

TREFFTZ FUNCTIONS FOR NON-STATIONARY PROBLEMS

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Different types of Trefftz functions for non-stationary linear and weakly nonlinear differential equations are presented. The Trefftz methods are defined and briefly described. Certain results for non-stationary problems of heat conduction (among others boundary temperature identification and thermal diffusivity estimation), for beam vibration, for thermoelasticity and for the wave equation (direct and inverse problem of membrane vibrations) are shown. In many cases, the FEM with Trefftz functions (FEMT) as probe functions is applied. Three kinds of FEMT are tested for direct and inverse non-stationary problems. Examples of the making use of T-functions for solving inverse problems and smoothing inaccurate input data are discussed.

Key words: non-stationary inverse problems, Trefftz method, FEMT

1. Introduction

The method known as “Trefftz method” was firstly presented in 1926 (Trefftz, 1926). An approximate solution of a problem is assumed to have form of a linear combination of functions that satisfy the governing differential linear equation (without sources). In the case of using finite elements, such an approximation is applied to each element. The unknown coefficients are determined based on approximate fulfilling the boundary, initial and other conditions. It leads to a system of algebraic equations. Such an approach to the approximate solving of a direct or inverse problem is referred to as indirect Trefftz method (Kita and Kamiya, 1995). In order to apply the Trefftz method, a complete system of the so-called T-functions (or T-complete functions or Trefftz base) has to be known.

Generally, Trefftz bases fall into two broad classes. The F-Trefftz method constructs its basis function space by allowing many source points outside the domain of interest to keep the basis functions regular. For each source point, the fundamental solution is adopted; therefore, this method is also called the method of fundamental solutions (MFS) (Liu *et al.*, 2006). The T-Trefftz bases consist of functions which fulfill identically the considered differential equation, (Ciałkowski and Frąckowiak, 2000; Grysa, 2010; Kita and Kamiya, 1995; Zieliński, 1995). Here we consider only the T-Trefftz bases.

Starting at the end of the 70s, the T-functions have been presented for both bounded and unbounded regions, for Laplace and Helmholtz equations (Herrera, 1984; Herrera and Sabina, 1978), biharmonic equation (Gurgeon and Herrera, 1981) and Stokes problem (Herrera and Gurgeon, 1982). For elasticity problems, T-functions have been shown by Jiroušek and Teodorescu (1982), Qin (2000). The basic sets of T-functions for stationary problems can be found by Zieliński (1995).

The non-stationary problem was simplified by getting rid of the time variable. A result of such an approach was the Helmholtz type equation.

In the late 90s, the T-functions for non-stationary problems have been published, at first, based on the paper Rosenbloom and Widder (1956), for parabolic equations (Futakiewicz, 1999;

Grysa, 2003; Hożejowski, 1999; Rakoczy, 1997), then for wave equation (Ciałkowski and Frąckowiak, 2000; Ciałkowski and Jarosławski, 2003; Maciąg, 2007), beam vibration (Al-Khatib *et al.*, 2008), and for plate vibration and thermoelasticity (Maciąg, 2009b).

The T-functions for the linear differential equation usually have been obtained by the method of variable separation. In 2000, two new methods of deriving the T-functions were published (Ciałkowski and Frąckowiak, 2000). The first one based on the expanding on the function into Taylor series is particularly simple and effective; the second is based on the so-called inverse operators (Ciałkowski and Frąckowiak, 2000, 2003).

During the last 10 years, the Trefftz method has been frequently applied to find approximate solutions of the direct and inverse problems (Grysa, 2010).

At first, a “global” approach was applied with good results for simple geometry and initial-boundary conditions. Next, the FEM with Trefftz functions as trial functions was investigated. Recently, the T-functions have been tested to smooth inaccurate internal data for the wave and for heat transfer equations with very good results (Grysa and Leśniewka, 2010; Maciąg, 2009a).

Here we present the basic sets of T-functions for chosen nonstationary linear and weakly nonlinear problems in bounded regions. Then, the Trefftz methods are briefly described with a special attention to the least square method in relation to nonstationary direct and inverse problems. Moreover, approximate solutions of problems of heat conduction and wave motion are shown.

2. Basic sets of T-functions for chosen differential equations

We confine our attention to non-stationary linear differential equations that model physical phenomena in bounded areas. Sets of T-functions for stationary problems are discussed e.g. in Ciałkowski and Grysa (2010), Zieliński (1995), and in the monograph by Grysa (2010). The sets of T-functions presented below are useful in approximate solving of direct and inverse problems. Below, in almost all cases (except for thermoelasticity problems), the governing equation are presented in dimensionless form.

2.1. Heat conduction equation

In 1D, for $(x, t) \in \Omega \times (0, T)$, $\Omega \subset R^1$, the T-functions (also called the heat polynomials) read (Hożejowski, 1999; Rosenbloom and Widder, 1956)

$$v_n(x, t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{x^{n-2k} t^k}{(n-2k)! k!} \quad (2.1)$$

where $\lfloor x \rfloor$ denotes the function floor(x).

In the polar system of coordinates (r, t) , the T-complete set of functions consists of two sets of functions

$$g_n(r, t) = \sum_{k=0}^n \frac{\left(\frac{r}{2}\right)^{2n-2k} t^k}{((n-k)!)^2 k!} \quad u_n(r, t) = -g_n(r, t) \ln r + q_n(r, t) \quad (2.2)$$

with

$$q_n(r, t) = \sum_{k=0}^n \frac{a_{n-k} \left(\frac{r}{2}\right)^{2n-2k} t^k}{((n-k)!)^2 k!}$$

and a_n defined by formulas

$$a_0 = 0 \quad a_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} \quad \text{for } n > 0$$

In 2D, T-functions are defined as (Ciałkowski and Frąckowiak, 2000)

$$V_m(x, y, t) = v_{n-k}(x, t)v_k(y, t) \quad n = 0, 1, \dots \quad k = 0, \dots, n \quad (2.3)$$

and $m = n(n+1)/2 + k$; $v_n(u, t)$ is given by (2.1).

