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NEW SOLITARY WAVES FOR THIN-FILM FERROELECTRIC MATERIAL EQUATION ARISING IN DIELECTRIC MATERIALS

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Abstract: In this paper, the thin-film ferroelectric material equation (TFFME), which enables the propagation of solitary polarisation in thin-film ferroelectric materials is investigated, will be expressed through the non-linear evolution models. Ferroelectrics are dielectric materials that explain wave propagation non-linear demeanors. The non-linear wave propagation form is administrated by TFFME. To investigate the characterisations of new waves and solitonic properties of the TFFME, the modified exponential Jacobi technique and rational $\exp(-\phi(\eta))$-expansion technique are used. Plenty of alternative responses may be achieved by employing individual formulas; each of these solutions is offered by some plain graphs. The validity of such schemes and solutions may be exhibited by assessing how well the relevant schemes and solutions match up. The effect of the free variables on the manner of acting of reached plots to a few solutions in the exact forms was also explored depending upon the nature of non-linearities. The descriptive characteristics of the reached results are presented and analysed by some density, two- and three-dimensional figures. We believe that our results would pave the way for future research generating optical memories based on non-linear solitons.

Key words: thin-film ferroelectric material equation, ferroelectrics are dielectric materials, modified exponential Jacobi method, rational $\exp(-\phi(\eta))$-expansion method, soliton solution

1. INTRODUCTION

Ferroelectric thin films have become a potential nominee in the field of ultraviolet photodetectors detection due to their wide bandgap and unique photovoltaic aspects. Additionally, ferroelectric thin films perform excellent dielectric, piezoelectric, pyroelectric, acousto-optic effects, etc. [1]. Xiao et al. [2] showed that the growth of ferroelectric layer on the original perovskite grains can reduce the formation of grain boundaries and hence minimise the recombination of electrons and holes at grain boundaries. The solitary wave dynamics of the thin-film ferroelectric material equation (TFFME) were investigated by Xiao et al. [2]. Umoh et al. [3] used nanoelectronic devices based on oxide films to require materials for exhibiting combined properties such as ferroelectricity, ferromagnetism and ferroelasticity at the same phase. An analysis of the thermal and ferroelectric properties of using a thin film in a transverse field extended for a higher spin within the quantum Monte Carlo method was provided by Tarnaoui et al. [4]. Ferroelectric thin films have demonstrated great potential in electrocaloric solid-state refrigeration on account of large adiabatic temperature changes [3]. Yang et al. [5] demonstrated a ferroelectric tunnel junction, whose conductivity varies linearly and symmetrically by judiciously combining ferroelectric domain switching and oxygen vacancy migration. The order parameter of ferrotoroidic order has been generated by a head-to-tail configuration of magnetic moment and has been theoretically proposed that one-dimensional dimerised and antiferromagnetic (AFM)-like spin chain hosts ferrotoroidicity [6]. Through the Landau–Ginzburg–Devonshire equation, the governing behaviour of the polarisation field in ferroelectric material was derived, and ferroelectric material is subjected to a standing electric field [7]. The modified simple equation method and the Riccati-Bernoulli Sub-ODE method were utilised for TFFME, which plays a vital role in optics to waves propagate through ferroelectric materials [8]. The optical soliton solutions of the thin-film ferroelectric materials equation through the Paul–Painlevé approach have been obtained [9]. In this paper, we are concerned with a wave polarisation for ferroelectric materials to the TFFME in one-dimensional form as follows:

$$\frac{m}{T^2}\frac{\partial^2 u}{\partial t^2}-[(p_2-2\mu)u+p_4u^3+p_6u^5]-J\Delta u=0, \tag{1}$$

where mass and charge density are m and $\mathrm{T}, \mathrm{p}_2, \mathrm{p}_4, \mathrm{p}_6$ showing temperature and pressure. Also, J is related to the space inhomo-

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geneity coefficient and μ is the reciprocal of the electric susceptibility [7]. In the last decades, researchers have developed numerous methods. Gu et al. [10] investigated the generalised (2 + 1)-dimensional shallow water wave equation, which enables a unidirectional propagation of shallow water waves by using the bilinear method and semi-inverse variational principle scheme. Some novel analytical solutions, including 2-lump-type, the interaction between 2-lump and one kink, and other forms were obtained for the (3 + 1)-dimensional Burger system [11]. The modified auxiliary equation approach and the generalised projective Riccati equation method were used for the first time to solve the Zoomeron equation [12]. In paper [13], experts applied the bilinear method on models arising in variable coefficient Caudrey–Dodd–Gibbon–Kotera–Sawada equation. In paper [14], the third-order evolution equation was investigated by the Hirota bilinear method, which arises the propagation of long waves over shallow water. Rao et al. [15] studied the extended homoclinic breather wave solutions to the non-linear vibration and dispersive wave systems. The solutions of (3 + 1)-dimensional Burgers system via Lie symmetry analysis have been investigated by Alimirzaluo et al. [16]. In paper [17], Xiao at al. studied the inverse scattering transform for the coupled modified Korteweg–de Vries equation with non-zero boundary conditions. The cubic B-splines and linear triangular elements were used for a test problem including the motion of a single solitary solution of Benjamin–Bona–Mahony (BBM) equation coupled BBM-system by finite element method [18]. M-soliton and N-soliton solutions have been discovered for variable coefficient-generalised non-linear wave equation arising in liquid with gas bubbles [19]. Hirota's bilinear operator has been used for the generalised Hietarinta equation [20] and to study the variable coefficients of generalised shallow water wave equation [21]. The exact solutions to the generalised non-linear Schrodinger equation by means of the extended sinh–Gordon equation expansion method, $\tan(\Gamma(\varpi))$-expansion method and the improved $\cos(\Gamma(\varpi))$ function method were obtained [22]. In paper [23], Singh and co-workers focussed on describing the evolution of water waves with higher order temporal dispersion by characterising the dynamics of lump and soliton waves on different spatially varying backgrounds to an integrable (3 + 1)-dimensional non-linear model. Non-linear partial differential equations arise in many different branches of social and basic sciences and engineering. They have gained prominence in recent years due to their crucial role in a variety of domains involving complicated physical processes, from control theory and electrical circuits to wave propagation. Especially, they appear in electrodynamics of complex medium, electrical networks, signal and image processing, electrodynamics, including porous flow, surface water flow, land sliding, faulting, circled fuel reactor, wave motion and distribution, transmission lines. In order to compute these solutions and better understand the fundamental characteristics of physical structures in varied contexts, several authors have employed a variety of techniques. As a result, the analytical methods have been developed and it has been shown that no single technique can be used to solve all types of non-linear problems with precision. Therefore, many different methods have emerged, some of which are sub-equation methods. High-dimensional partial differential equations have attracted academics' curiosity greatly in recent years. They also appear in modelling many phenomena in biology, chemistry, physics, engineering, mechanics, economy and many different branches [24, 25, 26, 27]. Soliton theory is a very efficient and competent way to describe non-linear features. Soliton theory has two basic routes to study and explain non-linear features. Solitons have the most remarkable properties of particles and waves simultaneously that reflect non-linear features in a well-organised and competent way. To study nature by framing non-linear evolution equations, along with their soliton solutions is immediate and unquestionable. Solitons keep their velocities, shapes and amplitudes unchanged even after interacting with others due to their perfectly elastic interaction. In this paper, some solutions including soliton, bright soliton, singular soliton, periodic wave solutions by the modified exponential Jacobi technique and rational $\exp(-\phi(\eta))$-expansion technique were also obtained. These good results show that the auxiliary methods are a powerful mathematical tool to handle non-linear integrable equations from nature. The complex integrable Kuralay governing system was offered to study the new auxiliary equation method for discovering multiple types of solitons [28]. Faridi et al. [29] studied the non-linear integrable model, namely the generalised Kadomtsev–Petviashvili modified equal width–Burgers equation, which utilised a weakly non-linear restoring forces, dispersion, small damping and non-linear media with dissipation to narrate the long wave propagation in chemical theory. Faridi et al. [30] analysed the dimensional elliptic non-linear Schrödinger equation under the influence of three different fractional operators and found the generalised fractional soliton solutions and propagation of magnetohydrodynamics fluid in sort of solition. The propagation of optical pulses in optical fibres and plasma has been examined for the Chen–Lee–Liu dynamical equation using the extended direct algebraic technique [31]. The combo of $\frac{G'}{G^2}$-expansion method and new extended direct algebraic method were used to find soliton solutions like periodic patterns with anti-peaked crests and anti-troughs, singular solution and mixed complex solitary shock solution to the fractional TFFME [32]. The solitonic patterns of the considered model were successfully surveyed by using two integrated analytical techniques, new extended direct algebraic and expansion method, to investigate the system of cold bosonic atoms in zig-zag optics lattices [33]. Chen and Li [34] constructed the optical soliton solutions of the well-known TFFME, which describes the propagation of polarisation in thin-film materials. The bifurcation, phase portrait and travelling wave solution of time-fractional TFFME with beta fractional derivative were studied [35]. The Zoomeron model was applied to various types of solitons arising in fluid mechanics, laser optics and non-linear physics [36]. New optical soliton solutions for the coupled conformable Fokas–Lenells equation with spatio-temporal dispersion were obtained via Atangana's derivative operator [37]. The resonant non-linear Schrödinger equation with Kerr law non-linearity considering inter-modal dispersion and spatio-temporal was investigated using the new extended direct algebraic method [38]. The variational iterative method with the Laplace transform was used to solve non-linear evolution problems of a simple pendulum and mass-spring oscillator, which represents the Duffing equation [39]. The natural decomposition method and Laplace decomposition method were studied to solve the second-order Painlevé equation [40]. An optimal Galerkin-homotopy asymptotic method was applied to the non-linear second-order boundary value problems (BVPs) [41]. An analytical analysis to solve the fractional differential equations has been offered by Manafian and Allahverdiyeva [42]. Different forms of optical soliton solutions to the Kudryashov's quintuple self-phase modulation were obtained by Li et al. [43]. New soliton solutions to the Van der Waals model through the improved $\exp(-\Omega)$-expansion method (IEFM) and extended sinh-Gordon equation expansion method (EShGEEM) were attained [44]. Researchers worked on the network governance step-by-step method [46] and the neural network method [47]. Based on the invariant subspace method, the Lie symmetries including Riemann–Liouville and Erdelyi–Kober fractional derivatives of the time-fractional form of the Gardner equation have been studied [48]. Numerical analysis of bioconvective heat and mass transfer across a non-linear stretching sheet with hybrid nanofluids was investigated by Aneja et al. [49]. Inspired by the previous work, the motivation of the paper is to investigate the solitons and other forms of solutions by the modified exponential Jacobi technique and rational $\exp(-\phi(\eta))$-expansion technique. The outline of the paper is as follows. In Section 2, we transform the thin-film ferroelectric material (TFFM) equation to a non-linear equation of the non-linear ordinary differential equation. Furthermore, in Sections 3–6, different forms of solitary wave solutions have been established by the modified exponential Jacobi method (MEJM) and rational $\exp(-\phi(\eta))$-expansion method. Finally, the conclusions are provided in Section 7.

2. TRANSFORMING PDE TO ODE

For Eq. (1), x, t show the longitudinal and transverse coordinates. Using the next wave transformation, $u(x,t) = u(\eta), \eta = x - \lambda t$, where λ is arbitrary constants to be determined through the method's steps, leads to the ODE as follows:

$$\frac{\partial^2 u}{\partial t^2} = \lambda^2 \frac{d^2 u}{d\eta^2}, \quad J\Delta u = J\frac{\partial^2 u}{\partial x^2} = J\frac{d^2 u}{d\eta^2}, \quad 4cm \tag{2}$$

By substituting $\frac{d^2u}{d\eta^2} = u''$ and simplifying Eq. (2), we get,

$$\left(\frac{m\lambda^2}{T^2} - J\right)u'' - [(p_2 - 2\mu)u + p_4 u^3 + p_6 u^5 = 0, \tag{3}$$

where wave speed is denoted by λ. By using the balance principle to the terms of Eq. (4) between ψ'' and ψ^5 we get $k + 2 = 5k$, which leads to $k = 1/2$. By utilising the transformation $u(\eta) = \sqrt{\psi(\eta)}$ in Eq. (3) once, with respect to η and zero-integration constant, leads to

$$\left(\frac{m\lambda^2}{T^2} - J\right)\left(-\frac{1}{4}\psi'^2 + \frac{1}{2}\psi\psi''\right) - (p_2 - 2\mu)\psi^2 - p_4\psi^3 - p_6\psi^4 = 0. \tag{4}$$

Deponing up the balance principle to the terms of Eq. (4) between ψ'' and ψ^4, we get $k = 1$.

