathcal{P}) \to \mathcal{H}$ be a stochastic process such that*

$$E\left(\|F\|_{k,T}^2 + \int_0^T\int_0^T |\mathbb{D}_s^{H_k} F_t|^2 ds dt\right) < \infty.$$

Then

$$\hat{E}^{H_k}\left(\int_t^T F_s dB_s^{H_k} | \mathcal{F}_t\right) = 0.$$

3. BACKWARD STOCHASTIC VARIATIONAL INEQUALITIES

Assume that

(H_1) $\sigma_{ik} : [0,T] \to \mathbb{R}$, $i = 1,\ldots,d$, $k = 1,\ldots,m$ are deterministic continuous functions, differentiable and such that $\sigma_{ik}(t) \neq 0$, for all $t \in [0,T]$. For a given vector constant $\eta_0 = (\eta_0^1, \ldots, \eta_0^d)$ consider

$$\eta_t^i = \eta_0^i + \sum_{k=1}^m \int_0^t \sigma_{ik}(s) dB_s^{H_k}, \quad t \in [0,T], i = 1,\ldots m.$$

Note that, since

$$\langle \sigma_{ik}, \sigma_{jk}\rangle_{k,t} = H_k(2H_k - 1)\int_0^t\int_0^t |v-u|^{2H_k-2}\sigma_{ik}(v)\sigma_{jk}(u) dv du,$$

we have

$$\begin{aligned}\frac{d}{dt}\left(\langle \sigma_{ik}, \sigma_{jk}\rangle_{k,t}\right) &= 2H_k(2H_k-1)\sigma_{ik}(t)\int_0^t |t-u|^{2H_k-2}\sigma_{jk}(u)du \\ &= 2\sigma_{ik}(t)\hat{\sigma}_{jk}(t) > 0,\end{aligned}$$

where

$$\hat{\sigma}_{jk}(t) = \int_0^t \phi_k(t-u)\sigma_{jk}(u)du.$$

We suppose that

(H_2) $\xi = h(\eta_T)$ for some function h with bounded derivative and such that $E|\xi|^2 < \infty$,
(H_3) $f\colon [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is a continuous function such that there exists a positive constant L satisfying for all $t \in [0,T]$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^m$,

$$|f(t,x,y,z) - f(t,x,y',z')| \leq L(|y-y'| + \|z-z'\|)$$

and

$$E\left(\int_0^T |f(t,\eta_t,0,0)|^2 dt\right) < \infty.$$

Now consider the set

$$\mathcal{V}_T = \Big\{ Y = \phi(\cdot,\eta) : \phi \in C^{1,3}_{pol}([0,T]\times\mathbb{R}) \text{ and } \frac{\partial\phi}{\partial t} \text{ is bounded} \Big\}.$$

By $\tilde{\mathcal{V}}^H_T$ denote the completition of the set of processes from $\mathcal{V}_T$ with the following norm:

$$\|Y\|^2_H = E\int_0^T t^{2H-1}|Y_t|^2 dt = E\int_0^T t^{2H-1}|\phi(t,\eta_t)|^2 dt.$$

Additionally consider

(H_4) a function $\varphi : \mathbb{R} \to (-\infty,\infty]$ which is proper, convex and lower semi-continuous such that $\varphi(y) \geq \varphi(0) = 0$

and denote

$$\begin{aligned}
&\partial\varphi(y) = \{\hat{y}\in\mathbb{R} : \ \ \hat{y}\cdot(v-y)+\varphi(y)\leq\varphi(v) \text{ for all } v\in\mathbb{R}\},\\
&\mathrm{Dom}\varphi = \{y\in\mathbb{R} : \ \ \varphi(y)<\infty\}, \quad \mathrm{Dom}(\partial\varphi) = \{y\in\mathbb{R} : \ \ \partial\varphi(y)\neq\emptyset\},\\
&\langle y,\hat{y}\rangle \in \partial\varphi \Leftrightarrow y\in\mathrm{Dom}(\partial\varphi),\ \hat{y}\in\partial\varphi(y).
\end{aligned}$$

Remark 3.1. $\partial\varphi$ is maximal in this sense that

$$(\hat{y}-\hat{u})(y-u)\geq 0, \quad (y,\hat{y}),(u,\hat{u})\in\partial\varphi.$$

We will consider the following backward stochastic variational inequality driven by multidimensional fBm:

$$\begin{cases} dY_t + f(t,\eta_t,Y_t,Z_t)dt - \sum_{k=1}^m Z^k_t dB^{H_k}_t \in \partial\varphi(Y_t)dt, \\ Y_T = \xi = h(\eta_T). \end{cases} \tag{3.1}$$

Definition 3.2. A solution of a backward stochastic variational inequality (BSVI) driven by multidimensional fBm (3.1) associated with data (ξ, f) is a triple $(Y_t,Z_t,U_t)_{t\in[0,T]}$ of processes satisfying

$$Y_t = \xi + \int_t^T f(s,\eta_s,Y_s,Z_s)ds - \sum_{k=1}^m \int_t^T Z^k_s dB^{H_k}_s - \int_t^T U_s ds, \quad t\in[0,T] \tag{3.2}$$

and such that $Y,U \in \tilde{\mathcal{V}}^{H_1}_T \cap \tilde{\mathcal{V}}^{H_2}_T \cap \ldots \cap \tilde{\mathcal{V}}^{H_m}_T$, $(Y_t,U_t)\in\partial\varphi$, $t\in[0,T]$ and $Z = (Z^1,\ldots,Z^k)$, where $Z^k \in \tilde{\mathcal{V}}^{H_1+H_k-1/2}_T \cap \tilde{\mathcal{V}}^{H_2+H_k-1/2}_T \cap \ldots \cap \tilde{\mathcal{V}}^{H_m+H_k-1/2}_T$.

