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THE CUBIC DICE IN THE ELEMENTARY MATHEMATICAL EDUCATION

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Abstract. The main issue of the article is geometry, arithmetic and stochastics of the dice in children's mathematics. It is a well-known object having mathematical features which can be used for developing mathematical skills at an elementary level. We consider not only geometrical and arithmetical aspects of mathematical education but also the combinatorial and stochastic ones.

1. The die and its W_K feature as a mathematical discovery

Today's die was created from a cube by placing different numbers of spots on its faces. The spots go in numbers from 1 to 6 and are placed on the cube in such a way that spots on opposite faces add up to 7. Thanks to this W_K feature the die has a rich mathematical structure. This feature is often unseen in school textbooks (see [8], p. 17). Throwing the dice is a random experiment.

Let us start with a simple trick. A student throws the die (so that the teacher does not see the result) and remembers the number of spots s_1. Then the student turns the die over and adds the number of spots on the opposite side to the previous one. He does not reveal the sum s_2. Now the student rolls the die again and adds the number of spots to the sum s_2. He does not reveal the new sum s_3 either. Now the teacher comes to the die, throws it, thinks for a while and then gives the exact sum s_3. Surprised students start asking questions: How is that possible? How can we explain this? And all of them lead to an important one: Was it just a coincidence or maybe there is a rule behind this? The W_K feature becomes the center of further argumentation.

Task 1. Roll the die and note down the number of spots you see. Turn the die over and check the number of spots on the opposite side of it. Repeat the throwing a couple of times, each time check up the opposite side spots. Can you see anything unusual?

Creating student's own dice from the cubic grid can also be a mathematical activity.

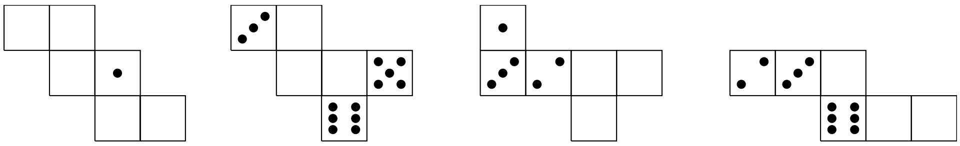


Fig. 1.

Task 2. There are some spots in some squares of each grid in Fig. 1. Fill in the missing spots in such a way that will enable you to get the die out of each grid.

Task 3. Each side of the die has four neighbouring sides and one opposite side. Which squares will become neighbours ith the side with four spots after we glue up the grids from Fig. 1? Which will become the opposite side?

2. The dice arithmetic

The W_K feature of the die enables us to count the spots on it quickly. Each pair of opposite sides has 7 spots together, there are 3 such pairs, so the sum of the spots equals the product of $3 \cdot 7$.

Task 4. Roll the die and guess the number of spots on the side touching the table.

Task 5. Roll two (three) dice. How many spots are there on all the sides touching the table?

Task 6. Place two (three) dice on the table in such a way that the number of spots on their bottom sides is the highest.

Task 7. Every student builds a tower of two dice on the table. The student who gets the biggest number of spots on the side walls of the rower wins the game. Did you manage to show the winner? Is it possible to win such game if the students build their towers of three dice? Why?

Task 8. Explain the fact that there is no winner in the game described above no matter how many dice the students use.

Task 9. Can you place the dice on a table in such a way that the spots on side walls add up to 18? Why?

Task 10. Roll the die and count the spots on the four side walls. Repeat that as many times as you like. Is it a coincidence that the spots on the die side walls always add up to 14? Can you explain this?

Among the four side walls of the dice there are two pairs of opposites, so the sum of spots on them always equals $2 \cdot 7$. The number of spots on the side walls of a tower does not depend on the dice placing, but only on the number of “floors”.

Task 11. Some of the dice in Fig. 2 touch three walls of the room. How many sides of the die are invisible in each case? How many spots are there altogether on the invisible sides of the die in Fig. 2a, how many in Fig. 2b and how many in Fig. 2c?

Task 12. Roll four dice and quickly (without turning them over) join them in a 2×2 combination. How many spots are there altogether on the invisible sides of the dice?