In the cylindrical system of coordinates (r, z, t) , the T-complete set of functions consists of two sets of functions

$$V_{1m}(r, z, t) = g_{n-k}(r, t)v_k(z, t) \quad V_{2m}(r, z, t) = u_{n-k}(r, t)v_k(z, t) \quad (2.4)$$

with $n = 0, 1, \dots$; $k = 0, \dots, n$ and $m = n(n+1)/2 + k$.

In 3D, we have (Ciałkowski and Frąckowiak, 2000)

$$V_{nkl}(x, y, z, t) = v_{n-k-l}(x, t)v_{k-l}(y, t)v_l(z, t) \quad (2.5)$$

Here $n = 0, 1, \dots$; $k = 0, \dots, n$; $l = 0, \dots, k$ and $v_n(u, t)$ is given by (2.1).

2.2. Wave equation

In 2D, for $(x, y, t) \in \Omega \times (0, T)$, $\Omega \subset R^2$, the variable separation method leads to the following recurrent formulas for the T-functions (wave polynomials) $P_{mn0}(x, y, t)$, $P_{mn1}(x, y, t)$, $Q_{mn0}(x, y, t)$ and $Q_{mn1}(x, y, t)$ (Maciąg, 2007)

$$\begin{aligned} P_{000} &= 1 & Q_{000} &= 0 \\ P_{(n-k)k0} &= -\frac{1}{n} \left(xQ_{(n-k-1)k0} + yQ_{(n-k)(k-1)0} + tQ_{(n-k-2)k1} + tQ_{(n-k)(k-2)1} \right) \\ P_{(n-k-1)k1} &= -\frac{1}{n} \left(xQ_{(n-k-2)k1} + yQ_{(n-k-1)(k-1)1} + tQ_{(n-k-1)k0} \right) \\ Q_{(n-k)k0} &= \frac{1}{n} \left(xP_{(n-k-1)k0} + yP_{(n-k)(k-1)0} + tP_{(n-k-2)k1} + tP_{(n-k)(k-2)1} \right) \\ Q_{(n-k-1)k1} &= \frac{1}{n} \left(xP_{(n-k-2)k1} + yP_{(n-k-1)(k-1)1} + tP_{(n-k-1)k0} \right) \end{aligned} \quad (2.6)$$

for $n = 1, 2, \dots$, $k = 0, 1, \dots, n$. If any subscript is negative, the T-function with such a subscript is equal to zero.

In 3D, for $n = 1, 2, \dots$, $k = 0, 1, \dots, n$, $l = 0, 1, \dots, n$ and $m = 0, 1$ and for $P_{0000} = 1$, $Q_{0000} = 0$ the T-functions $P_{(n-k-l-m)klm}(x, y, z, t)$ and $Q_{(n-k-l-m)klm}(x, y, z, t)$ may be obtained from the following recurrent formulas (Maciąg, 2009b)

$$\begin{aligned} P_{(n-k-l)kl0} &= -\frac{1}{n} \left(xQ_{(n-k-l-1)kl0} + yQ_{(n-k-l)(k-1)l0} + zQ_{(n-k-l)k(l-1)0} \right. \\ &\quad \left. + tQ_{(n-k-l-2)kl1} + tQ_{(n-k-l)(k-2)l1} + tQ_{(n-k-l)k(l-2)1} \right) \\ P_{(n-k-l-1)kl1} &= -\frac{1}{n} \left(xQ_{(n-k-l-2)kl1} + yQ_{(n-k-l-1)(k-1)l1} + yQ_{(n-k-l-1)k(l-1)1} \right. \\ &\quad \left. + tQ_{(n-k-l-1)kl0} \right) \\ Q_{(n-k-l)kl0} &= \frac{1}{n} \left(xP_{(n-k-l-1)kl0} + yP_{(n-k-l)(k-1)l0} + zP_{(n-k-l)k(l-1)0} \right. \\ &\quad \left. + tP_{(n-k-l-2)kl1} + tP_{(n-k-l)(k-2)l1} + tP_{(n-k-l)k(l-2)1} \right) \\ Q_{(n-k-l-1)kl1} &= \frac{1}{n} \left(xP_{(n-k-l-2)kl1} + yP_{(n-k-l-1)(k-1)l1} + yP_{(n-k-l-1)k(l-1)1} \right. \\ &\quad \left. + tP_{(n-k-l-1)kl0} \right) \end{aligned} \quad (2.7)$$

2.3. Beam vibration equation

For the beam vibration equation

$$\left(\frac{\partial^4}{\partial x^4} + \frac{\partial^2}{\partial t^2}\right)u = 0 \quad (x, t) \in \Omega \times (0, T) \quad \Omega \subset R^1$$

the T-function set can be derived considering the Taylor series for a solution of the equation. In order to eliminate the derivatives with respect to variable x of the order higher than four, the following form of the governing equation will be used

$$\frac{\partial^{4\nu} u}{\partial x^{4\nu}} = (-1)^\nu \frac{\partial^{2\nu} u}{\partial t^{2\nu}} \quad \nu = 1, 2, \dots$$