Theorem 3.3. *Assume* (H_1)*–*(H_4)*. There exists a unique solution of* (3.2).

The proof of the above theorem is deferred to Section 5.

4. A PRIORI ESTIMATES

Theorem 4.1. *Assume* (H_1)–(H_4) *and let* (Y, Z, U) *be a solution of* (3.2). *Then for all* $t \in [0, T]$,

$$E\left(|Y_t|^2 + \sum_{k=1}^{m} \int_t^T s^{2H_k-1} |Z_s^k|^2 ds\right) \le CE\left(|\xi|^2 + \int_t^T |f(s, \eta_s, 0, 0)|^2 ds\right) = C\Theta(t, T).$$

Proof. By C we will denote a constant which may vary from line to line. From the Itô formula,

$$\begin{aligned} |Y_t|^2 &= |\xi|^2 - \int_t^T d|Y_s|^2 \\ &= |\xi|^2 - 2\int_t^T Y_s dY_s - 2\sum_{k=1}^{m} \int_t^T \mathbb{D}_s^{H_k} Y_s Z_s^k ds \\ &= |\xi|^2 + 2\int_t^T Y_s f(s, \eta_s, Y_s, Z_s) ds - 2\sum_{k=1}^{m} \int_t^T Y_s Z_s^k dB_s^{H_k} \\ &\quad - 2\int_t^T Y_s U_s ds - 2\sum_{k=1}^{m} \int_t^T \mathbb{D}_s^{H_k} Y_s Z_s^k ds. \end{aligned} \tag{4.1}$$

It is known (see [12, 15]) that

$$\mathbb{D}_s^{H_k} Y_s = \int_0^T \phi_k(s - r) D_r^{H_k} Y_s dr = \frac{\hat{\sigma}_k(s)}{\sigma_k(s)} Z_s^k.$$

Moreover, by Remark 6 in [15], there exists $M_k > 0$ such that for all $t \in [0, T]$,

$$\frac{s^{2H_k-1}}{M_k} \le \frac{\hat{\sigma}_k(s)}{\sigma_k(s)} \le M_k s^{2H_k-1}. \tag{4.2}$$

Denote $M = \max_{1 \le k \le m} M_k$. By the above, integrating (4.1) we have

$$\begin{aligned} &E|Y_t|^2 + \frac{2}{M} \sum_{k=1}^{m} \int_t^T s^{2H_k-1} |Z_s^k|^2 ds \\ &\le E|Y_t|^2 + 2\sum_{k=1}^{m} \int_t^T \frac{\hat{\sigma}_k(s)}{\sigma_k(s)} |Z_s^k|^2 ds \\ &= E|\xi|^2 + 2E\int_t^T Y_s f(s, \eta_s, Y_s, Z_s) ds - 2E\int_t^T Y_s U_s ds. \end{aligned} \tag{4.3}$$

By Remark 3.1, the last component of the right hand side is non-positive. By the Lipschitz continuity of f and the simple inequality $2ab \leq a/\varepsilon + \varepsilon b$, we have

$$\begin{aligned} 2yf(s,\eta,y,z) &\leq 2L|y|\left(|y| + \|z\|\right) + 2|y||f(s,\eta,0,0)| \\ &\leq (2L+1)\,|y|^2 + |f(s,\eta,0,0)|^2 + 2L\sqrt{\sum_{k=1}^{m}|y|^2|z^k|^2} \\ &\leq (2L+1)\,|y|^2 + |f(s,\eta,0,0)|^2 + \sum_{k=1}^{m} 2L|y||z^k| \\ &\leq \left(2L+1+\sum_{k=1}^{m}\frac{ML^2}{s^{2H_k-1}}\right)|y|^2 + |f(s,\eta,0,0)|^2 + \frac{1}{M}\sum_{k=1}^{m} s^{2H_k-1}|z^k|^2 \end{aligned}$$

and therefore

$$\begin{aligned} E|Y_t|^2 + \frac{2}{M}\sum_{k=1}^{m}\int_t^T s^{2H_k-1}|Z_s^k|^2 ds &\leq E\Bigg(|\xi|^2 + \int_t^T |f(s,\eta_s,0,0)|^2 ds\Bigg) \\ &+ E\int_t^T\Bigg(2L+1+\sum_{k=1}^{m}\frac{ML^2}{s^{2H_k-1}}\Bigg)|Y_s|^2 ds + \frac{1}{M}\sum_{k=1}^{m} E\int_t^T s^{2H_k-1}|Z_s^k|^2 ds. \end{aligned}$$

Denoting

$$\Theta(t,T) = E\left(|\xi|^2 + \int_t^T |f(s,\eta_s,0,0)|^2\right)$$

we can write

$$\begin{aligned} E\left(|Y_t|^2 + \frac{1}{M}\sum_{k=1}^{m}\int_t^T s^{2H_k-1}|Z_s^k|^2 ds\right) &\leq \Theta(t,T) \\ &+ E\int_t^T\left(2L+1+\sum_{k=1}^{m}\frac{ML^2}{s^{2H_k-1}}\right)|Y_s|^2 ds. \end{aligned} \tag{4.4}$$