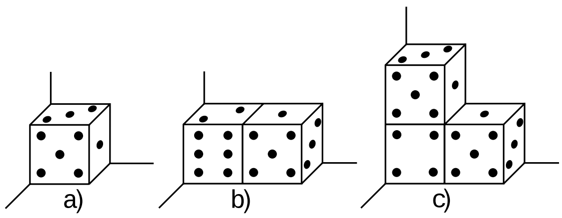


Fig. 2.

We create buildings using many dice placing them in such a way that their sides touch one another. Those touching sides are invisible to us, so they are the back walls and some of the side walls. It is an arithmetic activity to add up all the spots on the invisible sides of all the dice forming a tower.

Task 13. How many spots are there on the invisible sides of dice in the building A in Fig. 3, how many in the building B and how many in the building C? Why is it not possible to state this number in the building D?

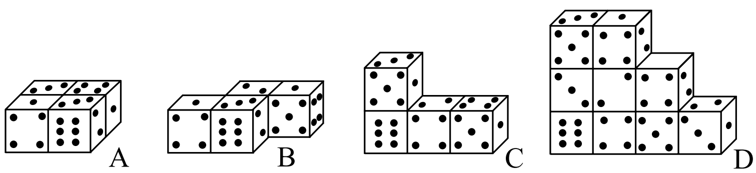


Fig. 3.

3. Odd and even numbers and elements of logic in the dice arithmetic

In the following tasks some elements of logic and deduction appear in the context of the dice and odd and even numbers.

Task 14. How many spots must appear on the die to give an odd number on the bottom side?

From the W_K feature we know that the number of spots on the bottom side of the die is the difference between 7 and the number of spots on the upper side. If the number of spots showing on the die is an odd number, then the number of spots on its bottom side must be – as a difference of two odd numbers – an even one.

Task 15. Casper threw three dice. Each of them showed an odd number of spots. Casper claims that on the bottom sides of all the dice the numbers of spots are even. Is he right? Why?

Task 16. Melchior threw three dice. Each of them showed an odd number of spots. Melchior claims the sum of the spots on the bottom sides of the dice is an even number. Is he correct? How do we explain this?

The W_K feature causes the number of the bottom sides of the dice being a difference of 21 ($21 = 3 \cdot 7$) and the sum of the spots showing on the upper sides of the dice. The number $3 \cdot 7$ is an odd one. The number of spots showing is also odd (being the sum of three odd numbers), so the difference between them (as a difference of two odd numbers) is even. Melchior is right.

Task 17. Baltasar threw three dice and said that the number of spots he got was odd. He also said that the number of spots on the bottom sides of the dice is even. Is he correct?

Task 18. Artie claimed that the tasks 16 and 17 were in fact one task (they both deal with the same problem). Why is he wrong?

Task 19. Bart threw four dice and said that the number of spots he got was odd. He also said that (as in task 17 with three dice) the number of spots on the bottom sides of his dice was even. Is he right?

Task 20. Artie invited Bart to a game: *– On my mark we both roll one die each. Then we count how many spots our dice show together. If the number is odd – I win, if it is even – you win,* said Artie. The boys threw their dice, but they did not reveal the scores.

1. Both dice showed odd numbers. Who won?
2. The number of the spots was more than 6 and dividable by 5. Who won?
3. Is it possible to state the winner if all we know is that the number of spots was dividable by 5? Why?

4. The dice in space geometry

The die will be moved in steps. Each step means turning the die to one of the side walls. Each move directs the die into one of the four sides. Let us implement the compass names to distinguish the sides and assume the movements occur in snow and the spots leave marks in it.

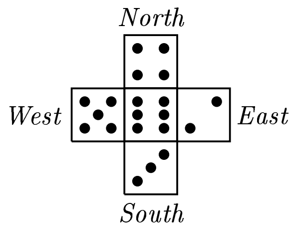


Fig. 4.

If the bottom side of the die has 6 spots, the front side has 3 spots and the right hand side has 2 spots (Fig. 5), then we can move in the following ways:
— to the South, then the bottom side has 3 spots,
— to the North, the bottom side has 4 spots,
— to the West, the bottom side has 5 spots,
— to the East, then the bottom side has 2 spots.
All the possible markings after taking the first step are shown in Fig. 4.