3. THE MEJM

This part introduces the general properties of a new MEJM, which has been proposed by Aldhabani et al. [45]. The necessary steps for using this method are summarised as follows. Handling the inquired into model through the MEJM gets the following steps as mentioned earlier:

Step 1.

$$\mathcal{S}_1(F, F_x, F_t, F_{xx}, F_{tt}, \dots) = 0, \tag{5}$$

where $\mathcal{S}$ is a polynomial of F and its partial derivatives.

Step 2. Firstly, by utilising travelling wave transformation,

$$\xi = x - \lambda t, \tag{6}$$

where λ is the non-zero arbitrary value, allows to diminish Eq. (61) to an ODE of $F = F(\xi)$ in the below form

$$\mathcal{S}_2(F, F', -\lambda F', F'', \lambda^2 F'', \dots) = 0. \tag{7}$$

Step 3. The generated solutions of Eq. (61) are:

$$F(\xi) = A_0 + \sum_{i=1}^{a} A_i \left(\frac{\Omega'(\xi)}{\Omega(\xi)}\right)^i + \sum_{i=1}^{a} B_i \left(\frac{\Omega(\xi)}{\Omega'(\xi)}\right)^i, \tag{8}$$

where $\Omega'(\xi) = d\Omega(\xi)/d\xi$ and

$$\Omega(\xi) = \frac{\sigma_1 cn(\xi,k) + \sigma_2 sn(\xi,k)}{\sigma_3 cn(\xi,k) + \sigma_4 sn(\xi,k)}, \quad 5.9\ cm \tag{9}$$

where $cn(\xi, k)$ and $sn(\xi, k)$ are the Jacobi elliptic functions of index, k, and we obtain a few interesting and important relationships as follows:

$$sn^2(\xi,k) = 1 - cn^2(\xi,k), \tag{10}$$

$$dn^2(\xi,k) = 1 - sn^2(\xi,k),$$

$$\frac{d}{d\xi} cn(\xi,k) = -sn(\xi,k)dn(\xi,k),$$

$$\frac{d^2}{d\xi^2} cn(\xi,k) = -cn(\xi,k)dn^2(\xi,k) + kcn(\xi,k)sn^2(\xi,k),$$

$$\frac{d^3}{d\xi^3} cn(\xi,k) = -k^2 dn(\xi,k)sn^3(\xi,k) + 4k^2 cn^2(\xi,k)sn(\xi,k)dn(\xi,k) + sn(\xi,k)dn^3(\xi,k),$$

$$sn(\xi,0) = \sin(\xi), \quad cn(\xi,0) = \cos(\xi), \quad dn(\xi,0) = 1,$$

$$sn(\xi,1) = \tanh(\xi), \quad cn(\xi,1) = sech(\xi), \quad dn(\xi,1) = sech(\xi).$$

By utilising the balance tenet on Eq. (63), we can discover the value of a.

Step 4. Substituting Eq. (9) in Eq. (63) and collecting the coefficients of disparate orders in terms of $cn(\xi, k)$, $sn(\xi, k)$ and $dn(\xi, k)$ make a set of non-linear algebraic equations.

Step 5. In the next step, we solve the non-linear algebraic equations and get the needed results.

4. APPLICATION OF MEJM

It can be seen that the above governing differential equation is highly non-linear, and such non-linearity imposes some difficulties in the development of exact analytical techniques to generate closed-form solutions for the equation. Therefore, a modified exponential Jacobi scheme is used in this work. The MEJM, which is an analytical scheme for providing analytical solutions to non-linear ordinary differential equations, is adopted. Upon constructing the transformation and a new function, the following categories of solutions can be expressed:

The set of categories of solutions:

4.1. Set I

$$J = -\frac{4T^2 {A_1}^2 p_6 - 3\lambda^2 m}{3T^2}, \quad A_0 = \left(-1 + \sqrt{-k^2 + 1}\right)A_1,$$

$$p_2 = -\frac{4}{3} {A_1}^2 p_6\left(k^2 + 3\sqrt{1 - k^2} - 2\right) + 2\mu, \tag{11}$$

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$A_1 = A_1, \quad B_1 = 0, \; p_4 = -\frac{8}{3}\left(-1+\sqrt{-k^2+1}\right)A_1 p_6,$
$\sigma_2 = \sigma_3 = 0,$

$$\psi_1 = \frac{A_1\left(\left(-1+\sqrt{-k^2+1}\right)\left(sn(\xi,k)\right)^2 - dn(\xi,k)\right)}{sn(\xi,k)cn(\xi,k)},$$

$$A_1 = \frac{9\,p_4+\sqrt{-192\,k^2p_2p_6+384\,k^2p_6-192\,p_2p_6+81\,p_4{}^2+384\,p_6}}{16p_6(k^2+1)}.$$

As a result, the exact soliton solution is given by

$$u_1(x,t) = \left\{\frac{\frac{9\,p_4+\sqrt{-192\,k^2p_2p_6+384\,k^2p_6-192\,p_2p_6+81\,p_4{}^2+384\,p_6}}{16p_6(k^2+1)}\left(\left(-1+\sqrt{-k^2+1}\right)\left(sn(\xi,k)\right)^2-dn(\xi,k)\right)}{sn(\xi,k)cn(\xi,k)}\right\}^{\frac{1}{2}}, \quad (12)$$

$$\xi = x - \frac{\sqrt{3}T}{3m}\sqrt{m\left(\frac{\left(9\,p_4+\sqrt{-192\,k^2p_2p_6+384\,k^2p_6-192\,p_2p_6+81\,p_4{}^2+384\,p_6}\right)^2}{64\,p_6(k^2+1)^2}+3\,J\right)}\;t.$$

Some subgroups for relation (12):
Supposing $k = 0$ in Eq. (12) provides

$$u_2(x,t) = \left\{\frac{-\frac{9\,p_4+\sqrt{-192\,p_2p_6+81\,p_4{}^2+384\,p_6}}{16p_6}}{\sin(\xi)\cos(\xi)}\right\}^{\frac{1}{2}}, \quad (13)$$

$$\xi = x - \frac{\sqrt{3}T}{3m}\sqrt{m\left(\frac{\left(9\,p_4+\sqrt{-192\,p_2p_6+81\,p_4{}^2+384\,p_6}\right)^2}{64\,p_6}+3\,J\right)}\;t.$$

Supposing $k = 1$ in Eq. (12) yields

$$u_3(x,t) = \left\{\frac{\frac{9\,p_4+\sqrt{-192p_2p_6+384p_6-192\,p_2p_6+81\,p_4{}^2+384\,p_6}}{32p_6}\left(-(\tanh(\xi))^2-sceh(\xi)\right)}{\tanh(\xi)sech(\xi)}\right\}^{\frac{1}{2}}, \quad (14)$$

$$\xi = x - \frac{\sqrt{3}T}{3m}\sqrt{m\left(\frac{\left(9\,p_4+\sqrt{-384\,p_2p_6+81\,p_4{}^2+768\,p_6}\right)^2}{256\,p_6}+3\,J\right)}\;t.$$

The effect of analysis periodic solution when plots of u are given as in Fig. 1 with the following amounts:

$$p_2 = 1, p_4 = 2, p_6 = 3, m = 2, T = 3,$$
$$\mu = 2, J = 2, k = 0, \quad (15)$$

$$u = \sqrt{\frac{1}{\sin(3\sqrt{3}t-x)\cos(3\sqrt{3}t-x)}}. \quad (16)$$

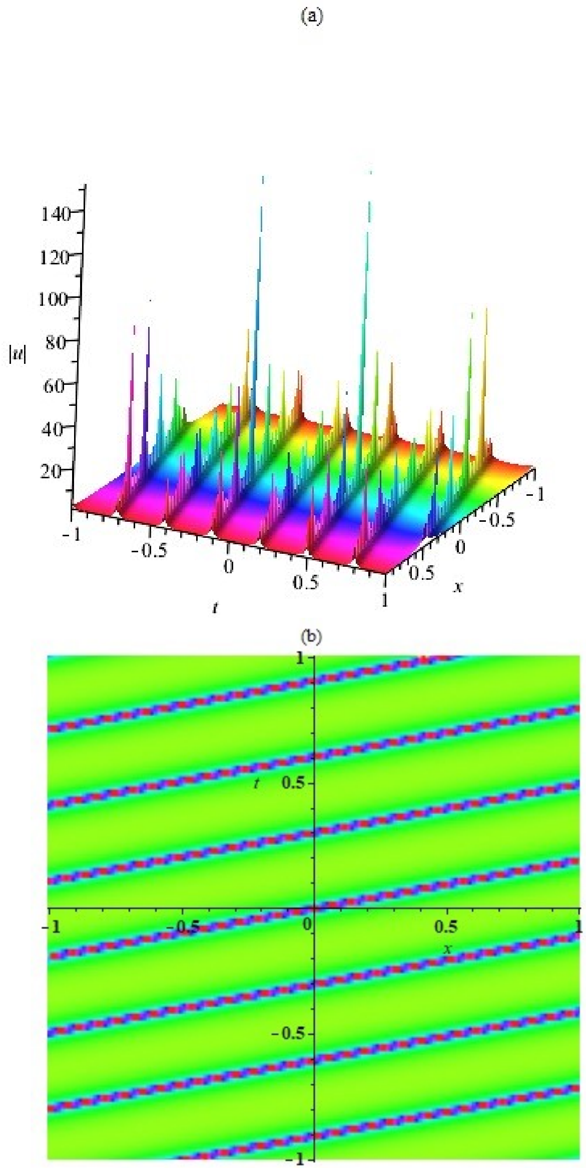
for Eq. (16). We investigate the behaviour of general periodic and the periodic received from the mentioned technique, which is presented in Fig. 1. From the graph, it is ostensible that the periodic structure exhibits a stable propagation for the generalised non-local non-linearity as offered in Fig. 1. Also, the effect of analysis of the periodic solution when plots of u are given in Fig. 2 with the following amounts

$$p_2 = 1, p_4 = 2, p_6 = 3, m = 2,$$
$$T = 3, \mu = 2, J = 2, k = 1, \quad (17)$$

$$u = 1/24\sqrt{-\frac{(108+36\sqrt{41})(\tanh(\zeta)sech\zeta - sech(\zeta))}{\tanh(\zeta)sech(\zeta)}},$$

$$\zeta = \frac{\sqrt{3}\sqrt{6\,(18+6\,\sqrt{41})^2+27648t}}{96} - x. \quad (18)$$

for Eq. (18).