By the Gronwall inequality,

$$E|Y_t|^2 \leq \Theta(t,T)\exp\left\{(2L+1)(T-t) + ML^2\sum_{k=1}^{m}\frac{T^{2-2H_k} - t^{2-2H_k}}{2-2H_k}\right\}$$

and by (4.4) also

$$E\left(\sum_{k=1}^{m}\int_t^T s^{2H_k-1}|Z_s^k|^2 ds\right) \leq C\Theta(t,T). \qquad \square$$

Proposition 4.2. *Assume* (H_1)–(H_4) *and let* (Y, Z, U) *and* $(\tilde{Y}, \tilde{Z}, \tilde{U})$ *be two solutions of* (3.2) *with data* (ξ, f) *and* $(\tilde{\xi}, \tilde{f})$, *respectively. Then*

$$\begin{aligned}
&E\bigg(|Y_t - \tilde{Y}_t|^2 + \sum_{k=1}^{m} \int_t^T s^{2H_k-1} |Z_s^k - \tilde{Z}_s^k|^2 ds\bigg) \\
&\leq CE\bigg(|\xi - \tilde{\xi}|^2 + \int_t^T |f(s, \eta_s, Y_s, Z_s) - \tilde{f}(s, \eta_s, Y_s, Z_s)|^2 ds\bigg).
\end{aligned}$$

Proof. By the Itô formula, computing similarly as in the previous theorem

$$\begin{aligned}
&|Y_t - \tilde{Y}_t|^2 + \frac{2}{M} \sum_{k=1}^{m} \int_t^T s^{2H_k-1} |Z_s^k - \tilde{Z}_s^k|^2 ds \\
&\leq |\xi - \tilde{\xi}|^2 + 2 \int_t^T (Y_s - \tilde{Y}_s)\big(f(s, \eta_s, Y_s, Z_s) - \tilde{f}(s, \eta_s, \tilde{Y}_s, \tilde{Z}_s)\big) ds \\
&\quad - 2 \int_t^T \sum_{k=1}^{m} (Y_s - \tilde{Y}_s)(Z_s^k - \tilde{Z}_s^k) dB_s^{H_k} - 2 \int_t^T (Y_s - \tilde{Y}_s)(U_s - \tilde{U}_s) ds.
\end{aligned}$$

From assumptions we get

$$\begin{aligned}
2(y - \tilde{y})(f(s, \eta, y, z) - \tilde{f}(s, \eta, \tilde{y}, \tilde{z})) &\leq 2(y - \tilde{y})(f(s, \eta, y, z) - \tilde{f}(s, \eta, y, z)) \\
&\quad + \left(2L + \sum_{k=1}^{m} \frac{L^2 M}{s^{2H_k-1}}\right) |y - \tilde{y}|^2 + \sum_{k=1}^{m} \frac{s^{2H_k-1}}{M} |z^k - \tilde{z}^k|^2 \\
&\leq |f(s, \eta, y, z) - \tilde{f}(s, \eta, y, z)|^2 \\
&\quad + \left(2L + 1 + \sum_{k=1}^{m} \frac{L^2 M}{s^{2H_k-1}}\right) |y - \tilde{y}|^2 + \frac{1}{M} \sum_{k=1}^{m} s^{2H_k-1} |z^k - \tilde{z}^k|^2.
\end{aligned}$$

Since $U_t \in \partial\varphi(Y_t)$ and $\tilde{U}_t \in \partial\varphi(\tilde{Y}_t)$,

$$\begin{aligned}
(U_t - \tilde{U}_t)(Y_t - \tilde{Y}_t) &= U_t(Y_t - \tilde{Y}_t) + \tilde{U}_t(\tilde{Y}_t - Y_t) \\
&\geq \varphi(Y_t) - \varphi(\tilde{Y}_t) + \varphi(\tilde{Y}_t) - \varphi(Y_t) = 0.
\end{aligned}$$

Therefore, we obtain

$$E\Bigg(|Y_t-\tilde{Y}_t|^2+\frac{1}{M}\int\limits_t^T\sum_{k=1}^m s^{2H_k-1}|Z_s^k-\tilde{Z}_s^k|^2ds\Bigg)$$

$$\le E\Bigg(|\xi-\tilde{\xi}|^2+\int\limits_t^T\big|f(s,\eta_s,Y_s,Z_s)-\tilde{f}(s,\eta_s,Y_s,Z_s)\big|ds\Bigg)$$

$$+E\int\limits_t^T\Bigg(2L+1+L^2M\sum_{k=1}^m\frac{1}{s^{2H_k-1}}\Bigg)|Y_s-\tilde{Y}_s|^2ds.$$

Using the Gronwall Lemma we get the required inequality. □

5. PENALIZATION SCHEME

We will approximate the function φ by a sequence of convex, C^1 class functions φ_ε, $\varepsilon>0$, defined by

$$\varphi_\varepsilon(y)=\inf\left\{\frac{1}{2\varepsilon}|y-v|^2+\varphi(v):v\in\mathbb{R}\right\}=\frac{1}{2\varepsilon}|y-J_\varepsilon(y)|^2+\varphi(J_\varepsilon(y)),\tag{5.1}$$

where $J_\varepsilon(y)=y-\varepsilon\nabla\varphi_\varepsilon(y)$.