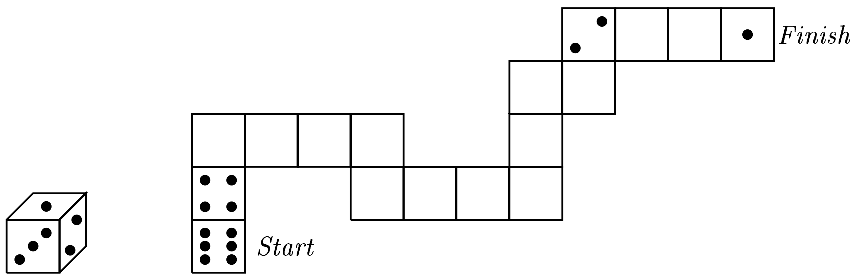


Fig. 5.

Task 21. Place the dice which you see in Fig. 5 on the "start" square. On the bottom side of the die there are 6 spots. You will move it by turning to its sides along the way shown in Fig. 5. Before you actually turn the die, write down the number of spots which will show in the snow after your move in the following squares of the route. Will it be the mark with one spot in the "finish" square, as shown in Fig. 5?

We can write down the markings in the snow left by the die moving along the route from Fig. 5 as a following sequence:

6-4-1-2-6-5-4-1-3-6-5-1-4-2-6-5-1.

Task 22. In the "start" square the bottom side of the die had 6 spots, the front side had 3 spots and the right hand side had 2 spots. The code of the die markings is the sequence: 6-3-2-4-5-3-2-4-1-5-4-2-3. Draw the die route.

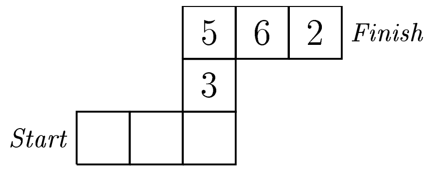


Fig. 6.

Task 23. In the "start" square the upper side of the die has one spot, the front side had 3 spots and the right hand side had 2 spots. Figure 6 shows part of the die movements (we cannot see the first three steps). Reconstruct the obliterated steps.

5. Natural numbers as sums of two other natural numbers

Task 24. Place two dice, the white one and the red one, in a row in such a way, that the numbers of spots showing add up to 7. How many spots are there on the bottom sides of the dice?

Figure 7 shows top views of four of the possible rows.

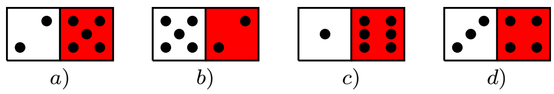
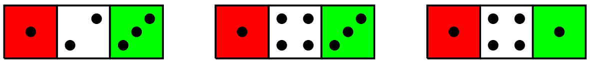


Fig. 7.

Task 25. Place three dice, red one, white one and green one, in a row in such a way that the number of spots showing adds up to 11. How many spots are there on the bottom sides of the dice? Is it possible to arrange three dice in a row in such a way, that the sum of the spots showing equals 19?

Task 26. Paul threw the dice twice, but he did not reveal how many spots he scored in the first or the second roll. He only said that in the first trial he got 5 spots more than in the second one. What was Paul's final score?

Task 27. Gaweł rolled two dice, the white one and the red one, but he did not reveal the scores. He said that on the white die he got 3 spots more than on the red one, and that his final score was 7. How many spots did he get on the white and how many on the red die?



a) b) c)

Fig. 8.

Task 28. Place three dice, red one, green one and white one, on a table (the red one first, then the white one and the green one third) in such a way, that:

a) the spots on the white and green dice add up to the number of spots on the red one,
b) on the upper side of the white (middle) die there are more spots than on the upper side of the red (left) one and more than on the green (right) one – is it so in Fig. 8a?
c) on the upper side of the red die there are less spots than on the white one, and on the white die – less spots than on the green one,
d) the number of spots showing on the red die is a sum of the numbers of spots on the white and green dice,
e) on the upper side of the white (middle) die there are twice as many spots as on the red one and on the upper side of the green die there are twice as many spots as on the white one.

Task 29. Paul placed red, white and green dice in a row, but covered them with his hands. He said that the white one shows two spots more than the red one, the green one shows two spots more than the white one, and the total score is 5. How many spots are there on the upper side of each die in this row?

6. The dice combinatorial aspect – different ways of showing the same problem

Task 30. Let us get back to placing two dice (the white and red one) in a row in a way that the total number of spots showing equals 7 (task 24). How many ways of doing so are there?