The sums of coefficients accompanying the same derivatives stand for the T-functions. Finally, the T-functions for the beam vibration equation for $n = 1, 2, \dots$ read (Al-Khatib *et al.*, 2008)

$$S_n(x, t) = \sum_{j=0}^{[A_n/2]} \frac{(-1)^j t^{A_n-2j} x^{B_n+4j}}{(A_n - 2j)!(B_n + 4j)!} \quad (2.8)$$

where

$$A_n = n - 3 \left[\frac{n-2}{4} \right] - 3 \quad B_n = 4 \left[\frac{n-2}{4} \right] - n + 5$$

2.4. Plate vibration equation

For the beam vibration equation

$$\left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} + \frac{\partial^2}{\partial t^2}\right)u = 0 \quad (x, y, t) \in \Omega \times (0, T) \quad \Omega \subset R^2$$

the T-functions $P_{kl}(x, y, t)$ and $Q_{kl}(x, y, t)$ for $n = 1, 2, \dots$, and for $P_{00} = 1$, $Q_{00} = t$ may be obtained from the following recurrent formulas (Maciąg, 2009b)

$$\begin{aligned} P_{n1} &= yP_{(n-1)0} & Q_{n1} &= yQ_{(n-1)0} \\ P_{n(n-1)} &= xP_{(n-1)(n-1)} & Q_{n(n-1)} &= xQ_{(n-1)(n-1)} \\ P_{(n+2)0} &= \frac{2x(n+1)P_{(n+1)0} - x^2P_{n0} - 4t^2P_{(n-1)0} - 2tQ_{(n-2)0}}{(n+1)(n+2)} \\ Q_{n0} &= \frac{2x(n+1)Q_{(n-1)0} - x^2Q_{(n-2)0} - 4t^2Q_{(n-4)0} + 2tP_{n0}}{(n+1)(n+2)} \\ P_{n(k+1)} &= \frac{(n-k+1)P_{n(k-1)} - xP_{(n-1)(k-1)} + yP_{(n-1)k}}{k+1} & 1 \leq k \leq n-1 \\ Q_{n(k+1)} &= \frac{(n-k+1)Q_{n(k-1)} - xQ_{(n-1)(k-1)} + yQ_{(n-1)k}}{k+1} & 1 \leq k \leq n-1 \end{aligned} \quad (2.9)$$

2.5. Thermoelasticity

The set of thermoelasticity equations read (Grysa and Maciąg, 2011; Maciąg, 2009b)

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \text{grad div } \mathbf{u} = \rho \ddot{\mathbf{u}} + \gamma \text{grad } T \quad \frac{1}{\kappa} \frac{\partial T}{\partial t} = \nabla^2 T \quad (2.10)$$

Here \mathbf{u} represents the displacement vector, ∇ stands for nabla operator, μ , λ denote Lamé constants, ρ means mass density, κ is the coefficient of thermal diffusivity. Applying Lamé substitution $\mathbf{u} = \text{grad } \phi + \text{rot } \Psi$ in Eq. (2.10)₁ leads to the following wave equations for the potentials ϕ and Ψ

$$\left(\nabla^2 - \frac{1}{c_1^2} \frac{\partial^2}{\partial t^2}\right)\phi = mT \quad \left(\nabla^2 - \frac{1}{c_2^2} \frac{\partial^2}{\partial t^2}\right)\Psi = \mathbf{0} \quad (2.11)$$

with $c_1^2 = (\lambda + 2\mu)/\rho$, $c_2^2 = \mu/\rho$ standing for the velocities of waves and $m = \gamma/(c_1^2\rho)$. After rewriting equations (2.11) in dimensionless form, the T-functions for them have form (2.6) in 2D and form (2.7) in 3D. However, the T-functions are defined only for homogeneous equations. Hence, for Eq. (2.11)₁ one has to find a particular solution for known T . For heat transfer equation (2.10)₂, the T-functions are defined by (2.1), (2.3) or (2.5) in 1D, 2D and 3D, respectively.

Sets of T-functions for stationary problems are discussed e.g. in Ciałkowski and Grysa (2010), Zieliński (1995) and in the monograph by Grysa (2010).

2.6. Weakly nonlinear problems

Consider a nonlinear partial differential equation $Lu = f$ in Ω with the condition $Bu = g$ on a set of points \mathbf{K} being a boundary $\partial\Omega$ in the case of the direct problem or a part of the boundary and some points inside Ω in the case of an inverse problem. Let $L = L_{lin} + N$; then we have $L_{lin}u = f - Nu$ with L_{lin} standing for the linear part of the operator L and N being its nonlinear part. We assume that the nonlinearity is weak what means that its influence on the solution is, say, of the order of 5-10% of the linearized problem solution.

Assume that for the homogeneous equation $L_{lin}\varphi = 0$ the T-complete functions $\{\varphi_n^*\}$ are known. Then the equation $L_{lin}u = f - Nu$ is a quasilinear equation in implicit form. To solve the equation $L_{lin}u = f - Nu$, the Picard iterations can be implemented

$$L_{lin}u^{(k)} = f - Nu^{(k-1)} \quad \text{in } \Omega \quad (\text{for } k = 0 \quad L_{lin}u^{(0)} = f) \quad (2.12)$$

and

$$Bu^{(k)} = g \quad \text{on } \mathbf{K} \quad k = 0, 1, \dots, K \quad (2.13)$$

with $u^{(k)} = u_{part}^{(k)} + u_{gen}^{(k)}$. Here $u^{(k)}$ is a linear combination of T-functions, $k = 0, 1, \dots, K$. Its form is obtained in the following way. Because $u_{gen}^{(k)}$ is a general solution to the homogeneous equation $L_{lin}u_{gen}^{(k)} = 0$, it can be expressed as a linear combination of a finite number of the T-functions $\{\varphi_n^*\}$. Let us approximate the source function f and the nonlinear term $Nu^{(k-1)}$ as $f \approx \sum_j b_j P_j$ and $Nu^{(k-1)} \approx \sum_i c_i P_i$ with P_i, P_j being monomials (f and $Nu^{(k-1)}$ are then approximated by polynomials). A particular solution to equation (2.12) is then

$$u_{part}^{(k)} = L_{lin}^{-1} \left(\sum_j b_j P_j + \sum_i c_i P_i \right) \quad (2.14)$$