Here are some properties of φ_ε (see [1] or [6]):

$$\nabla\varphi_\varepsilon(y)=\frac{y-J_\varepsilon(y)}{\varepsilon}\in\partial\varphi(J_\varepsilon(y)),\tag{5.2}$$

$$|J_\varepsilon(y)-J_\varepsilon(v)|\le|y-v|\quad\text{and}\quad\lim_{\varepsilon\searrow0}J_\varepsilon(y)=\pi_{\overline{\mathrm{Dom}\varphi}}(y),\tag{5.3}$$

$$0\le\varphi_\varepsilon(y)\le y\nabla\varphi_\varepsilon(y),\tag{5.4}$$

where by $\pi_{\overline{\mathrm{Dom}\varphi}}(y)$ we denote the projection of y on the closure of the set $\mathrm{Dom}\varphi$.

Note that if for some $a\le0$ we define convex indicator function

$$\varphi(y)=\begin{cases}0, & y\ge a,\\ \infty, & y<a,\end{cases}$$

then $\nabla\varphi_\varepsilon(y)=-\frac{1}{\varepsilon}(y-a)^-$, where $x^-=\max(-x,0)$.

Consider a sequence of BSDEs

$$Y_t^\varepsilon=\xi+\int\limits_t^T f(s,\eta_s,Y_s^\varepsilon,Z_s^\varepsilon)ds-\sum_{k=1}^m\int\limits_t^T Z_s^{k,\varepsilon}dB_s^{H_k}-\int\limits_t^T\nabla\varphi_\varepsilon(Y_s^\varepsilon)ds,\quad t\in[0,T].\tag{5.5}$$

Since $\nabla\varphi_\varepsilon$ is Lipschitz continuous function, then by [5] (5.5) has a unique solution $(Y^\varepsilon,Z^\varepsilon)$.

Proposition 5.1. *Let assumptions (H_1)–(H_4) hold. Then*

$$E\bigg(|Y_t^\varepsilon|^2+\sum_{k=1}^m\int_t^T s^{2H_k-1}|Z_s^{k,\varepsilon}|^2ds+\int_t^T Y_s^\varepsilon\nabla\varphi_\varepsilon(Y_s^\varepsilon)ds\bigg)$$
$$\le CE\bigg(|\xi|^2+\int_t^T\left(|f(s,\eta_s,0,0)|^2\right)ds\bigg)=C\Theta(t,T).$$

Proof. Similarly as in the proof of Theorem 4.1, we have

$$E\bigg(|Y_t^\varepsilon|^2+\frac{1}{M}\sum_{k=1}^m\int_t^T s^{2H_k-1}|Z_s^{k,\varepsilon}|^2ds\bigg)$$
$$\le E\bigg(|\xi|^2+\int_t^T|f(s,\eta_s,0,0)|^2ds\bigg)$$
$$+E\int_t^T\bigg(2L+1+\sum_{k=1}^m\frac{ML^2}{s^{2H_k-1}}\bigg)|Y_s^\varepsilon|^2ds-2E\int_t^T Y_s^\varepsilon\nabla\varphi_\varepsilon(Y_s^\varepsilon)ds.$$

By (5.4), the last component of the right hand side is non-positive, so we obtain

$$E\bigg(|Y_t^\varepsilon|^2+\frac{1}{M}\sum_{k=1}^m\int_t^T s^{2H_k-1}|Z_s^{k,\varepsilon}|^2ds+2\int_t^T Y_s^\varepsilon\nabla\varphi_\varepsilon(Y_s^\varepsilon)ds\bigg)$$
$$\le\Theta(t,T)+E\int_t^T\bigg(2L+1+\sum_{k=1}^m\frac{ML^2}{s^{2H_k-1}}\bigg)|Y_s^\varepsilon|^2ds.$$

Now using similar arguments as in the proof of Theorem 4.1 we finish the proof. □

Proposition 5.2. *Under assumptions (H_1)–(H_4) there exists a positive constant C such that for any $t\in[0,T]$*

(a) $E\sum_{k=1}^m\int_t^T s^{2H_k-1}|\nabla\varphi_\varepsilon(Y_s^\varepsilon)|^2ds\le C\Theta_2(t,T)$,

(b) $E\sum_{k=1}^m t^{2H_k-1}\left|Y_t^\varepsilon-J_\varepsilon(Y_t^\varepsilon)\right|^2\le\varepsilon\cdot C\Theta_2(t,T)$,

(c) $E\sum_{k=1}^m t^{2H_k-1}\varphi\left(J_\varepsilon(Y_t^\varepsilon)\right)\le C\Theta_2(t,T)$,

(d) $E\sum_{k=1}^m\int_t^T s^{2H_k-1}\left|Y_s^\varepsilon-J_\varepsilon(Y_s^\varepsilon)\right|^2ds\le\varepsilon^2C\Theta_2(t,T)$, *where*

$$\Theta_2(t,T) = E\sum_{k=1}^{m}\left(T^{2H_k-1}\varphi(\xi) + \int_t^T s^{2H_k-1}\left(|Y_s^\varepsilon|^2 + |Z_s^{k,\varepsilon}|^2 + |f(s,\eta_s,0,0)|^2\right)\right).$$