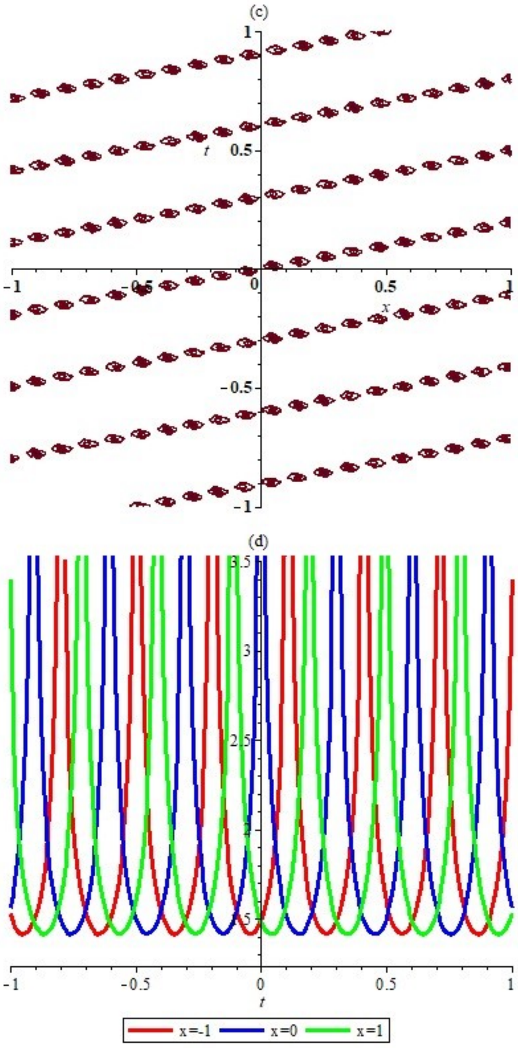
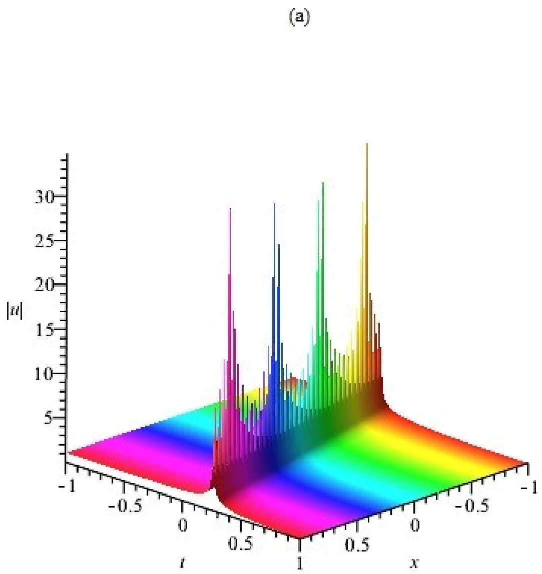


Fig. 1. Plots of real solution (16) (a [3D plot], b [density plot], c [contour plot], d [2D plot]) for Graph u_1



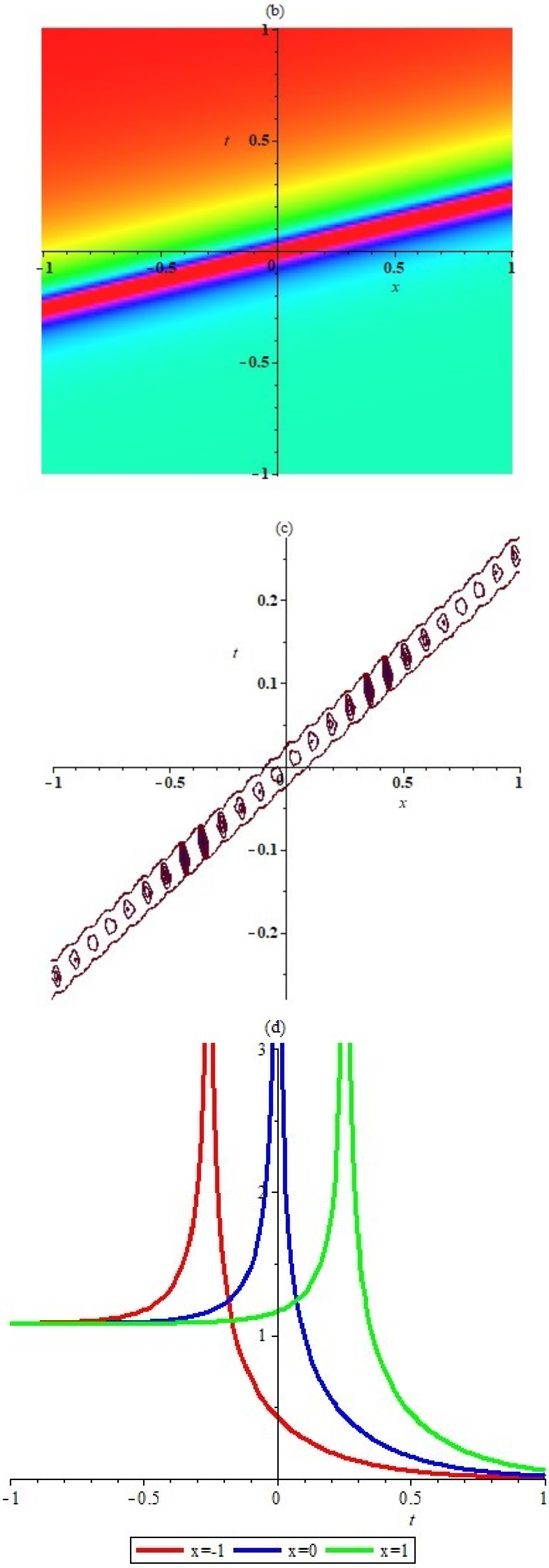


Fig. 2. Plots of real solution (18) (a [3D plot], b [density plot], c [contour plot], d [2D plot]) for Graph u_2

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4.2. Set II

$$J=-1/3\,\frac{-3\,k^4\lambda^2 m+4\,T^2{B_1}^2p_6}{T^2k^4},$$
$$A_0=\frac{\left(-1+\sqrt{-k^2+1}\right)B_1}{k^2},\quad A_1=0,\qquad(19)$$
$$B_1=B_1,\; p_4=-8/3\,\frac{\left(-1+\sqrt{-k^2+1}\right)B_1p_6}{k^2},\quad \sigma_2=\sigma_3=0,$$
$$p_2=-2/3\,\frac{2\,{B_1}^2p_6\left(k^4-10\,k^2+10\right)-3\,k^6\mu+2\,\sqrt{-k^2+1}\left(-3\,k^4\mu+5\,k^2{B_1}^2p_6-10\,{B_1}^2p_6\right)+6\,k^4\mu}{k^4\left(k^2+2\,\sqrt{-k^2+1}-2\right)},$$
$$\psi_1=\frac{B_1}{dn(\xi,k)}\left(\frac{\left(-1+\sqrt{-k^2+1}\right)dn(\xi,k)}{k^2}-sn(\xi,k)cn(\xi,k)\right).$$

As a result, the exact soliton solution is given by

$$u_1(x,t)=\left\{\frac{B_1}{dn(\xi,k)}\left(\frac{\left(-1+\sqrt{-k^2+1}\right)dn(\xi,k)}{k^2}-sn(\xi,k)cn(\xi,k)\right)\right\}^{\frac{1}{2}},\qquad(20)$$
$$\xi=x-\frac{\sqrt{3}\sqrt{m\left(3\,Jk^4+4\,{B_1}^2p_6\right)T}}{3mk^2}\,t.$$

Some subgroups for relation (20):
Supposing $k=1$ in Eq. (20) provides

$$u_2(x,t)=\left\{B_1\left[-1-\tanh\left(x-\frac{\sqrt{3}\sqrt{m\left(4\,{B_1}^2p_6+3\,J\right)T}}{3m}t\right)\right]\right\}^{\frac{1}{2}}.\qquad(21)$$

4.3. Set III

$$J=-1/3\,\frac{-3\,k^4\lambda^2 m+4\,T^2{B_1}^2p_6}{T^2k^4},$$
$$A_0=\frac{\left(-1+\sqrt{-k^2+1}\right)B_1}{k^2},\quad A_1=0,\qquad(22)$$
$$B_1=B_1,\; p_4=-8/3\,\frac{\left(1+\sqrt{-k^2+1}\right)B_1p_6}{k^2},\quad \sigma_2=\sigma_3=0,$$
$$p_2=2/3\,\frac{-2\left(1+\sqrt{-k^2+1}\right)\left(-3\,k^4\mu+5\,k^2{B_1}^2p_6-10\,{B_1}^2p_6\right)-\left(3\,k^4\mu-2\,k^2{B_1}^2p_6+10\,{B_1}^2p_6\right)k^2}{k^4\left(-k^2+2\,\sqrt{-k^2+1}+2\right)},$$
$$\psi_1=\frac{B_1}{dn(\xi,k)}\left(\frac{\left(1+\sqrt{-k^2+1}\right)dn(\xi,k)}{k^2}-sn(\xi,k)cn(\xi,k)\right).$$

As a result, the exact soliton solution is given by

$$u_1(x,t)=\left\{\frac{B_1}{dn(\xi,k)}\left(\frac{\left(1+\sqrt{-k^2+1}\right)dn(\xi,k)}{k^2}-sn(\xi,k)cn(\xi,k)\right)\right\}^{\frac{1}{2}},\qquad(23)$$
$$\xi=x-\frac{\sqrt{3}\sqrt{m\left(3\,Jk^4+4\,{B_1}^2p_6\right)T}}{3mk^2}\,t.$$

Some subgroups for relation (23):
Supposing $k=1$ in Eq. (23) provides

$$u_2(x,t)=\left\{\frac{B_1}{sech(x-\lambda t)}\left(sech(x-\lambda t)-\tanh(x-\lambda t)sech(x-\lambda t)\right)\right\}^{\frac{1}{2}}$$
$$\lambda=\frac{\sqrt{3}\sqrt{m\left(3\,J+4\,{B_1}^2p_6\right)T}}{3m}.\qquad(24)$$

4.4. Set IV

$$J=-1/3\,\frac{4\,T^2{A_1}^2p_6-3\,\lambda^2m}{T^2},\; A_0=2\,A_1,\quad B_1=k^2A_1,\qquad(25)$$
$$p_2=-4/3\,{A_1}^2k^2p_6+\frac{20\,{A_1}^2p_6}{3}+2\,\mu,$$
$$p_4=-16/3\,A_1p_6,\;\sigma_2=\sigma_3=0,\quad A_1=-4\,\frac{2\,\mu-p_2}{p_4(k^2-5)},$$
$$\psi_1=-4\,\frac{(2\,\mu-p_2)\left(\left(sn(\xi,k)\right)^4k^2+2\,sn(\xi,k)cn(\xi,k)dn(\xi,k)-1\right)}{p_4(k^2-5)sn(\xi,k)cn(\xi,k)dn(\xi,k)},$$
$$\xi=x-1/3\,\frac{\sqrt{3}T}{m}\sqrt{m\left(3\,\frac{2\,\mu-p_2}{k^2-5}+3\,J\right)}\,t.$$

As a result the exact soliton solution is given by

$$u_1(x,t)=\left\{-4\,\frac{(2\,\mu-p_2)\left(\left(sn(\xi,k)\right)^4k^2+2\,sn(\xi,k)cn(\xi,k)dn(\xi,k)-1\right)}{p_4(k^2-5)sn(\xi,k)cn(\xi,k)dn(\xi,k)}\right\}^{\frac{1}{2}},\qquad(26)$$
$$\xi=x-\frac{\sqrt{3}T}{3m}\sqrt{m\left(3\,\frac{2\,\mu-p_2}{k^2-5}+3\,J\right)}\,t.$$

Some subgroups for relation (26):
Supposing $k=0$ in Eq. (26) provides

$$u_2(x,t)=\left\{4\,\frac{(2\,\mu-p_2)(2\sin(x-\lambda t)\cos(x-\lambda t)-1)}{5p_4\sin(x-\lambda t)\cos(x-\lambda t)}\right\}^{\frac{1}{2}},$$
$$\lambda=\frac{\sqrt{3}T}{3m}\sqrt{m\left(3\,J-3\,\frac{2\,\mu-p_2}{5}\right)}.\qquad(27)$$

Supposing $k=1$ in Eq. (26) supplies

$$u_3(x,t)=\left\{\frac{(2\,\mu-p_2)\left(\left(\tanh(x-\lambda t)\right)^4+2\tanh(x-\lambda t)sech^2(x-\lambda t)-1\right)}{p_4\tanh(x-\lambda t)sech^2(x-\lambda t)}\right\}^{\frac{1}{2}},\qquad(28)$$
$$\lambda=\frac{\sqrt{3}T}{3m}\sqrt{m\left(3\,J-3\,\frac{2\,\mu-p_2}{4}\right)}.$$

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The effect of analysis periodic solution when plots of u are given in Fig. 3 with the following amounts

$$p_2 = 5, p_6 = 1, m = 2, T = 3,$$

$$\mu = 2, J = 2, k = 0, \tag{29}$$

$$u = \frac{\sqrt{10}}{10}\sqrt{2\,\frac{\sqrt{15}\left(-1-\sin\left(3/5\,\sqrt{110}t-x\right)\right)}{\sin\left(3/5\,\sqrt{110}t-x\right)\left(\left(\cos\left(3/10\,\sqrt{110}t-x\right)\right)^2+\left(\sin\left(3/10\,\sqrt{110}t-x\right)\right)^2\right)}}, \tag{30}$$

for Eq. (30). We investigate the behaviour of general periodic and periodic received from the mentioned technique, which is presented in Fig. 3. From the graph, it is ostensible that the periodic structure exhibits a stable propagation for the generalised non-local non-linearity as offered in Fig. 3. Also, the effect of analysis periodic solution when plots of u are given in Fig. 4 with the following amounts

$$p_2 = 5, p_6 = 1, m = 2, T = 3, \mu = 2, J = 2, k = 1, \tag{31}$$

$$u = 1/2\sqrt{\frac{\sqrt{3}\left(\left(\tanh\left(9/4\,\sqrt{2}t-x\right)\right)^4-2\tanh\left(9/4\,\sqrt{2}t-x\right)\left(sech\left(9/4\,\sqrt{2}t-x\right)\right)^2-1\right)}{\tanh\left(9/4\,\sqrt{2}t-x\right)\left(sech\left(9/4\,\sqrt{2}t-x\right)\right)^2}}, \tag{32}$$

for Eq. (32).