The stopping criterion for the Picard iteration can be adopted in the form

$$E = \max_{\mathbf{x}_m \in \Omega} \sqrt{\sum_{m=1}^M \left(u^{(k)}(\mathbf{x}_m) - u^{(k-1)}(\mathbf{x}_m) \right)^2} < \varepsilon \quad (2.15)$$

with $\{\mathbf{x}_m\}_{m=1}^M$ standing for a set of trial points; M and ε are chosen arbitrarily.

As an example, consider a 1D dimensionless heat conduction problem in Ω with an unknown conductivity coefficient that depends on temperature

$$\left(\frac{\partial}{\partial x}\lambda(u)\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)u = f \quad \lambda(u) = \lambda_0 + \alpha u$$

α being a small number.

Assume that $u|_{\partial\Omega}$ is known, $\partial\Omega = \{(x_1, t), (x_2, t)\}$, $\Omega \times (0, t_e)$, $\Omega \subset R$. Then

$$L = \frac{\partial}{\partial x}\lambda(u)\frac{\partial}{\partial x} - \frac{\partial}{\partial t} = L_{lin} + N$$

with

$$L_{lin} = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t} \quad N = \frac{\partial}{\partial x}\alpha u\frac{\partial}{\partial x}$$

For the homogeneous equation, the T-functions are known, see (2.1). A general solution to the homogeneous equation has then a form $u_{gen}^{(k)} = \sum_n a_n^{(k)} v_n$. With the polynomial approximation of the source term f and of the nonlinear term $Nu^{(k-1)}$, it is easy to find a particular solution $u_{part}^{(k)}$, (2.14), to equation (2.12), making use of the following formulas

$$\begin{aligned} w_{0,k} &= L_{lin}^{-1}(t^k) = -\frac{t^{k+1}}{k+1} & w_{1,k} &= L_{lin}^{-1}(xt^k) = -\frac{xt^{k+1}}{k+1} \\ w_{m,k} &= L_{lin}^{-1}(x^m t^k) = \frac{1}{k+1} (m(m-1)w_{m-2k+1} - x^m t^{k+1}) \\ k &= 0, 1, \dots & m &= 2, 3, \dots \end{aligned}$$

With $u^{(k)} = u_{part}^{(k)} + u_{gen}^{(k)}$ it is easy to find the coefficients $a_n^{(k)}$ and – as a consequence – a solution in the k -th iteration to equation (2.12) with conditions (2.13). Finally, stopping criterion (2.15) shows whether the next iteration has to be calculated.

Convergence of the Picard iteration is here an open question. In the case when it fails, any other iterating procedure can be adopted.

3. Trefftz methods

The Trefftz methods have not received a precise definition, although this terminology has had wide acceptance. Herreras definition of what is meant by a Trefftz method is (Herrera, 1984):

Given a region of an Euclidean space or some partitions of that region, a “Trefftz Method” is any procedure for solving initial boundary value problems of partial differential equations or systems of such equations, on such region, using solutions of that differential equation or its adjoint, defined in its subregions.

The Trefftz methods are generally divided into two groups: direct methods and indirect methods. In the direct method, the governing equation is to be fulfilled in a weak formulation with the trial function being a linear combination of T-functions. In the indirect method, the solution itself is approximated as a linear combination of the T-functions.

Among the indirect methods the most popular ones seem to be the least square method (LSM) and collocation method. Different formulation of the indirect methods are presented in Kita and Kamiya (1995). The LSM seems also to be useful when dealing with the inverse problems.

Let us consider a problem formulated as follows:

$$\begin{aligned} Lu = f & \quad \text{in} \quad \Omega \times (0, t_e) \\ u = g_1 & \quad \text{on} \quad S_D \times (0, t_e) & p \frac{\partial u}{\partial n} = g_2 & \quad \text{on} \quad S_N \times (0, t_e) \\ & & p \frac{\partial u}{\partial n} + \alpha u = g_3 & \quad \text{on} \quad S_R \times (0, t_e) \\ u = g_4 & \quad \text{on} \quad S_{int} \times T_{int} & u = h & \quad \text{on} \quad \Omega \quad \text{for} \quad t = 0 \end{aligned}$$

where $Lu = f$ is a parabolic linear differential equation, S_{int} stands for a set of points inside the considered region $(S_D \cup S_N \cup S_R) \subset \partial\Omega$, $T_{int} \subset (0, t_e)$ is a set of moments of time and the functions g_i , $i = 1, 2, 3, 4$ and h are of proper class of differentiability in the domains in which they are determined. In order to solve such a problem in an approximate way, let us consider the following objective functional

$$I(v) = \int_{\Omega \times (0, t_e)} (Lv - f)^2 d\Omega dt + w_1 \int_{S_D \times (0, t_e)} (v - g_1)^2 dS dt + w_2 \int_{S_N \times (0, t_e)} \left(p \frac{\partial v}{\partial n} - g_2 \right)^2 dS dt + w_3 \int_{S_R \times (0, t_e)} \left(p \frac{\partial v}{\partial n} + \alpha v - g_3 \right)^2 dS dt + w_4 \sum_{S_{int} \times T_{int}} (v - g_4)^2 + w_5 \int_{\Omega} (v - h)^2 d\Omega \quad (3.1)$$

Here w_i , $i = 1, \dots, 5$, stand for constant positive weighting parameters which are to preserve the numerical equivalence between the terms, and $v \in V$. The problem may be formulated as follows: find a function $u \in V$ such that

$$I(u) = \min_{\forall v \in V} I(v) \quad (3.2)$$

Such an approach leads to the LSM.