Proof. In the proof below we will use similar arguments as in the proof of Proposition 2.2 in [20] and in the proof of Proposition 11 in [16]. Since $\nabla\varphi_\varepsilon(Y_r^\varepsilon) \in \partial\varphi(J_\varepsilon(Y_r^\varepsilon))$,

$$\nabla\varphi_\varepsilon(Y_r^\varepsilon)\cdot(Y_s^\varepsilon - Y_r^\varepsilon) \le \varphi_\varepsilon(Y_s^\varepsilon) - \varphi_\varepsilon(Y_r^\varepsilon)$$

and

$$\varphi_\varepsilon(Y_s^\varepsilon) \ge \nabla\varphi_\varepsilon(Y_r^\varepsilon)\cdot(Y_s^\varepsilon - Y_r^\varepsilon) + \varphi_\varepsilon(Y_r^\varepsilon).$$

Now for any $k\in\{1,2,\ldots,m\}$ and $s>r\ge 0$

$$\begin{aligned} s^{2H_k-1}\varphi_\varepsilon(Y_s^\varepsilon) &\ge s^{2H_k-1}\varphi_\varepsilon(Y_r^\varepsilon) + s^{2H_k-1}\nabla\varphi_\varepsilon(Y_r^\varepsilon)\cdot(Y_s^\varepsilon - Y_r^\varepsilon) \\ &\ge r^{2H_k-1}\varphi_\varepsilon(Y_r^\varepsilon) + s^{2H_k-1}\nabla\varphi_\varepsilon(Y_r^\varepsilon)\cdot(Y_s^\varepsilon - Y_r^\varepsilon). \end{aligned}$$

Take $s = t_{i+1}\wedge T$, $r = t_i\wedge T$, where $0 = t_0 < t_1 < \ldots < t\wedge T$ and $t_{i+1} - t_i = 1/n$. Summing up over i and passing to the limit as $n\to\infty$, we deduce

$$T^{2H_k-1}\varphi_\varepsilon(Y_T^\varepsilon) \ge t^{2H_k-1}\varphi_\varepsilon(Y_t^\varepsilon) + \int_t^T s^{2H_k-1}\nabla\varphi_\varepsilon(Y_s^\varepsilon)dY_s^\varepsilon.$$

Therefore,

$$\begin{aligned} t^{2H_k-1}\varphi_\varepsilon(Y_t^\varepsilon) &\le T^{2H_k-1}\varphi_\varepsilon(\xi) - \int_t^T s^{2H_k-1}\nabla\varphi_\varepsilon(Y_s^\varepsilon)dY_s^\varepsilon \\ &= T^{2H_k-1}\varphi_\varepsilon(\xi) + \int_t^T s^{2H_k-1}\nabla\varphi_\varepsilon(Y_s^\varepsilon)f(s,\eta_s,Y_s^\varepsilon,Z_s^\varepsilon)ds \\ &\quad - \int_t^T s^{2H_k-1}\nabla\varphi_\varepsilon(Y_s^\varepsilon)\sum_{j=1}^m Z_s^{j,\varepsilon}dB_s^{H_j} - \int_t^T s^{2H_k-1}\left|\nabla\varphi_\varepsilon(Y_s^\varepsilon)\right|^2 ds \end{aligned}$$

and

$$\begin{aligned} &t^{2H_k-1}\varphi_\varepsilon(Y_t^\varepsilon) + \int_t^T s^{2H_k-1}\left|\nabla\varphi_\varepsilon(Y_s^\varepsilon)\right|^2 ds \\ &\le T^{2H_k-1}\varphi_\varepsilon(\xi) + \int_t^T s^{2H_k-1}\nabla\varphi_\varepsilon(Y_s^\varepsilon)f(s,\eta_s,Y_s^\varepsilon,Z_s^\varepsilon)ds \qquad (5.6)\\ &\quad - \int_t^T s^{2H_k-1}\nabla\varphi_\varepsilon(Y_s^\varepsilon)\sum_{j=1}^m Z_s^{j,\varepsilon}dB_s^{H_j}. \end{aligned}$$

Note that

$$\begin{aligned}\nabla\varphi_\varepsilon(y)f(s,\eta,y,z) &\le |\nabla\varphi_\varepsilon(y)|\left(L|y|+L\|z\|+|f(s,\eta,0,0)|\right)\\ &\le \frac{1}{2}|\nabla\varphi_\varepsilon(y)|^2+\frac{3}{2}\left(L^2|y|^2+L^2\|z\|^2+|f(s,\eta,0,0)|^2\right).\end{aligned}$$

Using the fact that $\varphi_\varepsilon(\xi)\le\varphi(\xi)$ and integrating the inequality (5.6) we get

$$\begin{aligned}&Et^{2H_k-1}\varphi_\varepsilon(Y_t^\varepsilon)+\frac{1}{2}E\int_t^T s^{2H_k-1}\left|\nabla\varphi_\varepsilon(Y_s^\varepsilon)\right|^2 ds\\ &\le ET^{2H_k-1}\varphi(\xi)+\frac{3}{2}E\int_t^T s^{2H_k-1}\left(L^2|Y_s^\varepsilon|^2+L^2\|Z_s^\varepsilon\|^2+|f(s,\eta_s,0,0)|^2\right)ds.\end{aligned}\tag{5.7}$$