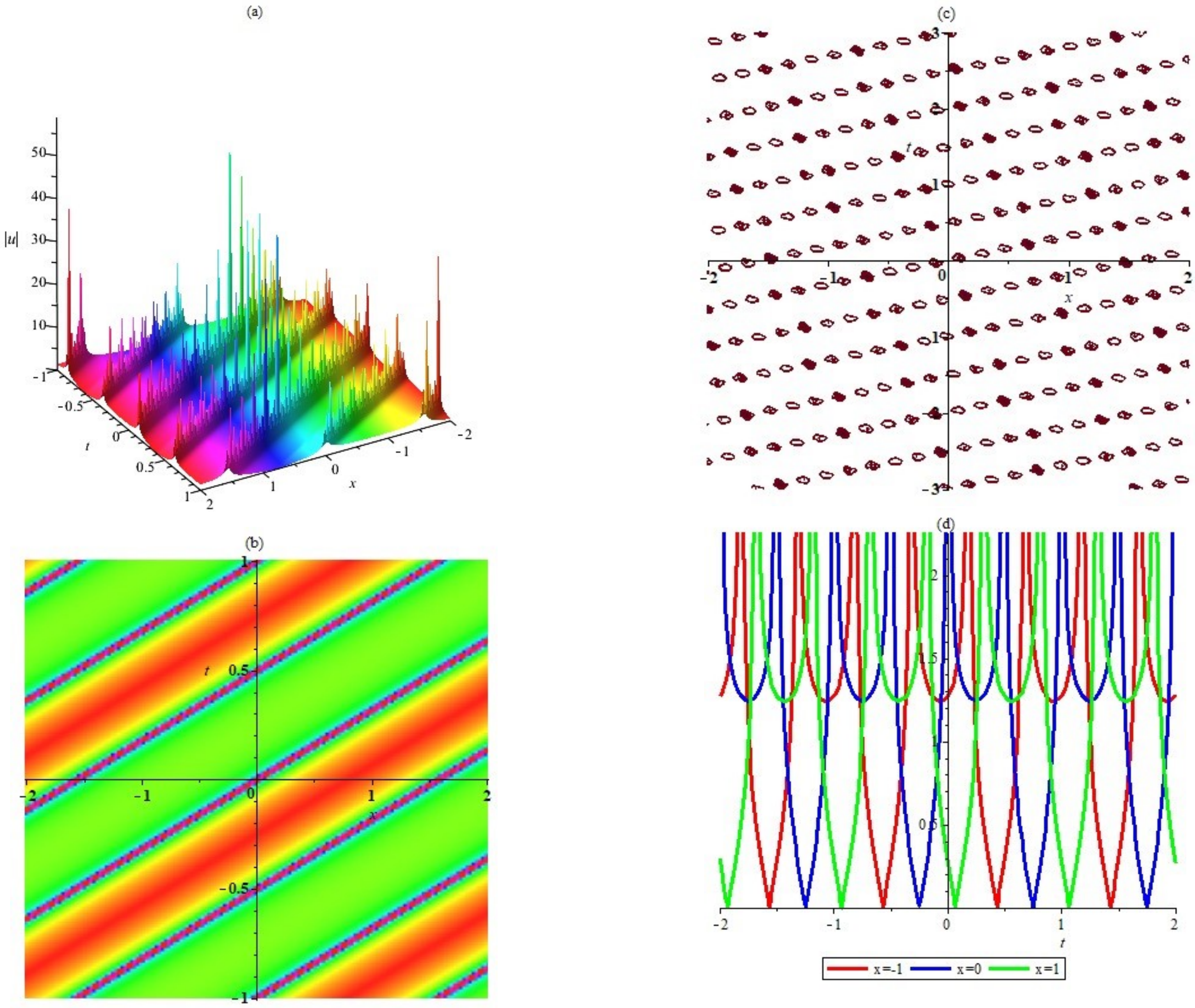

Fig. 3. Plots of real solution (30) (a [3D plot], b [density plot], c [contour plot], d [2D plot]) for Graph u_1.

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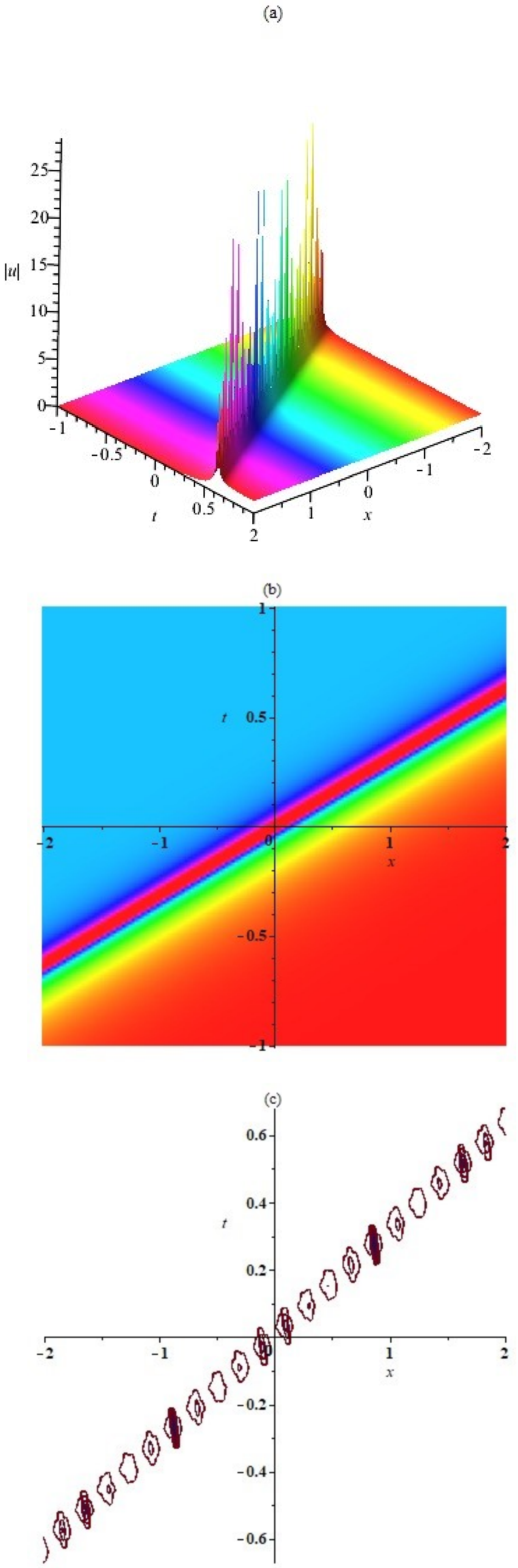


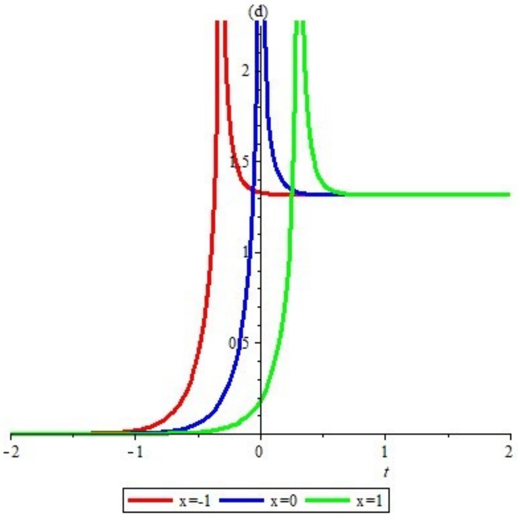


Fig. 4. Plots of real solution (32) (a [3D plot], b [density plot], c [contour plot], d [2D plot]) for Graph u_2

4.5. Set V

$$J=-\frac{1}{3}\frac{4T^2{A_1}^2p_6-3\lambda^2m}{T^2},\;\; A_0=-2\,A_1,\;\; B_1=k^2A_1, \tag{33}$$

$$p_2=-4/3\,{A_1}^2k^2p_6+\frac{20\,{A_1}^2p_6}{3}+2\,\mu,\;\; p_4=16/3\,A_1p_6,$$

$$\sigma_2=\sigma_3=0,\quad A_1=4\,\frac{2\,\mu-p_2}{p_4(k^2-5)},$$

$$\psi_1=4\,\frac{(2\,\mu-p_2)\left(-(sn(\xi,k))^4k^2+1+2\,sn(\xi,k)cn(\xi,k)dn(\xi,k)\right)}{p_4(k^2-5)sn(\xi,k)cn(\xi,k)dn(\xi,k)}.$$

As a result, the exact soliton solution is given by

$$u_1(x,t)=\left\{4\,\frac{(2\,\mu-p_2)\left(-(sn(\xi,k))^4k^2+1+2\,sn(\xi,k)cn(\xi,k)dn(\xi,k)\right)}{p_4(k^2-5)sn(\xi,k)cn(\xi,k)dn(\xi,k)}\right\}^{\frac{1}{2}}. \tag{34}$$

Some subgroups for relation (34):

Supposing $k=0$ in Eq. (34) yields

$$u_2(x,t)=\left\{-4\,\frac{(2\,\mu-p_2)(2\sin(x-\lambda t)\cos(x-\lambda t)+1)}{5p_4\sin(x-\lambda t)\cos(x-\lambda t)}\right\}^{\frac{1}{2}}. \tag{35}$$

Supposing $k=1$ in Eq. (34) yields

$$u_3(x,t)=\left\{-\frac{(2\,\mu-p_2)\left(-(\tanh(x-\lambda t))^4+2\tanh(x-\lambda t)sech^2(x-\lambda t)+1\right)}{p_4\tanh(x-\lambda t)sech^2(x-\lambda t)}\right\}^{\frac{1}{2}}. \tag{36}$$

4.6. Set VI

$$J=-1/3\,\frac{4T^2{A_1}^2p_6-3\lambda^2m}{T^2},\;\; A_0=2k\,A_1,\;\; B_1=k^2A_1, \tag{37}$$

$$p_2=4/3\,{A_1}^2p_6(5\,k^2-1)+2\,\mu,\;\; p_4=-16/3k\,A_1p_6,$$

$$\sigma_2=\sigma_3=0,\quad A_1=4\,\frac{(2\,\mu-p_2)k}{p_4(k^2-5)},$$

$$\psi_1=4\,\frac{(2\,\mu-p_2)k\left((sn(\xi,k))^4k^2+2\,ksn(\xi,k)cn(\xi,k)dn(\xi,k)-1\right)}{p_4(k^2-5)sn(\xi,k)cn(\xi,k)dn(\xi,k)}.$$

As a result, the exact soliton solution is given by

$$u_1(x,t)=$$
$$\left\{4\,\frac{(2\,\mu-p_2)k\left((sn(\xi,k))^4k^2+2\,ksn(\xi,k)cn(\xi,k)dn(\xi,k)-1\right)}{p_4(k^2-5)sn(\xi,k)cn(\xi,k)dn(\xi,k)}\right\}^{\frac{1}{2}}. \quad (38)$$

Some subgroups for relation (38):
Supposing $k=1$ in Eq. (38) provides

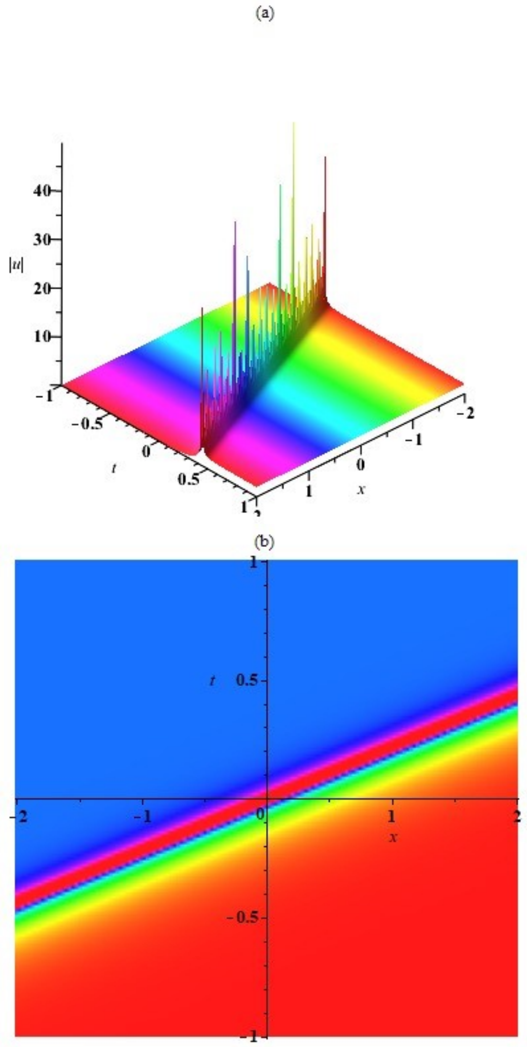
$$u_2(x,t)=$$
$$\left\{-\frac{(2\,\mu-p_2)\left((\tanh(x-\lambda t))^4+2\tanh(x-\lambda t)sech^2(x-\lambda t)-1\right)}{p_4\tanh(x-\lambda t)sech^2(x-\lambda t)}\right\}^{\frac{1}{2}}. \quad (39)$$

The effect of analysis soliton solution when plots of u are given in Fig. 5 with the following amounts

$$p_2=5, p_6=1, m=2, T=3, \mu=2, J=2, k=1, \quad (40)$$

$$u=$$
$$\frac{\left(\tanh(\sqrt{3}\sqrt{7}t-x)\right)^4-2\tanh(\sqrt{3}\sqrt{7}t-x)\left(sech(\sqrt{3}\sqrt{7}t-x)\right)^2-1}{\tanh(\sqrt{3}\sqrt{7}t-x)\left(sech(\sqrt{3}\sqrt{7}t-x)\right)^2\left(\left(sech(\sqrt{3}\sqrt{7}t-x)\right)^2+\left(\tanh(\sqrt{3}\sqrt{7}t-x)\right)^2\right)}, \quad (41)$$

for Eq. (41).