Let V be a space generated by a finite number of T-functions for the equation $Lu = 0$. If v is a linear combination of the Treffitz functions for the homogeneous equation plus v^{part} and v^{part} is known (the function v^{part} stands for a particular solution to equation $Lu = f$), the first term in equation (3.1) vanishes. If $f = 0$ then the function v^{part} is equal to zero.

Usually, the weighting parameters are equal to one. Moreover, the objective functional may be completed with a term or terms regularizing the solution (Ciałkowski *et al.*, 2007; Grysa *et al.*, 2008).

The advantage of the LSM is that the stiffness matrix in the system of algebraic equations is always symmetric and positive definite (or semi-definite) and that even complicated constraints of the solution can be easily incorporated with a numerical method. However, in the case of global approach, the disadvantage of LSM is that the condition number will increase significantly, compared with FEM Li *et al.*, 2007).

When the region $\Omega \times (0, t_e)$ has to be divided into subregions, one introduces the time-space elements (Grysa *et al.*, 2009b; Maciejewska, 2004). Then the FEM with T-functions (FEMT) is convenient to be applied. At least three kinds of FEMT may be used:

- (a) FEMT with the condition of continuity of temperature in the common nodes of elements
- (b) FEMT with no temperature continuity at any point between the elements
- (c) Nodeless FEMT (substructuring). Instead, in each finite element the temperature is approximated with a linear combination of the T-functions.

Hence, depending of the kind of FEMT, the T-functions are used as trial functions or the approximate solution being a linear combination of the T-functions is constructed in each element separately (Grysa and Leńiewska, 2010). The elements may be greater than in the classic FEM because the approximate solution fulfill the governing equation identically. Moreover, the number of base functions can be smaller than in the classic FEM.

Usually, on the borders between the elements, one considers continuity of the approximate solution and its normal derivative. Here, we only require minimization of the difference of values v^+ and v^- and of values $\partial v / \partial n^+$ and $\partial v / \partial n^-$. Therefore, the objective functional is completed with terms

$$\int_{\Gamma \times (0, t_e)} (v^+ - v^-)^2 dS dt \quad \int_{\Gamma \times (0, t_e)} \left(\frac{\partial v}{\partial n^+} - \frac{\partial v}{\partial n^-} \right)^2 dS dt$$

accompanied with positive weights. Here Γ stands for the border between the elements v^+ and v^- (n^+ and n^- denote values of the function v (normal vector n) on both sides of Γ , respectively). Equation (3.2) always leads to a system of algebraic equations for the coefficients of the T-functions combination.

In the case of operator L being a hyperbolic one the analysis is similar.

4. Examples of the problems solved using Trefftz method

4.1. 2D heat conduction problem

Consider a transient 2D heat conduction problem in dimensionless coordinates in a square. On three sides of the square, the heat flux is prescribed. The fourth thermal condition is known inside the square on a line distant at $\delta_b \in (0, 0.99)$ from the fourth side of the square. For $\delta_b = 0$ the initial boundary problem is considered (a direct problem). For $\delta_b > 0$ the considered problem is called an inverse one. Hence, the heat conduction equation is considered

$$\Delta T = \frac{\partial T}{\partial t} \quad (x, y) \in \Omega = (0, 1) \times (0, 1) \quad t \in (0, t_K) \quad (4.1)$$

with the conditions

$$\begin{aligned} T(x, y, t) \Big|_{t=0} &= T_0(x, y) & T(x, y, t) \Big|_{x=0} &= h_1(y, t) = e^{y+2t} \\ \frac{\partial T}{\partial y}(x, y, t) \Big|_{y=1} &= h_2(x, t) = e^{x+1+2t} & \frac{\partial T}{\partial y}(x, y, t) \Big|_{y=0} &= h_3(x, t) = e^{x+2t} \\ T(1 - \delta_b, y_i, t_k) &= T_{ik} \end{aligned} \quad (4.2)$$

Here T_{ik} stand for temperatures measured in points distant at $\delta_b \in (0, 0.99)$ from the side $x = 1$ of the square, $i = 1, \dots, I$; $k = 1, \dots, K$.

In order to solve the problem, the time interval is divided into subintervals. In each subinterval, the domain Ω is divided into 4 finite subdomains Ω_j , $j = 1, 2, 3, 4$ and in each subdomain the temperature is approximated as a linear combination of the T-functions $V_m(x, y, t)$, $T^j(x, y, t) \approx \tilde{T}^j(x, y, t) = \sum_{m=1}^N c_m^j V_m(x, y, t)$, given by formula (2.3). In the case of the first time subinterval, the initial condition is known. For the next subintervals, the initial condition is understood as the temperature field in the subdomain Ω_j at the final moment of time in the previous subinterval. The LSM is used to minimize the inaccuracy of the approximate solution on the boundary, in the initial moment of time and on the borders between the elements. This way the unknown coefficients of the combination are calculated. Moreover, on the border between the elements the heat flux jumps are minimized (Grysa and Leśniewska, 2010).