Since H_k was arbitrary, we can write similar inequalities for every $k=1,\ldots,m$. Now, summing up (5.7) over k we have

$$\begin{aligned}&E\sum_{k=1}^m t^{2H_k-1}\varphi_\varepsilon(Y_t^\varepsilon)+E\sum_{k=1}^m\int_t^T s^{2H_k-1}\left|\nabla\varphi_\varepsilon(Y_s^\varepsilon)\right|^2 ds\\ &\le C\sum_{k=1}^m E\bigg(T^{2H_k-1}\varphi(\xi)+\int_t^T s^{2H_k-1}\left(L^2|Y_s^\varepsilon|^2+L^2\|Z_s^\varepsilon\|^2+|f(s,\eta_s,0,0)|^2\right)ds\bigg)\\ &= C\Theta_2(t,T).\end{aligned}$$

From the above inequality (a) is clear. Condition (c) follows additionally from inequality $\varphi(J_\varepsilon(y))\le\varphi_\varepsilon(y)$. From $|y-J_\varepsilon(y)|^2\le 2\varepsilon\varphi_\varepsilon(y)$ follows (b). Finally (d) we get from $y-J_\varepsilon(y)=\varepsilon\nabla\varphi_\varepsilon(y)$. □

Proposition 5.3. *Let assumptions (H_1)–(H_4) be satisfied. Then $(Y^\varepsilon,Z^\varepsilon)$ is a Cauchy sequence, i.e. for $\varepsilon,\delta>0$*

$$\begin{aligned}&\sum_{k=1}^m E\bigg(t^{2H_k-1}|Y_t^\varepsilon-Y_t^\delta|^2+\int_t^T s^{2H_k-2}|Y_s^\varepsilon-Y_s^\delta|^2ds\\ &\qquad+\int_t^T s^{2H_k-1}\sum_{j=1}^m s^{2H_j-1}|Z_s^{j,\varepsilon}-Z_s^{j,\delta}|^2ds\bigg)\\ &\le C\cdot(\varepsilon+\delta)\cdot\Theta_2(t,T).\end{aligned}$$

Proof. Put $\check{Y} = Y^\varepsilon - Y^\delta$ and $\check{Z} = Z^\varepsilon - Z^\delta$. For any $k \in \{1, 2, \ldots, m\}$, we have

$$\begin{aligned} &t^{2H_k-1}|\check{Y}_t|^2 + \int_t^T (2H_k - 1)s^{2H_k-2}|\check{Y}_s|^2 ds \\ &= T^{2H_k-1}|\check{Y}_T|^2 - 2\int_t^T s^{2H_k-1}\check{Y}_s d\check{Y}_s - 2\int_t^T s^{2H_k-1}\sum_{j=1}^m \frac{\hat{\sigma}_j(s)}{\sigma_j(s)}|\check{Z}_s^j|^2 ds. \end{aligned}$$

Therefore,

$$\begin{aligned} &E\Bigg(t^{2H_k-1}|\check{Y}_t|^2 + \int_t^T (2H_k - 1)s^{2H_k-2}|\check{Y}_s|^2 ds + \frac{2}{M}\int_t^T s^{2H_k-1}\sum_{j=1}^m s^{2H_j-1}|\check{Z}_s^j|^2 ds\Bigg) \\ &\leq 2E\int_t^T s^{2H_k-1}\check{Y}_s \left(f(s,\eta_s,Y_s^\varepsilon,Z_s^\varepsilon) - f(s,\eta_s,Y_s^\delta,Z_s^\delta)\right) ds \\ &\quad - 2E\int_t^T s^{2H_k-1}\check{Y}_s \left(\nabla\varphi_\varepsilon(Y_s^\varepsilon) - \nabla\varphi_\delta(Y_s^\delta)\right) ds. \end{aligned}$$

Note that

$$\begin{aligned} 2s^{2H_k-1}\check{y}\cdot\left(f(s,\eta,y^\varepsilon,z^\varepsilon) - f(s,\eta,y^\delta,z^\delta)\right) \leq s^{2H_k-1}&\left(2L + \sum_{j=1}^m \frac{L^2M}{s^{2H_j-1}}\right)|\check{y}|^2 \\ &+ \frac{1}{M}s^{2H_k-1}\sum_{j=1}^m s^{2H_j-1}|\check{z}^j|^2. \end{aligned}$$

Moreover, by the definition of φ_ε we get

$$\begin{aligned} 0 &\leq \left(\nabla\varphi_\varepsilon(Y_s^\varepsilon) - \nabla\varphi_\delta(Y_s^\delta)\right)\cdot\left(J_\varepsilon(Y_s^\varepsilon) - J_\delta(Y_s^\delta)\right) \\ &= \left(\nabla\varphi_\varepsilon(Y_s^\varepsilon) - \nabla\varphi_\delta(Y_s^\delta)\right)\cdot\left(Y_s^\varepsilon - Y_s^\delta - \varepsilon\nabla\varphi_\varepsilon(Y_s^\varepsilon) + \delta\nabla\varphi_\delta(Y_s^\delta)\right) \\ &= \left(\nabla\varphi_\varepsilon(Y_s^\varepsilon) - \nabla\varphi_\delta(Y_s^\delta)\right)\cdot\left(Y_s^\varepsilon - Y_s^\delta\right) - \varepsilon|\nabla\varphi_\varepsilon(Y_s^\varepsilon)|^2 - \delta|\nabla\varphi_\delta(Y_s^\delta)|^2 \\ &\quad + (\varepsilon+\delta)\nabla\varphi_\varepsilon(Y_s^\varepsilon)\cdot\nabla\varphi_\delta(Y_s^\delta) \end{aligned}$$

and then

$$\left(\nabla\varphi_\varepsilon(Y_s^\varepsilon) - \nabla\varphi_\delta(Y_s^\delta)\right)\cdot\left(Y_s^\varepsilon - Y_s^\delta\right) \geq -(\varepsilon+\delta)\nabla\varphi_\varepsilon(Y_s^\varepsilon)\cdot\nabla\varphi_\delta(Y_s^\delta).$$