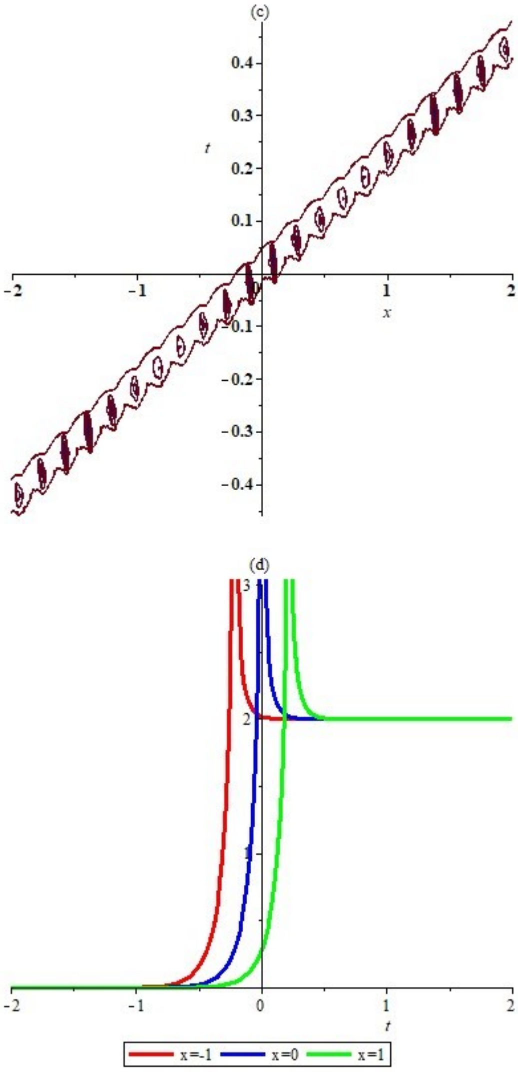


Fig. 5. Plots of real solution (39) (a [3Dplot], b [density plot], c [contour plot], d [2D plot]) for Graph u_2

4.7. Set VII

$$J=-1/3\,\frac{4\,T^2A_1{}^2p_6-3\,\lambda^2m}{T^2},\ A_0=-2k\,A_1,\quad B_1=k^2A_1, \quad (42)$$
$$p_2=4/3\,A_1{}^2p_6(5\,k^2-1)+2\,\mu,\quad p_4=16/3k\,A_1p_6,$$
$$\sigma_2=\sigma_3=0,\quad A_1=-4\,\frac{(2\,\mu-p_2)k}{p_4(k^2-5)},$$
$$\psi_1=-4\,\frac{(2\,\mu-p_2)k\left((sn(\xi,k))^4k^2-2\,ksn(\xi,k)cn(\xi,k)dn(\xi,k)-1\right)}{p_4(k^2-5)sn(\xi,k)cn(\xi,k)dn(\xi,k)}.$$

As a result, the exact soliton solution is given by

$$u_1(x,t)=$$
$$\left\{-4\,\frac{(2\,\mu-p_2)k\left((sn(\xi,k))^4k^2-2\,ksn(\xi,k)cn(\xi,k)dn(\xi,k)-1\right)}{p_4(k^2-5)sn(\xi,k)cn(\xi,k)dn(\xi,k)}\right\}^{\frac{1}{2}}. \quad (43)$$

Some subgroups for relation (43):
Supposing $k=1$ in Eq. (43) supplies

$$u_2(x,t)=$$
$$\left\{\frac{(2\,\mu-p_2)\left((\tanh(x-\lambda t))^4-2\tanh(x-\lambda t)sech^2(x-\lambda t)-1\right)}{p_4\tanh(x-\lambda t)sech^2(x-\lambda t)}\right\}^{\frac{1}{2}}. \quad (44)$$

4.8. Set VIII

$$J=-1/3\ \frac{4\,T^2{A_1}^2p_6-3\,\lambda^2 m}{T^2},\ A_0=2ik\,A_1,$$
$$B_1=-k^2A_1,\ i=\sqrt{-1}, \qquad (45)$$
$$p_2=-4/3\,{A_1}^2p_6(4\,k^2+1)+2\,\mu,$$
$$p_4=16/3ik\,A_1p_6,\quad \sigma_2=\sigma_3=0,\quad A_1=\frac{-4\,ik(2\,\mu-p_2)}{p_4(4\,k^2+1)},$$
$$\psi_1=\frac{-4\,ik(2\,\mu-p_2)\left(-(sn(\xi,k))^4k^2+2\,ik sn(\xi,k)cn(\xi,k)dn(\xi,k)+2\,k^2(sn(\xi,k))^2-1\right)}{p_4(4\,k^2+1)sn(\xi,k)cn(\xi,k)dn(\xi,k)}.$$

As a result, the exact soliton solution is given by

$$u_1(x,t)=\left\{\frac{-4\,ik(2\,\mu-p_2)\left(-(sn(\xi,k))^4k^2+2\,ik sn(\xi,k)cn(\xi,k)dn(\xi,k)+2\,k^2(sn(\xi,k))^2-1\right)}{p_4(4\,k^2+1)sn(\xi,k)cn(\xi,k)dn(\xi,k)}\right\}^{\frac{1}{2}}. \qquad (46)$$

Some subgroups for relation (46):
Supposing $k=1$ in Eq. (46) yields

$$u_2(x,t)=\left\{\frac{-4\,i(2\,\mu-p_2)\left(-(\tanh(x-\lambda t))^4+2\,i\tanh(x-\lambda t)sech^2(x-\lambda t)+2\,(\tanh(x-\lambda t))^2-1\right)}{5p_4\tanh(x-\lambda t)sech^2(x-\lambda t)}\right\}^{\frac{1}{2}}. \qquad (47)$$

4.9. Set IX

$$J=-\frac{1}{3}\frac{4\,T^2{A_1}^2p_6-3\,\lambda^2m}{T^2},$$
$$A_0=2\sqrt{1-k^2}\,A_1,$$
$$B_1=-k^2A_1,\ i=\sqrt{-1}, \qquad (48)$$
$$p_2=-\frac{4}{3}{A_1}^2p_6(4\,k^2-5)+2\,\mu,$$
$$p_4=-\frac{16}{3}\sqrt{1-k^2}A_1p_6,$$
$$\sigma_2=\sigma_3=0,$$
$$A_1=-4\,\frac{\sqrt{-k^2+1}(2\,\mu-p_2)}{p_4(4\,k^2-5)},$$
$$\psi_1=-4\,\frac{\sqrt{1-k^2}(2\,\mu-p_2)\left(-(sn(\xi,k))^4k^2+2\,\sqrt{1-k^2}sn(\xi,k)cn(\xi,k)dn(\xi,k)+2\,k^2(sn(\xi,k))^2-1\right)}{p_4(4\,k^2-5)sn(\xi,k)cn(\xi,k)dn(\xi,k)}.$$

As a result, the exact soliton solution is given by

$$u_1(x,t)=\left\{\frac{4\sqrt{1-k^2}(2\mu-p_2)\left(2\sqrt{1-k^2}sn(\xi,k)cn(\xi,k)dn(\xi,k)-sn^4(\xi,k)k^2+2k^2sn^2(\xi,k)-1\right)}{p_4(5-4\,k^2)sn(\xi,k)cn(\xi,k)dn(\xi,k)}\right\}^{\frac{1}{2}}. \qquad (49)$$

Some subgroups for relation (49):
Supposing $k=0$ in Eq. (49) provides

$$u_2(x,t)=\left\{4\,\frac{(2\,\mu-p_2)(2\sin(x-\lambda t)\cos(x-\lambda t)-1)}{5p_4\sin(x-\lambda t)\cos(x-\lambda t)}\right\}^{\frac{1}{2}}. \qquad (50)$$

4.10. Set X

$$J=-\frac{4\,T^2{A_1}^2p_6-3\,\lambda^2m}{3T^2},\ A_0=\left(-1+\sqrt{1-k^2}\right)A_1$$
$$B_1=0,\ p_2=-\frac{4}{3}{A_1}^2p_6\left(k^2+3\sqrt{1-k^2}-2\right)+2\,\mu, \qquad (51)$$
$$p_4=-8/3\left(-1+\sqrt{-k^2+1}\right)A_1p_6,\quad \sigma_1=\sigma_4=0,$$
$$A_1=-2\,\frac{2\,\mu\sqrt{-k^2+1}-p_2\sqrt{-k^2+1}-2\,\mu+p_2}{p_4\left(k^2+3\sqrt{-k^2+1}-2\right)},$$
$$\psi_1=-2\,\frac{\left(2\,\mu\sqrt{-k^2+1}-p_2\sqrt{-k^2+1}-2\,\mu+p_2\right)\left(\left(-1+\sqrt{-k^2+1}\right)sn(\xi,k)cn(\xi,k)+dn(\xi,k)\right)}{p_4\left(k^2+3\sqrt{-k^2+1}-2\right)cn(\xi,k)sn(\xi,k)}.$$

As a result, the exact soliton solution is given by

$$u_1(x,t)=\left\{-2\,\frac{\left(2\,\mu\sqrt{-k^2+1}-p_2\sqrt{-k^2+1}-2\,\mu+p_2\right)\left(\left(-1+\sqrt{-k^2+1}\right)sn(\xi,k)cn(\xi,k)+dn(\xi,k)\right)}{p_4\left(k^2+3\sqrt{-k^2+1}-2\right)cn(\xi,k)sn(\xi,k)}\right\}^{\frac{1}{2}}. \qquad (52)$$

Some subgroups for relation (52):
Supposing $k=0$ in Eq. (52) yields

$$u_2(x,t)=\left\{-2\,\frac{(2\,\mu-p_2-2\,\mu+p_2)}{p_4\cos(x-\lambda t)\sin(x-\lambda t)}\right\}^{\frac{1}{2}}. \qquad (53)$$

Supposing $k=1$ in Eq. (52) provides

$$u_2(x,t)=\left\{2\ \frac{(-2\ \mu+p_2)\left(-\tanh(x-\lambda t)sech(x-\lambda t)+sech(x-\lambda t)\right)}{p_4 sech(x-\lambda t)\tanh(x-\lambda t)}\right\}^{\frac{1}{2}}. \tag{54}$$