The objective functional reads

$$\begin{aligned} J = & \sum_i \int_{D_i} \left(\tilde{T}_i(x, y, 0) - T_0(x, y) \right)^2 dD + \sum_i \int_{t_{m-1}}^{t_m} \int_{\Gamma_i} \left(\tilde{T}_i(0, y, t) - h_1(y, t) \right)^2 d\Gamma dt \\ & + \sum_i \int_{t_{m-1}}^{t_m} \int_{\Gamma_i} \left(\frac{\partial \tilde{T}_i}{\partial y}(x, 1, t) - h_2(x, t) \right)^2 d\Gamma dt + \sum_i \int_{t_{m-1}}^{t_m} \int_{\Gamma_i} \left(\frac{\partial \tilde{T}_i}{\partial y}(x, 0, t) - h_3(x, t) \right)^2 d\Gamma dt \\ & + \sum_{i,j} \int_{t_{m-1}}^{t_m} \int_{\Gamma_{ij}} \left(\tilde{T}_i - \tilde{T}_j \right)^2 d\Gamma dt + \sum_i \sum_{k=1}^K \left(\tilde{T}_j(1 - \delta_b, y_i, t_k) - T_{ik} \right)^2 \\ & + \sum_{i,j} \int_{t_{m-1}}^{t_m} \int_{\Gamma_{ij}} \left(\frac{\partial \tilde{T}_i}{\partial x} - \frac{\partial \tilde{T}_j}{\partial x} \right)^2 d\Gamma dt + \sum_{i,j} \int_{t_{m-1}}^{t_m} \int_{\Gamma_{ij}} \left(\frac{\partial \tilde{T}_i}{\partial y} - \frac{\partial \tilde{T}_j}{\partial y} \right)^2 d\Gamma dt \end{aligned} \quad (4.3)$$

where Γ_{ij} is a common boundary of the elements Ω_i and Ω_j . The dimensionless time interval is $(0, 0.01]$, i.e. $t_0 = 0$. The temperatures are known at points $(1 - \delta_b, y_i)$ with $y_i \in \{0.1, 0.2, 0.3, 0.4, 0.6, 0.7, 0.8, 0.9\}$ for $t \in \{0.0025, 0.0050, 0.0075, 0.01\}$ and $\delta_b \in (0, 0.99)$. It means that $I = 8$, $K = 4$ and $m = 1, 2, 3, 4$. For $\delta_b = 0$, the problem becomes a direct one. Temperatures inside the square (the internal temperature responses, abbr. ITRs) were generated from the accurate solution and then interfered with a random noise $N(0, 0.05)$. The noisy “measurements” have been next approximated with the linear combination of the T-functions due to the formula

$$T(1 - \delta_b, y, t) \approx \tilde{T}_{\delta_b}(y, t) = \sum_{s=1}^S a_s V_s(1 - \delta_b, y, t) \quad (4.4)$$

For the calculation, we have taken $S = 18$ T-functions. Because the exact solution is known (Grysa and Leśniewska, 2010), the error of the approximate solution in the square in the i -th time subinterval can be described using the norm

$$\delta L_2 = \sqrt{\frac{\int_D (\tilde{T}(x, y, t) - T(x, y, t))^2 dD}{\int_D (T(x, y, t))^2 dD}} \cdot 100\% \quad D = \Omega \times (t_{i-1}, t_i) \quad (4.5)$$

The error on the part of the boundary where the temperature has been calculated in the m -th time subinterval reads

$$\delta L_2|_{x=1} = \sqrt{\frac{\int_{t_{m-1}}^{t_m} dt \int_0^1 (\tilde{T}(1, y, t) - T(1, y, t))^2 dy}{\int_{t_{m-1}}^{t_m} dt \int_0^1 (T(1, y, t))^2 dy}} \cdot 100\% \quad (4.6)$$

The results obtained with the nodeless FEMT for noisy and smoothed data are presented in Table 1.

Table 1. Norm δL_2 and $\delta L_2|_{x=1}$ as functions of δ_b and ITRs

δ_b	Noisy ITRs		Smoothed ITRs		Exact ITRs	
	δL_2 [%]	$\delta L_2 _{x=1}$ [%]	δL_2 [%]	$\delta L_2 _{x=1}$ [%]	δL_2 [%]	$\delta L_2 _{x=1}$ [%]
0.1	0.7204	1.4844	0.04318	0.0745	0.0315	0.07938
0.3	1.0455	0.6235	0.04470	0.0833	0.0478	0.08107
0.5	0.6929	0.4371	0.04069	0.0991	0.0469	0.07197
0.7	1.1008	0.9198	0.04463	0.0932	0.0459	0.08005
0.9	0.7263	0.4261	0.04339	0.0890	0.0464	0.07696
0.99	0.4265	0.3864	0.04573	0.0884	0.0517	0.08119

Comparing the errors of the unknown boundary identification for the exact and smoothed noisy data, one observes a slight difference between their values. For the exact ITRs, the results are sometimes a bit worse than those for the smoothed data. It may be a result of approximating the solution with a small number of T-functions.