Therefore,

$$E\Big(t^{2H_k-1}|\check{Y}_t|^2 + \int\limits_t^T (2H_k-1)s^{2H_k-2}|\check{Y}_s|^2 ds + \frac{1}{M}\sum_{j=1}^m \int\limits_t^T s^{2H_k-1}s^{2H_j-1}|\check{Z}_s^j|^2 ds\Big)$$
$$\leq E\int\limits_t^T \left(2L + \sum_{j=1}^m \frac{L^2 M}{s^{2H_j-1}}\right) s^{2H_k-1}|\check{Y}_s|^2 ds \tag{5.8}$$
$$+ 2(\varepsilon+\delta)E\int\limits_t^T s^{2H_k-1}\nabla\varphi_\varepsilon(Y_s^\varepsilon)\,\nabla\varphi_\delta(Y_s^\delta)ds.$$

By the Gronwall Lemma and by the simple inequality $ab \leq a^2/2 + b^2/2$,

$$Et^{2H_k-1}|\check{Y}_t|^2 \leq C(\varepsilon+\delta)E\int\limits_t^T s^{2H_k-1}\nabla\varphi_\varepsilon(Y_s^\varepsilon)\,\nabla\varphi_\delta(Y_s^\delta)ds$$
$$\leq C(\varepsilon+\delta)E\int\limits_t^T s^{2H_k-1}\left(|\nabla\varphi_\varepsilon(Y_s^\varepsilon)|^2 + |\nabla\varphi_\delta(Y_s^\delta)|^2\right)ds.$$

From the above and from (5.8)

$$E\left(t^{2H_k-1}|\check{Y}_t|^2 + \int\limits_t^T s^{2H_k-2}|\check{Y}_s|^2 ds + \int\limits_t^T s^{2H_k-1}\sum_{j=1}^m s^{2H_j-1}|\check{Z}_s^j|^2 ds\right)$$
$$\leq C(\varepsilon+\delta)E\int\limits_t^T s^{2H_k-1}\left(|\nabla\varphi_\varepsilon(Y_s^\varepsilon)|^2 + |\nabla\varphi_\delta(Y_s^\delta)|^2\right)ds.$$

Summing up over k, $k = 1, 2, \ldots, m$ and using Proposition 5.2 a) we get the result. □

Now we can give a proof of Theorem 3.3.

Proof of Theorem 3.3. First, we show the uniqueness. From the proof of Proposition 4.2 it follows that for (Y, Z, U) and (Y', Z', U') being two solutions of (3.2), we have

$$E\Big(|Y_t - Y_t'|^2 + \sum_{k=1}^m \int\limits_t^T s^{2H_k-1}|Z_s^k - {Z'}_s^k|^2 ds\Big) = 0,$$

and

$$E\Big(\int\limits_t^T (Y_s - Y_s')(U_s - U_s')ds\Big) \leq 0,$$

which means that the solution is unique.

Now we will show that the limit of $(Y^\varepsilon, Z^\varepsilon, \nabla\varphi_\varepsilon(Y^\varepsilon))$ converges to a solution of (3.2).

Since by Proposition 5.3 $(Y^\varepsilon, Z^\varepsilon)$ is a Cauchy sequence, there exists its limit, i.e. there exists a pair of processes (Y, Z) such that $Y \in \tilde{\mathcal{V}}_T^{H_1} \cap \tilde{\mathcal{V}}_T^{H_2} \ldots \cap \tilde{\mathcal{V}}_T^{H_m}$, $Z = (Z^1, \ldots, Z^k)$, where

$$Z^k \in \tilde{\mathcal{V}}_T^{H_1+H_k-1/2} \cap \tilde{\mathcal{V}}_T^{H_2+H_k-1/2} \ldots \cap \tilde{\mathcal{V}}_T^{H_m+H_k-1/2}$$

and

$$\lim_{\varepsilon \searrow 0} E\left(\sum_{k=1}^{m} \int_t^T s^{2H_k-1} |Y_s^\varepsilon - Y_s|^2 ds + \sum_{k,j=1}^{m} \int_t^T s^{2(H_k+H_j-1/2)-1} |Z_s^{j,\varepsilon} - Z_s^j|^2 ds \right) = 0.$$

From Proposition 5.2 c),

$$\lim_{\varepsilon \searrow 0} E \sum_{k=1}^{m} t^{2H_k-1} \left|Y_t^\varepsilon - J_\varepsilon(Y_t^\varepsilon)\right|^2 = 0,$$

and we have

$$\lim_{\varepsilon \searrow 0} J_\varepsilon(Y^\varepsilon) = Y \text{ in } \tilde{\mathcal{V}}_T^{H_1} \cap \tilde{\mathcal{V}}_T^{H_2} \ldots \cap \tilde{\mathcal{V}}_T^{H_m}.$$