4.11. Set XI

$$J=-1/3\ \frac{-3\ k^4\lambda^2 m+4\ T^2{B_1}^2 p_6}{T^2k^4},\quad A_0=\frac{\left(-1+\sqrt{-k^2+1}\right)B_1}{k^2},\quad A_1=0,\ \sigma_1=\sigma_4=0, \tag{55}$$

$$p_2=-2/3\ \frac{2\ {B_1}^2p_6(k^4-10\ k^2+10)-3\ k^4\mu\ (k^2-2)+2\ \sqrt{-k^2+1}\left(-3\ k^4\mu+5\ k^2{B_1}^2p_6-10\ {B_1}^2p_6\right)}{k^4\left(k^2+2\ \sqrt{-k^2+1}-2\right)},$$

$$p_4=-\frac{8}{3}\frac{\left(-1+\sqrt{1-k^2}\right)B_1p_6}{k^2},\quad B_1=-2\frac{k^2\left(-\sqrt{1-k^2}(k^2-4)(p_2-2\mu)-6k^2\mu+3p_2k^2+8\mu-4\ p_2\right)}{p_4\left(k^4+5(k^2-2)\sqrt{1-k^2}-10k^2+10\right)},$$

$$\psi_1=2\ \frac{(p_2-2\ \mu)\left(\sqrt{-k^2+1}(k^2-4)-3\ k^2+4\right)\left(sn(\xi,k)cn(\xi,k)k^2+dn(\xi,k)\sqrt{-k^2+1}-dn(\xi,k)\right)}{p_4\left(k^4+5\ (k^2-2)\sqrt{-k^2+1}+10\right)dn(\xi,k)}.$$

As a result, the exact soliton solution is given by

$$u_1(x,t)=\left\{\frac{2(p_2-2\ \mu)\left(\sqrt{1-k^2}(k^2-4)-3\ k^2+4\right)\left(sn(\xi,k)cn(\xi,k)k^2+dn(\xi,k)\sqrt{1-k^2}-dn(\xi,k)\right)}{p_4\left(k^4+5\ (k^2-2)\sqrt{1-k^2}+10\right)dn(\xi,k)}\right\}^{\frac{1}{2}}. \tag{56}$$

Some subgroups for relation (56):
Supposing $k=1$ in Eq. (56) yields

$$u_2(x,t)=\left\{2/11\ \frac{(p_2-2\ \mu)\left(\tanh(x-\lambda t)sech(x-\lambda t)-sech(x-\lambda t)\right)}{p_4 sech(x-\lambda t)}\right\}^{\frac{1}{2}}. \tag{57}$$

4.12. Set XII

$$J=-1/3\ \frac{-3\ k^4\lambda^2 m+4\ T^2{B_1}^2 p_6}{T^2k^4},\quad A_0=\frac{\left(1+\sqrt{-k^2+1}\right)B_1}{k^2},\quad A_1=0,\ \sigma_1=\sigma_4=0, \tag{58}$$

$$p_2=-2/3\ \frac{-2\ {B_1}^2p_6(k^4-10\ k^2+10)+3\ k^6\mu+2\ \sqrt{-k^2+1}\left(-3\ k^4\mu+5\ k^2{B_1}^2p_6-10\ {B_1}^2p_6\right)-6\ k^4\mu}{k^4\left(-k^2+2\ \sqrt{-k^2+1}+2\right)},$$

$$p_4=-8/3\ \frac{\left(1+\sqrt{-k^2+1}\right)B_1p_6}{k^2},\quad B_1=2\ \frac{k^2\left(-\sqrt{-k^2+1}(k^2-4)(p_2-2\ \mu)+6\ k^2\mu-3\ p_2k^2-8\ \mu+4\ p_2\right)}{p_4\left(-k^4+5\ (k^2-2)\sqrt{-k^2+1}+10\ k^2-10\right)},$$

$$\psi_1=\frac{2(2\ \mu-p_2)\left(\sqrt{1-k^2}(k^2-4)+3k^2-4\right)\left(sn(\xi,k)cn(\xi,k)k^2+dn(\xi,k)\sqrt{-k^2+1}+dn(\xi,k)\right)}{p_4\left(-k^4+5\sqrt{1-k^2}k^2+10k^2-10\sqrt{1-k^2}-10\right)dn(\xi,k)}.$$

As a result, the exact soliton solution is given by

$$u_1(x,t)=\left\{\frac{2(2\ \mu-p_2)\left(\sqrt{1-k^2}(k^2-4)+3k^2-4\right)\left(sn(\xi,k)cn(\xi,k)k^2+dn(\xi,k)\sqrt{1-k^2}+dn(\xi,k)\right)}{p_4\left(-k^4+5\sqrt{1-k^2}k^2+10k^2-10\sqrt{1-k^2}-10\right)dn(\xi,k)}\right\}^{\frac{1}{2}}. \tag{59}$$

Some subgroups for relation (59):
Supposing $k=1$ in Eq. (59) supplies

$$u_2(x,t)=\left\{-2\ \frac{(p_2-2\ \mu)\left(\tanh(x-\lambda t)sech(x-\lambda t)+sech(x-\lambda t)\right)}{p_4 sech(x-\lambda t)}\right\}^{\frac{1}{2}}. \tag{60}$$

5. THE $\exp(-\phi(\eta))$-EXPANSION METHOD

Handling the investigated model through the rational $\exp(-\phi(\eta))$-expansion method gets the following steps as mentioned earlier:
Step 1.

$$\mathcal{S}_1(\psi,\psi_x,\psi_t,\psi_{xx},\psi_{tt},\dots)=0, \tag{61}$$

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where $\mathcal{S}$ is a polynomial of ψ and its partial derivatives.

Step 2. Firstly, by utilising travelling wave transformation

$$\eta = x - \lambda t + \theta_0, \quad 8.5cm \tag{62}$$

where λ is the non-zero arbitrary value, allows to diminish Eq. (61) to an ODE of $\psi = \psi(\eta)$ in the below form,

$$\mathcal{S}_2(\psi, \kappa\psi', \omega\psi', \kappa^2\psi'', \omega^2\psi'', \dots) = 0. \tag{63}$$

Step 3. The generated solutions of (61) are:

$$\psi(\eta) = \frac{\sum_{i=0}^{N} A_i\left(e^{-\phi(\eta)}\right)^i}{\sum_{i=0}^{M} B_i\left(e^{-\phi(\eta)}\right)^i}, \tag{64}$$

where $A_i, B_i (0 \le j \le N, M)$, are the parameters to be determined $A_N, B_M \neq 0$, and, $\phi = \phi(\eta)$ satisfying the ODE given below

$$\phi' = w_1 e^{\phi} + e^{-\phi} + w_2, \quad \phi' = \frac{d\phi}{d\eta}. \tag{65}$$

The particular solutions of Eq. (65) will be read as:

Solution-1: When $w_1 \neq 0$ and $w_2^2 - 4w_1 > 0$, therefore we attain

$$\phi(\eta) = \ln\left(-\frac{\sqrt{w_2^2-4w_1}}{2w_1}\tanh\left(\frac{\sqrt{w_2^2-4w_1}}{2}(\eta+E)\right) - \frac{w_2}{2w_1}\right).$$

Solution-2: When $w_1 \neq 0$ and $w_2^2 - 4w_1 < 0$, therefore we attain

$$\phi(\eta) = \ln\left(\frac{\sqrt{-w_2^2+4w_1}}{2w_1}\tan\left(\frac{\sqrt{-w_2^2+4w_1}}{2}(\eta+E)\right) - \frac{w_2}{2w_1}\right).$$

Solution-3: When $w_1 = 0$, $w_2 \neq 0$, and $w_2^2 - 4w_1 > 0$, therefore we attain

$$\Phi(\eta) = -\ln\left(\frac{w_2}{\exp(w_2(\eta+E))-1}\right).$$

Solution-4: When $w_1 \neq 0$, $w_2 \neq 0$, and $w_2^2 - 4w_1 = 0$, therefore we attain

$$\phi(\eta) = \ln\left(-\frac{2w_2(\xi+E)+4}{w_2^2(\eta+E)}\right).$$

Solution-5: When $w_1 = 0$, $w_2 = 0$, and $w_2^2 - 4w_1 = 0$, therefore we attain $\phi(\eta) = \ln(\eta + E)$, where $A_j(0 \le j \le N), B_j(0 \le j \le M)$, E, w_2 and w_1 are also the constants to be explored later.

Step 4. Balancing the non-linear ODE can obtain the values M and N.

Step 5. By solving the algebraic equations, we can get to the mentioned values.

6. APPLICATION EFM

In this section, the innovative soliton wave solutions for the model under investigation are constructed through an analytical approach outlined.

The set of categories of solutions with $N = 1, M = 0$:

6.1. Set I

$$\mu = 1/32\,\frac{16\,p_2p_6 - 3\,{p_4}^2}{p_6},\ A_0 = 0,$$

$$A_1 = -\frac{B_0(JT^2-\lambda^2 m)}{T}\sqrt{-3\,(4\,JT^2p_6 - 4\,\lambda^2 mp_6)^{-1}}, \tag{66}$$

$$w_1 = 0, \quad w_2 = \sqrt{-3\,(4\,JT^2p_6 - 4\,\lambda^2 mp_6)^{-1}}\,p_4T.$$

As a result (Group 3), the kink soliton solution is given by

$$u_1(x,t) = \left\{\frac{3(JT^2-\lambda^2 m)\left(4\,JT^2p_6 - 4\,\lambda^2 mp_6\right)^{-1}p_4}{\exp\left(\sqrt{-3\,(4\,JT^2p_6-4\,\lambda^2 mp_6)^{-1}}\,p_4T(x-\lambda t+E)\right)-1}\right\}^{\frac{1}{2}}. \tag{67}$$

6.2. Set II

$$\lambda = \frac{T}{B_0}\sqrt{1/3\,\frac{3\,J{B_0}^2 + 4\,{A_1}^2p_6}{m}},\ \mu = 1/32\,\frac{16\,p_2p_6 - 3\,{p_4}^2}{p_6},$$

$$A_0 = 1/8\,\frac{4\,A_1p_6w_2 - 3\,B_0p_4}{p_6}, \tag{68}$$

$$w_1 = \frac{16\,{A_1}^2{p_6}^2{w_2}^2 - 9\,{B_0}^2{p_4}^2}{64\,{A_1}^2{p_6}^2},\ w_2^2 - 4w_1 = \frac{9\,{B_0}^2{p_4}^2}{16\,{A_1}^2{p_6}^2}.$$

As a result by (Group 1), the soliton solution is concluded by

$$u_1(x,t) = \left\{1/8B_0\,\frac{4\,A_1p_6w_2 - 3\,B_0p_4}{p_6} + \frac{A_1}{B_0}\times\frac{1}{-24\,\frac{B_0p_4A_1p_6}{(4\,A_1p_6w_2-3\,B_0p_4)(4\,A_1p_6w_2+3\,B_0p_4)}\tanh\left(3/8\,\frac{B_0p_4}{A_1p_6}(\eta+E)\right) - 32\,\frac{w_2{p_6}^2{A_1}^2}{(4\,A_1p_6w_2-3\,B_0p_4)(4\,A_1p_6w_2+3\,B_0p_4)}}\right\}^{\frac{1}{2}} \tag{69}$$

when $\eta = x - \frac{T}{B_0}\sqrt{1/3\,\frac{3\,J{B_0}^2+4\,{A_1}^2p_6}{m}}\,t$.