4.2. Thermal diffusivity estimation

In Grysa *et al.* (2009a), thermal diffusivity estimation based on temperature measurements in PTFE (teflon), ice and soil are reported. In the experiments, a thermal probe having a shape

of a thin tube with 16 sensing or heating elements inside was used. The probe was placed in a specimen of shape of a hollow cylinder with outer radius much greater than the outer radius of the probe. Detailed technical specifications referring to the probe can be found in Spohn *et al.* (2007). In each specimen every 310 s, one of the sensors was used as a heater (heating interval lasted 900 s) and next the responses were registered with 10 s sampling (Grysa *et al.*, 2009a). The coefficients of thermal diffusivity of PTFE, ice and soil were known, and the experiment was to verify the method of thermal diffusivity estimation in the specimen with unknown thermal properties (Sendek, 2007).

When the probe is in the mode of heating, the heat is dissipated to the surrounding medium in the radial direction. The dimensionless heat transfer equation in cylindrical coordinates

$$\frac{\partial^2 T}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial T}{\partial \xi} + \frac{\partial^2 T}{\partial \varsigma^2} = \frac{\partial T}{\partial \tau} \quad (4.7)$$

$$\xi \in (\xi_p, \xi_b) \quad \varsigma \in (0, \varsigma_k) \quad \tau > 0$$

with the boundary condition $T(\xi_b, \varsigma, \tau_m) = T_m$ is considered. Here ξ_p and ξ_b stand for the dimensionless outer radius of the probe (the border between the probe and the specimen) and the outer radius of the specimen, respectively, ς_k denotes length of the k -th sensor, $\tau = \kappa t / (r_p - r_b)$ is the dimensionless time. Moreover, κ denotes the thermal diffusivity, r_p and r_b stand for outer radius of the probe and outer radius of the specimen, respectively.

The problem of thermal diffusivity estimation was solved using T-functions. For cylindrical geometry, the T-functions for the heat conduction equation are presented in Section 2, see formula (2.2). An approximate solution $\theta(\xi, \varsigma, \tau)$ to the problem formulated above (note that only one boundary condition is prescribed and no initial condition is formulated!) has form of a linear combination of T-functions. The unknown coefficients of the combination are chosen to minimize the functional

$$J = \int_0^{\varsigma_k} \sum_{m=1}^M \left(\theta(\xi_p, \varsigma, \tau_m) - T_m \right) d\varsigma \quad (4.8)$$

for $k = 1, 2, \dots, 16$. Here M denotes the number of temperature measurements. Minimization of the functional J means that we demand $\theta(\xi_p, \varsigma, \tau)$ to fit the boundary condition as accurately as possible. The necessary condition for minimum of the functional leads to a system of algebraic linear equations for coefficients of the linear combination of T-functions, standing for the approximate solution $\theta(\xi, \varsigma, \tau)$.

The functional J can be calculated only for a given value of thermal diffusivity (here denoted as κ). In further consideration, we focus our attention on how J varies with change of κ . Our objective is to find such κ for which $J(\kappa)$ is minimal. First we need to choose a range of the parameter κ bracketing the minimum. Next, in order to find such a value of κ for which the functional J has the minimum value, a method called the golden section is used (Press *et al.*, 2007). The process of searching the value of κ continues until the bracketing interval is acceptably small. Of course, the smaller the sampling step for κ , the better the accuracy of the thermal diffusivity. Finally, the value of thermal diffusivity for the investigated medium (a sample) is taken as a mean value for the results obtained for each sensor.

In the considered problem, only 12 T-functions have been taken into account. Three intervals of heating were considered. The results are presented in Table 2.

As it can be seen, the obtained diffusivities are at least acceptable when compared to the reference values, presented in the first column. It is worth noting that such results have been achieved for only 12 T-functions and for only one boundary condition taken into account.

Table 2. Average values of thermal diffusivity

Medium / κ value	1st interval	2nd interval	3rd interval
PTFE / $\kappa = 1.13 \cdot 10^{-7}$	$1.23 \cdot 10^{-7}$	$1.89 \cdot 10^{-7}$	$1.62 \cdot 10^{-7}$
Ice / $\kappa = 1.6 \cdot 10^{-6}$	no measurements	$1.58 \cdot 10^{-6}$	$1.77 \cdot 10^{-6}$
Soil / $\kappa = 8.72 \cdot 10^{-7}$	$4.77 \cdot 10^{-7}$	$6.30 \cdot 10^{-7}$	$6.26 \cdot 10^{-7}$

4.3. Free vibrations of a square membrane

Consider free vibrations of a square membrane $\Omega = (0, 1) \times (0, 1)$ described by dimensionless wave equation in 2D (Grysa, 2010; Maciąg, 2009a), with the conditions

$$\begin{aligned}
 u(x, y, 0) &= x(x-1)y(y-1) & \frac{\partial u}{\partial t} \Big|_{(x,y,0)} &= 0 \\
 u(0, y, t) &= u(x, 0, t) = u(x, 1, t) = 0
 \end{aligned} \tag{4.9}$$

and with the displacement measured in three inner points (the internal responses, abbr. IRs)

$$u(1-\varepsilon, 0.25, t) = u_1(t) \quad u(1-\varepsilon, 0.50, t) = u_2(t) \quad u(1-\varepsilon, 0.75, t) = u_3(t) \tag{4.10}$$

If $\varepsilon > 0$ we have an inverse problem (the measured displacements are then called the internal responses, abbr. IRs) and we search for the solution in the whole domain but especially $u(1, y, t)$. In the case of $\varepsilon = 0$, we consider a direct (initial-boundary) problem.