Denoting $U^\varepsilon = \nabla\varphi_\varepsilon(Y^\varepsilon)$ from Proposition 5.2 a) we obtain

$$E \sum_{k=1}^{m} \int_0^T s^{2H_k-1} |U_s^\varepsilon|^2 ds \le C.$$

Hence there exist a subsequence $\varepsilon_n \searrow 0$ and process U such that

$$U^{\varepsilon_n} \to U \text{ weakly in } \tilde{\mathcal{V}}_T^{H_1} \cap \tilde{\mathcal{V}}_T^{H_2} \ldots \cap \tilde{\mathcal{V}}_T^{H_m}$$

and from the Fatou Lemma

$$E \sum_{k=1}^{m} \int_0^T s^{2H_k-1} |U_s|^2 ds \le C.$$

Passing now with ε to 0 in (5.5) we obtain (3.2).

Moreover, since $U_t^\varepsilon \in \partial\varphi(J_\varepsilon(Y_t^\varepsilon))$, for all $u \in \tilde{\mathcal{V}}_T^{H_1} \cap \tilde{\mathcal{V}}_T^{H_2} \ldots \cap \tilde{\mathcal{V}}_T^{H_m}$ we have

$$U_t^\varepsilon \cdot (u_t - J_\varepsilon(Y_t^\varepsilon)) + \varphi(J_\varepsilon(Y_t^\varepsilon)) \le \varphi(u_t).$$

Therefore we can deduce (passing to limes infimum) that

$$U_t \cdot (u_t - Y_t) + \varphi(Y_t) \le \varphi(u_t),$$

which means that $(Y_t, U_t) \in \partial\varphi$, $t \in [0, T]$. This completes the proof. □

REFERENCES

[1] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Ed. Academiei Române and Noordhoff International Publishing, 1976.

[2] C. Bender, *Explicit solutions of a class of linear fractional BSDEs*, Systems Control Lett. **54** (2005) 7, 671–680.

[3] F. Biagini, Y. Hu, B. Øksendal, A. Sulem, *A stochastic maximum principle for processes driven by fractional Brownian motion*, Stochastic Process. Appl. **100** (2002) 1, 233–253.

[4] D. Borkowski, K. Jańczak-Borkowska, *Generalized backward stochastic variational inequalities driven by a fractional Brownian motion*, Braz. J. Probab. Stat. **30** (2016) 3, 502–519.

[5] D. Borkowski, K. Jańczak-Borkowska, *BSDE driven by a multidimensional fractional Brownian motion*, submitted.

[6] H. Brézis, *Opérateurs maximaux monotones et semigroupes de contractions dans les spaces de Hilbert*, North-Holland Publ. Co., 1973.

[7] W. Dai, C.C. Heyde, *Itô formula with respect to fractional Brownian motion and its application*, J. Appl. Math. Stoch. Anal. **9** (1990), 439–448.

[8] L. Decreusefond, A.S. Üstünel, *Stochastic analysis of the fractional Brownian motion*, Potential Anal. **10** (1998), 177–214.

[9] T.E. Duncan, Y. Hu, B. Pasik-Duncan, *Stochastic calculus for fractional Brownian motions. I. Theory*, SIAM J. Control Optim. **38** (2000), 582–612.

[10] Y. Hu, *Integral transformations and anticipative calculus for fractional Brownian motions*, Mem. Amer. Math. Soc. **175** (2005) 825.

[11] Y. Hu, B. Øksendal, *Fractional white noise calculus and application to finance*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **6** (2003), 1–32.

[12] Y. Hu, S. Peng, *Backward stochastic differential equation driven by fractional Brownian motion*, Siam J. Control Optim. **48** (2009) 3, 1675–1700.

[13] K. Jańczak-Borkowska, *Generalized BSDEs driven by fractional Brownian motion*, Statist. Probab. Lett. **83** (2013) 3, 805–811.

[14] S.J. Lin, *Stochastic analysis of fractional Brownian motions*, Stochastics Stochastics Rep. **55** (1995), 121–140.

[15] L. Maticiuc, T. Nie, *Fractional backward stochastic differential equations and fractional backward variational inequalities*, J. Theoret. Probab. **28** (2015) 1, 337–395.

[16] L. Maticiuc, A. Răşcanu, *A stochastic approach to a multivalued Dirichlet–Neumann problem*, Stochastic Process. Appl. **120** (2010) 6, 777–800.

[17] J. Miao, X. Yang, *Solutions to BSDEs driven by multidimensional fractional Brownian motions*, Math. Probl. Eng. **2015** (2015), Article ID 481 842.

[18] D. Nualart, *The Malliavin Calculus and Related Topics*, 2nd ed., Springer, Berlin, 2010.

[19] É. Pardoux, S. Peng, *Adapted solutions of a backward stochastic differential equation*, Systems Control Lett. **14** (1990), 55–61.

[20] É. Pardoux, A. Răşcanu, *Stochastic differential equations, Backward SDEs, Partial differential equations*, Springer International Publishing, 2014.

[21] L.C. Young, *An inequality of the Hölder type connected with Stieltjes integration*, Acta Math. **67** (1936), 251–282.

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