The set of categories of solutions with $N = 2, M = 1$:

6.3. Set I

$$\lambda = \frac{T}{w_2}\sqrt{1/4\,\frac{4\,Jp_6{w_2}^2 + 3\,{p_4}^2}{mp_6}},\ \mu = \frac{16\,p_2p_6 - 3\,{p_4}^2}{32p_6},$$

$$A_0 = A_2 = w_1 = 0, \quad B_0 = \frac{w_2(4\,A_1p_6 + 3\,B_1p_4)}{3p_4}, \tag{70}$$

$$u_1(x,t)=\left\{3\ \frac{A_1 e^{-\phi(\eta)} p_4}{3\ B_1 e^{-\phi(\eta)} p_4+4\ A_1 p_6 w_2+3\ B_1 p_4 w_2}\right\}^{\frac{1}{2}},$$

$$\eta=x-\frac{T}{w_2}\sqrt{1/4\ \frac{4\ J p_6 {w_2}^2+3\ {p_4}^2}{m p_6}}\,t. \tag{71}$$

6.4. Set II

$$\lambda=\frac{T}{B_1}\sqrt{\frac{3\ J{B_1}^2 w_1+4\ {A_1}^2 p_6+3\ A_1 B_1 p_4}{3 m w_1}},\ \mu=\frac{16\ p_2 p_6-3\ {p_4}^2}{32 p_6},\ A_0=\sqrt{\frac{4\ {A_1}^2 p_6 w_1+3\ A_1 B_1 p_4 w_1}{4 p_6}}, \tag{72}$$

$$A_2=B_0=0,\quad w_2=1/2\ \frac{(8\ A_1 p_6+3\ B_1 p_4) w_1}{p_6}\frac{1}{\sqrt{\frac{A_1 w_1 (4\ A_1 p_6+3\ B_1 p_4)}{p_6}}},$$

$$u_2(x,t)=\left\{\frac{e^{\phi(\eta)}}{B_1}\left(\sqrt{1/4\ \frac{4\ {A_1}^2 p_6 w_1+3\ A_1 B_1 p_4 w_1}{p_6}}+A_1 e^{-\phi(\eta)}\right)\right\}^{\frac{1}{2}}, \tag{73}$$

where $\eta=x-\frac{T}{B_1}\sqrt{1/3\ \frac{3\ J{B_1}^2 w_1+4\ {A_1}^2 p_6+3\ A_1 B_1 p_4}{m w_1}}\,t.$

6.5. Set III

$$\lambda=\frac{T}{-{A_1}^2 w_1+{A_0}^2}\sqrt{1/4\ \frac{4\ J p_6\left(-{A_1}^2 w_1+{A_0}^2\right)^2+3\ {A_0}^2 {A_1}^2 {p_4}^2}{m p_6}},$$

$$\mu=1/32\ \frac{16\ p_2 p_6-3\ {p_4}^2}{p_6},\ A_2=0, \tag{74}$$

$$B_0=-1/3\ \frac{-A_1 w_1 (4\ A_1 p_6+3\ B_1 p_4)+4\ {A_0}^2 p_6}{A_0 p_4},\quad w_2=\frac{{A_1}^2 w_1+{A_0}^2}{A_0 A_1},$$

$$w_2^2-4w_1=\frac{\left(-{A_1}^2 w_1+{A_0}^2\right)^2}{{A_0}^2 {A_1}^2},$$

$$u_3(x,t)=\left\{3\ \frac{\left(A_0+A_1 e^{-\phi(\eta)}\right) A_0 p_4}{3\ B_1 e^{-\phi(\eta)} A_0 p_4+4\ {A_1}^2 p_6 w_1+3\ A_1 B_1 p_4 w_1-4\ {A_0}^2 p_6}\right\}^{\frac{1}{2}}, \tag{75}$$

where

$$\eta=x-\frac{T}{-{A_1}^2 w_1+{A_0}^2}\sqrt{1/4\ \frac{4\ J p_6\left(-{A_1}^2 w_1+{A_0}^2\right)^2+3\ {A_0}^2 {A_1}^2 {p_4}^2}{m p_6}}\,t.$$

6.6. Set IV

$$\lambda=\frac{T}{B_0}\sqrt{1/3\ \frac{3\ J{B_0}^2+{A_1}^2 p_6}{m}},$$

$$\mu=1/32\ \frac{16\ p_2 p_6-3\ {p_4}^2}{p_6},$$

$$A_0=-1/2\ \frac{B_0 (2\ A_1 p_6+3\ B_1 p_4)}{p_6 B_1},\ A_2=0, \tag{76}$$

$$w_1=1/2\ \frac{{B_0}^2 (2\ A_1 p_6+3\ B_1 p_4)}{p_6 {B_1}^2 A_1},\quad w_2=-1/2\ \frac{(4\ A_1 p_6+3\ B_1 p_4) B_0}{A_1 p_6 B_1},\quad w_2^2-4w_1=9/4\ \frac{{B_0}^2 {p_4}^2}{{A_1}^2 {p_6}^2},$$

$$u_4(x,t)=\left\{1/2\ \frac{2\ A_1 e^{-\phi\left(x-\frac{T}{B_0}\sqrt{1/3\ \frac{3\ J{B_0}^2+{A_1}^2 p_6}{m}}t\right)} p_6 B_1-2\ A_1 B_0 p_6-3\ B_0 B_1 p_4}{p_6 B_1\left(B_0+B_1 e^{-\phi\left(x-\frac{T}{B_0}\sqrt{1/3\ \frac{3\ J{B_0}^2+{A_1}^2 p_6}{m}}t\right)}\right)}\right\}^{\frac{1}{2}}. \tag{77}$$

6.7. Set V

$$\lambda=\frac{T}{B_0 (2\ A_1 p_6+B_1 p_4)}\sqrt{\frac{12\ J{B_0}^2 (2\ A_1 p_6+B_1 p_4)^2+{A_1}^2 p_6 (4\ A_1 p_6+3\ B_1 p_4)^2}{12 m}},$$

$$\mu=\frac{16\ p_2 p_6-3\ {p_4}^2}{32 p_6}, \tag{78}$$

$$A_0=-\frac{B_0 (A_1 p_6+B_1 p_4)}{B_1 p_6},\quad A_2=0,$$

$$w_1=\frac{{B_0}^2 (A_1 p_6+B_1 p_4)(4\ A_1 p_6+B_1 p_4)}{p_6 {B_1}^2 A_1 (4\ A_1 p_6+3\ B_1 p_4)},$$

$$w_2=-\frac{\left(8\ {A_1}^2 {p_6}^2+8\ A_1 B_1 p_4 p_6+3\ {B_1}^2 {p_4}^2\right) B_0}{p_6 B_1 A_1 (4\ A_1 p_6+3\ B_1 p_4)},\ w_2^2-4w_1=9\ \frac{{B_0}^2 {p_4}^2 (2\ A_1 p_6+B_1 p_4)^2}{{p_6}^2 {A_1}^2 (4\ A_1 p_6+3\ B_1 p_4)^2},$$

$$u_5(x,t)=\left\{\frac{A_1 e^{-\phi(\eta)} p_6 B_1-A_1 B_0 p_6-B_0 B_1 p_4}{p_6 B_1\left(B_0+B_1 e^{-\phi(\eta)}\right)}\right\}^{\frac{1}{2}}, \tag{79}$$

where $\eta=x-\frac{T}{B_0 (2\ A_1 p_6+B_1 p_4)}\sqrt{1/12\ \frac{12\ J{B_0}^2 (2\ A_1 p_6+B_1 p_4)^2+{A_1}^2 p_6 (4\ A_1 p_6+3\ B_1 p_4)^2}{m}}\,t.$

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6.8. Set VI

$$\lambda = \frac{T}{B_0(8A_1p_6+3B_1p_4)}\sqrt{\frac{3\,JB_0{}^2(8A_1p_6+3B_1p_4)^2+4A_1{}^2p_6(4A_1p_6+3B_1p_4)^2}{3m}},\quad \mu = \frac{16p_2p_6-3p_4{}^2}{32p_6}, \tag{80}$$

$$A_0 = -1/4\ \frac{B_0(4\,A_1p_6+3\,B_1p_4)}{B_1p_6},\quad A_2=0,\quad w_1 = \frac{B_0{}^2}{B_1{}^2},$$

$$\eta = x - \frac{T}{B_0(8\,A_1p_6+3\,B_1p_4)}\sqrt{1/3\ \frac{3\,JB_0{}^2(8\,A_1p_6+3\,B_1p_4)^2+4\,A_1{}^2p_6(4\,A_1p_6+3\,B_1p_4)^2}{m}}\,t.$$

$$w_2 = -1/4\ \frac{B_0(32\,A_1{}^2p_6{}^2+24\,A_1B_1p_4p_6+9\,B_1{}^2p_4{}^2)}{B_1p_6A_1(4\,A_1p_6+3\,B_1p_4)},\quad w_2^2 - 4w_1 = \frac{9\,B_0{}^2p_4{}^2(8\,A_1p_6+3\,B_1p_4)^2}{16\,p_6{}^2A_1{}^2(4\,A_1p_6+3\,B_1p_4)^2},$$

$$u_6(x,t) = \left\{1/4\ \frac{4\,A_1\mathrm{e}^{-\phi(\eta)}p_6B_1-4\,A_1B_0p_6-3\,B_0B_1p_4}{p_6B_1\left(B_0+B_1\mathrm{e}^{-\phi(\eta)}\right)}\right\}^{\frac{1}{2}}, \tag{81}$$

where

6.9. Set VII

$$\lambda = \frac{T}{B_1}\sqrt{1/3\ \frac{3\,JB_1{}^2+4\,A_2{}^2p_6}{m}},\ \mu = 1/32\ \frac{16\,p_2p_6-3\,p_4{}^2}{p_6},$$

$$A_0 = \frac{B_0(A_1B_1-A_2B_0)}{B_1{}^2}, \tag{82}$$

$$w_1 = 1/4\ \frac{(A_1B_1-A_2B_0)(4\,A_1B_1p_6-4\,A_2B_0p_6+3\,B_1{}^2p_4)}{A_2{}^2B_1{}^2p_6},$$

As a result (Group 1), the soliton solution is given by

$$u_7(x,t) = \left\{\frac{\left(-\frac{3}{2}\frac{B_1{}^3p_4A_2}{S(3\,B_1{}^2p_4+4\,p_6S)}\tanh\left(\frac{3B_1p_4}{8A_2p_6}(\eta+E)\right)-\frac{(3\,B_1{}^2p_4+8\,p_6S)A_2B_1}{2S(3\,B_1{}^2p_4+4\,p_6S)}\right)^{-1}A_2B_1+A_1B_1-A_2B_0}{B_1{}^2}\right\}^{\frac{1}{2}}, \tag{84}$$

where $\eta = x - \frac{T}{B_1}\sqrt{\frac{3\,JB_1{}^2+4\,A_2{}^2p_6}{3m}}t$ and $S = A_1B_1 - A_2B_0$.

$$w_2 = 1/4\ \frac{8\,p_6(A_1B_1-A_2B_0)+3\,B_1{}^2p_4}{A_2B_1p_6},\quad w_2^2 - 4w_1 = \frac{9\,B_1{}^2p_4{}^2}{16\,A_2{}^2p_6{}^2},$$

$$u_7(x,t) = \left\{\frac{\mathrm{e}^{-\phi\left(x-\frac{T}{B_1}\sqrt{\frac{3\,JB_1{}^2+4\,A_2{}^2p_6}{3m}}t\right)}A_2B_1+A_1B_1-A_2B_0}{B_1{}^2}\right\}^{\frac{1}{2}}. \tag{83}$$

$$A_1 = 1/4\ \frac{8\,A_2B_0p_6-3\,B_1{}^2p_4}{B_1p_6},\quad w_1 = \frac{A_0}{A_2},\ w_2 = 1/4\ \frac{8\,A_2B_0p_6-3\,B_1{}^2p_4}{A_2p_6B_1},$$

$$w_2^2 - 4w_1 = -\frac{1}{16}\frac{64\,A_2p_6{}^2(A_0B_1{}^2-A_2B_0{}^2)+48\,A_2B_0B_1{}^2p_4p_6-9\,B_1{}^4p_4{}^2}{A_2{}^2B_1{}^2p_6{}^2},$$

6.10. Set VIII

$$\lambda = \frac{T}{B_1}\sqrt{1/3\ \frac{3\,JB_1{}^2+4\,A_2{}^2p_6}{m}},\quad \mu = \frac{64\,A_2p_6{}^2(A_0B_1{}^2-A_2B_0{}^2)+48\,B_1{}^2p_6(A_2B_0p_4+B_1{}^2p_2)-9\,B_1{}^4p_4{}^2}{96\,p_6B_1{}^4}, \tag{85}$$

$$u_8(x,t) = \left\{\frac{4A_2\mathrm{e}^{-2\,\phi\left(\frac{T}{B_1}\sqrt{\frac{3\,JB_1{}^2+4\,A_2{}^2p_6}{3m}}\right)}p_6B_1+\mathrm{e}^{-\phi\left(\frac{T}{B_1}\sqrt{\frac{3\,JB_1{}^2+4A_2{}^2p_6}{3m}}\right)}(8\,A_2B_0p_6-3\,B_1{}^2p_4)+4\,A_0p_6B_1}{4p_6B_1\left(B_0+B_1\mathrm{e}^{-\phi\left(\frac{T}{B_1}\sqrt{\frac{3\,JB_1{}^2+4A_2{}^2p_6}{3m}}\right)}\right)}\right\}^{\frac{1}{2}}. \tag{86}$$

As a result (Groups 1 and 2), the soliton and periodic solutions are given, respectively by

$$u_8(x,t) = \left\{\frac{4\,A_2\Phi^2p_6B_1+\Phi(8\,A_2B_0p_6-3\,B_1{}^2p_4)+4\,A_0p_6B_1}{4p_6B_1(B_0+B_1\Phi)}\right\}^{\frac{1}{2}}, \tag{87}$$

$$\Phi = \left\{-\frac{\sqrt{S}}{8B_1p_6A_0}\tanh\left(\frac{\sqrt{S}}{8A_2p_6B_1}(\eta+E)\right)-\frac{8\,A_2B_0p_6-3\,B_1{}^2p_4}{8B_1p_6A_0}\right\}^{-1},$$