The point here is to show the number 7 as a sum of two other natural numbers, remembering that the order of the dice matters (we can easily distinguish them), and to state an exact number of ways of doing so. Four possibilities are shown in Fig. 7. The question of how many ways of doing so are there touches the theory of combinations. We need to get 7 spots on the upper sides of two dice. We can see that the real question here is in how many ways we can divide the set of 7 into two unempty sets. We are considering another, but equal, aspect of task 30.

$7 = 2 + 5$ $7 = 5 + 2$ $7 = 1 + 6$ $7 = 3 + 4$

Fig. 9.

Figure 9 shows seven spots in a row. In order to divide this set into two we need to place a line between any two neighbouring spots. The number of spots preceding the line represents the number of spots showing on the white die, and the number of spots following the line represents the number of spots on the red die. There are six possibilities of placing such a line, so there are six ways of placing two dice in a row and getting 7 spots on them both.

The interpretation of placing the dice in a row in a specific way (so that there are seven spots on the top) being the same as dividing the set of seven spots into two parts suggests some actions having to be taken (arranging the spots, placing the line). Those activities are different from placing the dice on a table chaotically as they lead the student to the solution of the problem.

Four examples of placing lines shown in Fig. 9 represent four solutions of the task shown in Fig. 7.

7. Dice tricks, the dice arithmetic and generalization as mathematical activity

Let us get back to the trick with one die. Let x means the number of spots scored in the first roll and y mean the number of spots scored in the second roll of the die. Let us consider the equation:

$$\boxed{x} + \boxed{7-x} + \boxed{y} = 7 + \boxed{y}.$$

Each of the left-side components represents an action taken: roll of the die (the number s_1), turning it over (the number s_2) and the second roll (the number s_3). After the arithmetic transformation the left-side sum becomes the sum of 7 and the score of the second roll, and the later we can still see, as the die is still unmoved after the second roll. So the number s_3, which was to be guessed, is really the number of spots showing on the die enlarged by 7.

An important argument is the W_K feature of the die, so our argumentation is of arithmetic nature.

But the trick described above may create another mathematical activity, which is a generalization and inferring by analogies. We can modify this trick by complicating it a bit.

Now a student rolls two dice, the red one and the white one, then he:

— counts all the spots (the sum s_1),

— turns the white die over and adds the number of spots showing to the sum s_1 (the sum s_2),

— rolls the white die again and adds the score to the sum s_2 (the sum s_3).

The sum s_3 may be guessed if we know the trick with the single die. If we add 7 to the number of spots scored on the white die (both sides), we will get the number guessed in the one-die trick. The sum s_3 guessed in this trick has one more component – the number of spots scored on the red die (and this number is still visible, as the red die is still on the table). So the sum s_3 is really the sum of all the spots visible on the red and white dice and the number 7.

Task 31. Suggest a similar trick with two dice (the red one and white one) in which both dice will be thrown, turned and thrown again.

Task 32. Prepare a similar trick using three dice in which after the initial roll:
a) only the white die will be turned over and thrown again,
b) the red and green dice will be turned over and thrown again.

In relation to the dice tricks described above an element of surprise appears as an inspiration of mathematical activities.

8. Dice snakes and the dice arithmetic

We will create snakes using one white die and some red ones. The white die will represent the snake head, and the red ones will form its tail. The tails will be of different length. In Fig. 10 we can see four such snakes. The upper, visible side of the white die will be the snake head top, the upper sides of the red dice will form the snake back, and the bottom, invisible sides of the red dice will form the snake belly.

When we form such snakes and check the characteristic features of their "body parts" a number of arithmetical tasks will appear, each of them connected to adding, subtracting and multiplying natural numbers. Some tasks will refer to representing a number as a sum of two other numbers.

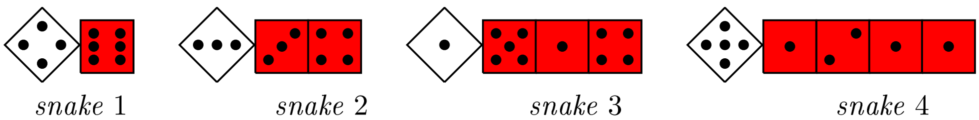


Fig. 10.

Task 33. For each snake from Fig. 10 count the number of spots on the head top, on the back and on the (invisible) belly. How many spots are there on all the dice forming the tail of each snake?