The exact solution to the problem for $u(1, y, t) = 0$ is known (Maciąg, 2009a), and is used to generate the IRs. The solution is approximated in the whole square according to formula (4.4), i.e. $u \approx w = \sum_{n=1}^N c_n V_n$ with V_n being T-functions (wave polynomials) for the wave equation in 2D (presented in Section 2, formulas (2.6)). The objective functional has the following form

$$\begin{aligned}
 I &= \int_0^1 \int_0^1 (w(x, y, 0) - x(x-1)y(y-1))^2 dx dy + \int_0^1 \int_0^1 \left(\frac{\partial w}{\partial t}(x, y, 0) \right)^2 dx dy \\
 &+ \int_0^{\Delta t} \int_0^1 (w(0, y, t))^2 dy dt + \int_0^{\Delta t} \int_0^1 (w(x, 0, t))^2 dx dt + \int_0^{\Delta t} \int_0^1 (w(x, 1, t))^2 dx dt \\
 &+ \int_0^{\Delta t} \left\{ \sum_{k=1}^3 (w(1-\varepsilon, 0.25k, t) - u_k(t))^2 \right\} dt
 \end{aligned} \tag{4.11}$$

The approximation of the solution with the linear combination of wave polynomials of the order up to 7 shows that the difference between the exact solution for the vibration as a function of time for the location $x = 1, y = 0.5$ and the approximation for $\varepsilon = 0, \varepsilon = 0.05$ and $\varepsilon = 0.3$ does not exceed 0.2% in the case when IRs are undisturbed.

Now let us disturb the internal response $u_{kl} = u(1-\varepsilon, 0.1(k-1), 0.1(l-1))$, $k, l = 1, 2, \dots, 11$ according to the formula $u_{kl}^d = u_{kl}(1 + \xi_{kl})$ where ξ_{kl} are random numbers of normal distribution with the expected value equal to zero and standard deviation equal to 0.01 ($N(0, 0.01)$). The disturbance ξ_{kl} has been generated separately for each u_{kl} using a random-number generator. In this case, the relative error

$$E = \sqrt{\frac{\int_0^{\Delta t} (u(1, 0.5, t) - w(1, 0.5, t))^2 dt}{\int_0^{\Delta t} (u(1, 0.5, t))^2 dt}} \tag{4.12}$$

has a value from circa 2000% for $\varepsilon = 0$ (direct problem) to circa 180000% for $\varepsilon = 0.8$.

It means that a very small disturbance causes a very bad approximation. However, smoothing the disturbed data by a linear combination of wave polynomials in a similar way like in example 1 (formula (4.4)) leads to much better results. The coefficients of the combination are chosen so as to minimize $\sum_{kl} \left(w_\varepsilon (1 - \varepsilon, (k - 1)0.1, (l - 1)0.1) - u_{kl}^d \right)^2$. After smoothing, the error is similar as for the exact data (Maciąg, 2009a).

5. Final conclusions

The object of this paper is to present the existing T-function bases for bounded regions for non-stationary problems. The applications of T-functions for solving non-stationary inverse problems show the usefulness of the functions. Many others are reported by Grysa (2010). The use of T-functions to solve weakly nonlinear direct and inverse problems is a new idea. A conclusion that follows the presented examples (and that should be checked by mathematicians) is that an approximate solution of a direct or inverse problem with the use of T-functions seems to be of better quality if the functions describing conditions and input data are formulated in the same subspace of the space generated by the Trefftz base. Increasing the number of T-functions leads initially to better results (Grysa and Leśniewska, 2010), but usually their number cannot be too large, because it leads to ill-conditioned problems. Therefore, the best seems to use them as basic functions in the FEM, since their number in an element is usually not too large.

There is still a lot of open problems left here. They concern some technical problems as well as mathematical background of the Trefftz methods.

For instance, when dealing with the FEM or FEMT with time-space elements, a question concerning the dependence between the length of the spatial and the “length” of the time dimensions of the element appears. Probably they may depend on the investigated signal propagation velocity. In the case of the heat conduction problems, the “velocity” of temperature propagation is difficult to define. Theoretically, it is equal to infinity, but practically, it depends strongly on the accuracy of temperature measurements.

Generally, in FEMT, big finite elements can be used. However, their size depends on the number of trial functions. The kind of the dependence is not known.

In the second example1 (and in many papers devoted to the inverse heat conduction problems), small numbers of points with input data (with ITRs) lead to good results. Also the incomplete data (e.g. lack of initial conditions (Sendek, 2007), or boundary conditions known only on a part of the boundary (Maciąg, 2009b)) lead to good (comparable with the accurate) results.

Many other questions arise when applying Trefftz methods. From the point of view of engineers, obtaining better results with an increasing number of T-functions is a good justification to apply the method. However, the mathematicians are kindly invited to check some ideas of the engineers. It would be useful to unify the studies of the two communities.

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Funkcje Trefftza dla problemów niestacjonarnych

Streszczenie

W pracy przedstawione są różne rodzaje funkcji Trefftza dla niestacjonarnych liniowych i słabo nieliniowych równań różniczkowych. Zdefiniowane są i krótko opisane metody Trefftza. Prezentowane są niektóre wyniki dla niestacjonarnych problemów przewodzenia ciepła (m.in. identyfikacja temperatury na brzegu oraz estymacja dyfuzyjności termicznej), drgań belki oraz dla termosprężystości i równania falowego (prosty i odwrotny problem drgań membrany). W pracy pokazany jest również przykład zastosowania metody elementów skończonych z funkcjami Trefftza (MEST) jako funkcjami próbnymi. Trzy rodzaje MEST są testowane na prostych i odwrotnych problemach niestacjonarnych. W pracy omawiane są również przykłady zastosowania T-funkcji w rozwiązywaniu problemów odwrotnych w połączeniu z wygładzaniem niedokładnych danych wejściowych.

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