$S > 0,$

$$\Phi = \left\{\frac{\sqrt{-S}}{8B_1p_6A_0}\tan\left(\frac{\sqrt{-S}}{8A_2p_6B_1}(\eta+E)\right)-\frac{8\,A_2B_0p_6-3\,B_1{}^2p_4}{8B_1p_6A_0}\right\}^{-1},$$

$S < 0,$

where $S = \left(8\,A_2B_0p_6 - 3\,B_1{}^2p_4\right)^2 - 64\,A_0A_2B_1{}^2p_6{}^2$

and $\eta = x - \frac{T}{B_1}\sqrt{\frac{3\,JB_1{}^2+4\,A_2{}^2p_6}{3m}}t.$

6.11. Set IX

$$\mu = \frac{16\,p_6{}^2(A_1B_1-2\,A_2B_0)^2+8\,B_1{}^2p_4p_6(A_1B_1-2\,A_2B_0)+16\,B_1{}^4p_2p_6-3\,B_1{}^4p_4{}^2}{32p_6B_1{}^4}, \tag{88}$$

$$A_0 = 1/4\,\frac{A_1{}^2}{A_2},\quad w_1 = 1/4\,\frac{A_1{}^2B_1{}^2p_6+4\,A_1A_2B_0B_1p_6-8\,A_2{}^2B_0{}^2p_6+3\,A_2B_0B_1{}^2p_4}{A_2{}^2B_1{}^2p_6},$$

$$\lambda = \frac{T}{B_1}\sqrt{1/3\,\frac{3\,JB_1{}^2+4\,A_2{}^2p_6}{m}},\ w_2 = \frac{8\,A_1B_1p_6-8\,A_2B_0p_6+3\,B_1{}^2p_4}{4A_2B_1p_6},$$

$$w_2^2-4w_1 = 3/16\,\frac{(4\,A_1B_1p_6-8\,A_2B_0p_6+3\,B_1{}^2p_4)(4\,A_1B_1p_6-8\,A_2B_0p_6+B_1{}^2p_4)}{A_2{}^2B_1{}^2p_6{}^2},$$

$$u_9(x,t) = \left\{1/4\,\frac{4\,A_2{}^2\mathrm{e}^{-2\,\phi\left(x-\frac{T}{B_1}\sqrt{\frac{3\,JB_1{}^2+4\,A_2{}^2p_6}{3m}}t\right)}+4\,A_1\mathrm{e}^{-\phi\left(x-\frac{T}{B_1}\sqrt{\frac{3\,JB_1{}^2+4\,A_2{}^2p_6}{3m}}t\right)}A_2+A_1{}^2}{A_2\left(B_0+B_1\mathrm{e}^{-\phi\left(x-\frac{T}{B_1}\sqrt{\frac{3\,JB_1{}^2+4\,A_2{}^2p_6}{3m}}t\right)}\right)}\right\}^{\frac{1}{2}}. \tag{89}$$

As a result (Groups 1 and 2), the soliton and periodic solutions are given, respectively, by

$$u_9(x,t) = \left\{1/4\,\frac{4\,A_2{}^2\Phi^2+4\,A_1\Phi A_2+A_1{}^2}{A_2(B_0+B_1\Phi)}\right\}^{\frac{1}{2}}, \tag{90}$$

$$\Phi = \left\{-1/2\,\frac{\sqrt{3}\sqrt{S}A_2B_1}{K}\tanh\left(1/8\,\frac{\sqrt{3}\sqrt{S}}{A_2B_1p_6}(\eta+E)\right)-\frac{(8\,p_6(A_1B_1-A_2B_0)+3\,B_1{}^2p_4)A_2B_1}{2K}\right\}^{-1},\quad S>0,$$

$$\Phi = \left\{1/2\,\frac{\sqrt{3}\sqrt{-S}A_2B_1}{K}\tan\left(1/8\,\frac{\sqrt{3}\sqrt{-S}}{A_2B_1p_6}(\eta+E)\right)-\frac{(8\,p_6(A_1B_1-A_2B_0)+3\,B_1{}^2p_4)A_2B_1}{2K}\right\}^{-1},\quad S<0,$$

where

$$\eta = x-\frac{T}{B_1}\sqrt{\frac{3\,JB_1{}^2+4\,A_2{}^2p_6}{3m}}t,\quad K = A_1{}^2B_1{}^2p_6+4\,A_1A_2B_0B_1p_6-8\,A_2{}^2B_0{}^2p_6+3\,A_2B_0B_1{}^2p_4,$$

$$S = (4\,A_1B_1p_6-8\,A_2B_0p_6+3\,B_1{}^2p_4)(4\,A_1B_1p_6-8\,A_2B_0p_6+B_1{}^2p_4).$$

6.12. Set X

$$\lambda = \frac{T}{B_1}\sqrt{\frac{3\,JB_1{}^2+4\,A_2{}^2p_6}{3m}},\quad \mu = \frac{16\,p_2p_6-3\,p_4{}^2}{32p_6},\quad A_0 = \frac{32\,A_2B_0p_6(8\,A_2B_0p_6-3\,B_1{}^2p_4)+9\,B_1{}^4p_4{}^2}{256\,B_1{}^2p_6{}^2A_2}, \tag{91}$$

$$A_1 = \frac{16\,A_2B_0p_6-3\,B_1{}^2p_4}{8B_1p_6},\quad w_1 = \frac{256\,A_2{}^2B_0{}^2p_6{}^2-9\,B_1{}^4p_4{}^2}{256\,A_2{}^2B_1{}^2p_6{}^2},\quad w_2 = 2\,\frac{B_0}{B_1},\quad w_2^2-4w_1 = \frac{9\,B_1{}^2p_4{}^2}{64\,A_2{}^2p_6{}^2},$$

$$u_{10}(x,t) = \left\{\frac{256A_2{}^2\mathrm{e}^{-2\phi(\eta)}B_1{}^2p_6{}^2+32\mathrm{e}^{-\phi(\eta)}A_2B_1p_6\Upsilon_1+32A_2B_0p_6(8A_2B_0p_6-3B_1{}^2p_4)+9B_1{}^4p_4^2}{256B_1{}^2p_6{}^2A_2(B_0+B_1\mathrm{e}^{-\phi(\eta)})}\right\}^{\frac{1}{2}}, \tag{92}$$

where $\eta = x-\frac{T}{B_1}\sqrt{\frac{3\,JB_1{}^2+4\,A_2{}^2p_6}{3m}}t$. As a result (Group 1), the soliton solution is given by

$$u_{10}(x,t) = \left\{\frac{256A_2{}^2\Phi^2B_1{}^2p_6{}^2+32\Phi A_2B_1p_6\Upsilon_1+32A_2B_0p_6(8A_2B_0p_6-3B_1{}^2p_4)+9B_1{}^4p_4^2}{256B_1{}^2p_6{}^2A_2(B_0+B_1\Phi)}\right\}^{\frac{1}{2}}, \tag{93}$$

$$\Upsilon_1 = (16A_2B_0p_6-3B_1{}^2p_4),$$

$$\Phi = \left\{-\frac{3}{8}\frac{B_1{}^3p_4A_2p_6}{2\,A_2{}^2B_0{}^2p_6{}^2-\frac{9\,B_1{}^4p_4{}^2}{128}}\tanh\left[\frac{3}{16}\frac{B_1p_4}{A_2p_6}\left(x-\frac{T}{B_1}\sqrt{\frac{3\,JB_1{}^2+4A_2{}^2p_6}{3m}}t\right)\right]-\frac{2\,B_0B_1A_2{}^2p_6{}^2}{2\,A_2{}^2B_0{}^2p_6{}^2-\frac{9\,B_1{}^4p_4{}^2}{128}}\right\}^{-1}.$$

6.13. Set XI

$$\lambda = \frac{T}{B_1}\sqrt{\frac{3\,JB_1{}^2+4\,A_2{}^2p_6}{3m}},\quad \mu = \frac{64\,p_2p_6-15\,p_4{}^2}{128p_6},\ A_0 = \frac{32\,A_2B_0p_6\left(8\,A_2B_0p_6-3\,B_1{}^2p_4\right)+9\,B_1{}^4p_4{}^2}{256\,B_1{}^2p_6{}^2A_2}, \tag{94}$$

$$A_1 = 1/8\,\frac{16\,A_2B_0p_6-3\,B_1{}^2p_4}{B_1p_6},\quad w_1 = \frac{256\,A_2{}^2B_0{}^2p_6{}^2+9\,B_1{}^4p_4{}^2}{256\,A_2{}^2B_1{}^2p_6{}^2},\quad w_2 = 2\,\frac{B_0}{B_1},\quad w_2^2-4w_1 = -\frac{9\,B_1{}^2p_4{}^2}{64\,A_2{}^2p_6{}^2},$$

$$u_{11}(x,t) = \left\{\frac{256A_2{}^2\mathrm{e}^{-2\phi(\eta)}B_1{}^2p_6{}^2+32\mathrm{e}^{-\phi(\eta)}A_2B_1p_6\Upsilon_1+32A_2B_0p_6(8A_2B_0p_6-3B_1{}^2p_4)+9B_1{}^4p_4^2}{256B_1{}^2p_6{}^2A_2\left(B_0+B_1\mathrm{e}^{-\phi(\eta)}\right)}\right\}^{\frac{1}{2}}, \tag{95}$$

where $\eta = x-\frac{T}{B_1}\sqrt{\frac{3\,JB_1{}^2+4\,A_2{}^2p_6}{3m}}\,t$. As a result (Group 2), the periodic solution is given by

$$u_{11}(x,t) = \left\{\frac{256A_2{}^2\Phi^2B_1{}^2p_6{}^2+32\Phi A_2B_1p_6\Upsilon_1+32A_2B_0p_6(8A_2B_0p_6-3B_1{}^2p_4)+9B_1{}^4p_4^2}{256B_1{}^2p_6{}^2A_2(B_0+B_1\Phi)}\right\}^{\frac{1}{2}}, \tag{96}$$

$$\Upsilon_1 = (16A_2B_0p_6-3B_1{}^2p_4),$$

$$\Phi = \left\{\frac{3}{8}\frac{B_1{}^3p_4A_2p_6}{2\,A_2{}^2B_0{}^2p_6{}^2+\frac{9\,B_1{}^4p_4{}^2}{128}}\tan\left[\frac{3}{16}\frac{B_1p_4}{A_2p_6}\left(x-\frac{T}{B_1}\sqrt{\frac{3\,JB_1{}^2+4A_2{}^2p_6}{3m}}\,t\right)\right]-\frac{2\,B_0B_1A_2{}^2p_6{}^2}{2\,A_2{}^2B_0{}^2p_6{}^2+\frac{9\,B_1{}^4p_4{}^2}{128}}\right\}^{-1}.$$

6.14. The graphical discussion and physical significance

By selecting the appropriate values for the parameter, we were able to generate the desired types of solutions that indicate wave discrepancy. The analytical solutions are coded in maple and the parametric and sensitivity analysis are carried out using the codes. The parametric results are presented in Figs. 1–4. The present results from the simulations show that through an inherent property of auxiliary parameters for the adjustment and control of region and rate of convergence of approximate series solutions, the MEJM and rational $\exp(-\phi(\eta))$-expansion method have proven to be very efficient and capable techniques in handling non-linear engineering problems in wider ranges of parameters. The importance of this study lies in the actuality that it can serve as a base for the experimental work that we want to undertake on the plasma physics and crystal lattice theory.

7. CONCLUSION

On the basis of the constructed auxiliary functions, the MEJM, the rational $\exp(-\phi(\eta))$-expansion method and the solitary wave solutions by utilising TFFME were inspected. The mentioned equation is non-integrable. The impact of wave motion in plasma on the physical parameters including speed and amplitudes of solitary waves has been focussed. Then, the general form rational solutions to TFFME containing soliton, kink soliton, singular soliton and periodic wave solutions were observed. We found plenty of exact solutions for two cases. The dynamical behaviour of results was investigated via graphical illustrations by using considered methods. Moreover, various important remarks about the physical meanings of solutions were presented. From these results, it may be seen that the MEJM and EEM are the power tools to solve such non-linear partial models arising in applied and engineering sciences. In the future, we can further study its soliton solutions, rogue wave solutions, solitary waves and symmetry, etc.

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