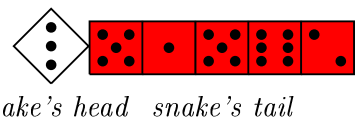


Fig. 11.

Task 34. Figure 11 shows the top view of a snake that was formed using one white die and five red ones. Its tail length equals 5. Count the spots on its back and then on its back and head top. How many spots are there on this snake's belly?

Task 35. Take five red dice and the white one. Form a snake in such a way that on its head top there are 6 spots and on its back there are six spots too. How many different snakes with six spots on their heads have tails with such backs? How many spots are there on such snake's belly?

Task 36. Take five red dice and the white one again. Form a snake that has the total number of spots on the back equal to the number of spots on the head top. Try several such snakes. What tail must a snake have if the number of spots on its head is 5? Is it possible for such a snake to have 4 spots on the head top? What other head tops are not possible for such a snake?

Task 37. Form a snake using five red dice and the white one in such a way that its back has 7 spots more than its head top.

Task 38. The snake tail is five dice long. There are 5 spots on its back. How many spots are there on its belly?

Task 39. The snake tail is six dice long. Is it possible for such a snake to have the same number of spots on the back and on the head top?

Task 40. Can a snake with a five (six) dice long tail have the same number of spots on the back and on the belly?

9. The dice versus certain, impossible and probable random events

A coincidence may also be a creator of "dice snakes".

Task 41. Roll two dice, the white and the red one. Form a snake without turning the dice. The white dice becomes the snake head and the red one – the snake tail.

1. How many different snakes with a tail of one die are there?
2. How many spots can there be on the head and tail of such a snake together? Is it possible to get a snake of 13 spots? What about 2?

The number of snakes we can get is the same as the number of possible scores of a two-dice roll – 36. It is worth to note that all the scores are equally probable. Let us connect the two-dice roll and form a snake with the following events:

$A = \{$*there are 14 spots on all the sides of snake's head*$\}$,
$B = \{$*there are 13 spots on both sides of snake's tail*$\}$,
$C = \{$*there is one spot on snake's head top*$\}$,
$D = \{$*there are 6 spots on one of the sides of the snake's head*$\}$.

It is easy to state that:

— we are sure to get the result A every time, so A is a *certain event*;
— it is not possible to get the result B at all, so B is an *impossible event*;
— getting the result C is not certain but it is not impossible either, so C is a *probable event*;
— it is also probable to get the result D.

Let us notice that whenever we get the event C we do not get the D at the same time, and the other way round. There is no such score of a two-dice roll that would be propitious for both events C and D. So C and D are *disjoint events.*

In the elementary mathematics education we will only decide whether two events are equally probable or not and if not, which of them is more possible to appear. We will also claim some events to be very likely to happen and some to have a slight possibility of happening.

Task 42. In a moment we will roll two dice, the red one and the white one, and form a snake. Before we do so, you can bet on the number of spots which the snake will have on both its head top and its back. If your bet is correct, you score a point. What numbers can you bet on? Does it matter what number you choose? Why?

In the game described above we bet on one of the events:

$$A_j = \{\textit{in a two-dice roll the total number of spots scored is } j\}$$

where $j = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$. Those events are not equally probable. Among them the event A_7 has the best chance of appearing, because there are 6 scores of the two-dice roll propitious for it (each of the remaining sums has less than 6 possible scores). This reasoning about "the chances of certain event happening" is correct as each score of the two-dice roll is equally probable. It is the most profitable to bet on the event A_7.

In the stochastics for everyone, we see two kinds of events as the most important:
— almost certain events which are very likely to happen,
— almost impossible events which are very unlikely to happen (see [4], p. 146).

Let us connect a six-dice roll with two random events:

$A = \{$*every number of spots will show*$\}$,

$B = \{$*one of the possible numbers of spots will not show on any dice*$\}$.

The event A is almost impossible, the event B – almost certain. Those stochastic features of the events A and B can be discovered *a posteriori* in the classroom. The scores of many repeated six-dice rolls are the qualitative (not quantitative) estimation of the probability of those events. The organization of the data gathered is discussed in [4] and [7].

Task 43. You have lost your die. What can you replaced it with? How? Can you “make” the die out of four coloured spheres?

The ways of replacing the die with an urn with three or four spheres of different colours are discussed in [4] (pp. 76-79).

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