Free Vibrations of Geometrically Nonlinear Column Locally Resting on the Winkler Elastic Foundation under the Specific Load

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Abstract

The results of theoretical and numerical studies concerning continuous system subjected to the follower force directed towards the positive pole, locally resting on Winkler elastic foundation are presented in this paper.

The load by follower force directed towards the positive pole is guaranteed by loading structure built of loading and receiving heads made of elements of circular outlines. Abovementioned heads are real constructions, used in experimental research of continuous systems.

Taking into account total mechanical energy of the system, the Hamilton's principle and the small parameter method, the differential equations of motion and boundary conditions of the considered column were determined. On the basis of a solution of the issue of dynamics of the system, an appropriate formulas were formulated and then the trajectory of curves on the plane frequency of free vibrations – the value of external load were calculated taking into considerations physical and geometrical parameters of the structure, including parameter of loading head and parameters describing Winkler elastic foundation.

Keywords: frequency of free vibrations, Winkler elastic foundation, slender systems

1. Introduction

The issue of stability and free vibrations of slender geometrically nonlinear bar systems [1] lying on the Winkler elastic foundation is the subject of many scientific publications, where an influence of elastic base parameters on the value of bifurcation load and the scope of changes in the frequency of free vibrations was analysed.

Taking into account the physical models, the following types of arrangement of elastic foundation are defined:
– along the full length of each rod of the system (total foundation – linear systems [2]),
– along the full length of selected rod of the system (partial foundation – geometrically nonlinear systems [3]),
– at a certain distance along the axis of the system (local foundation for the rod- linear structures [4] or for the selected rods - geometrically nonlinear structures [5]).

2. The Physical Model of the System

The physical model of geometrically nonlinear column (**NW**) subjected to the follower force directed towards the positive pole locally resting on the Winkler elastic foundation is presented in Figure 1.

The column (Figure 1a) is built of two external prismatic rods ①,② of length l_1 and l_2 and one prismatic internal rod mounted symmetrically in relation to the external rods. In order to model the local elastic foundation of stiffness K, the internal rod was divided into three segments ③,④,⑤ of constant flexural stiffness and lengths l_3, l_4, l_5 respectively.

Taking into consideration presented description, the following relations relating to the lengths l_i (i = 1…5), distribution of the flexural stiffness $(EJ)_i$, compressive stiffness $(EA)_i$, and mass per unit length $(\rho A)_i$ are assumed:

– in the case of external rods of the system:

$$l_1 = l_2 \tag{1}$$

$$(EJ)_1 = (EJ)_2 \tag{2}$$

$$(EA)_1 = (EA)_2 \tag{3}$$

$$(\rho A)_1 = (\rho A)_2 \tag{4}$$

– in the case of the internal rod:

$$l_1 = l_2 = l_3 + l_4 + l_5 \tag{5}$$

$$(EJ)_3 = (EJ)_4 = (EJ)_5 \tag{6}$$

$$(EA)_3 = (EA)_4 = (EA)_5 \tag{7}$$

$$(\rho A)_3 = (\rho A)_4 = (\rho A)_5 \tag{8}$$

The internal rod, which is supported on the elastic foundation (member ④) is characterised by lower flexural stiffness $(EJ)_3$ as compared to the flexural stiffness of external rods of the column, that is:

$$(EJ)_3 \leq (EJ)_1 + (EJ)_2 \tag{9}$$

Rods ①,②,③ are mounted rigidly (cantilevered) ($x_1 = x_2 = x_3 = 0$). The free ends of the rods ①,②,⑤ ($x_1 = l_1$, $x_2 = l_2$, $x_5 = l_5$) of the system are connected by the element of concentrated mass m that ensures the equality of longitudinal and transverse displacements as well as the equality of angles of deflection of these rods. The follower force directed towards the positive pole (see Figure 1a) is realised by loading ⑥ and receiving heads ⑦,⑧ of circular outlines (constant curvature) [6]. The direction of external load P ⑨ passes through a stationary point O lying on the non-deformed axis of the column and is tangential to the line of deflection of free end of the system. It is assumed in this paper that the elastic foundation does not effect the symmetry of the structure. The free end of the column is connected with the receiving head by the infinitely rigid element ⑧ of length l_0 that is a part of loading structure. Consideration of this element jest necessary due to real constructional solution of head realising the load [7]. The flexural stiffness of abovementioned element is many times higher than the flexural stiffness of the essential system. The pole O is located at a distance of (R-l_0) from the free end of the column.

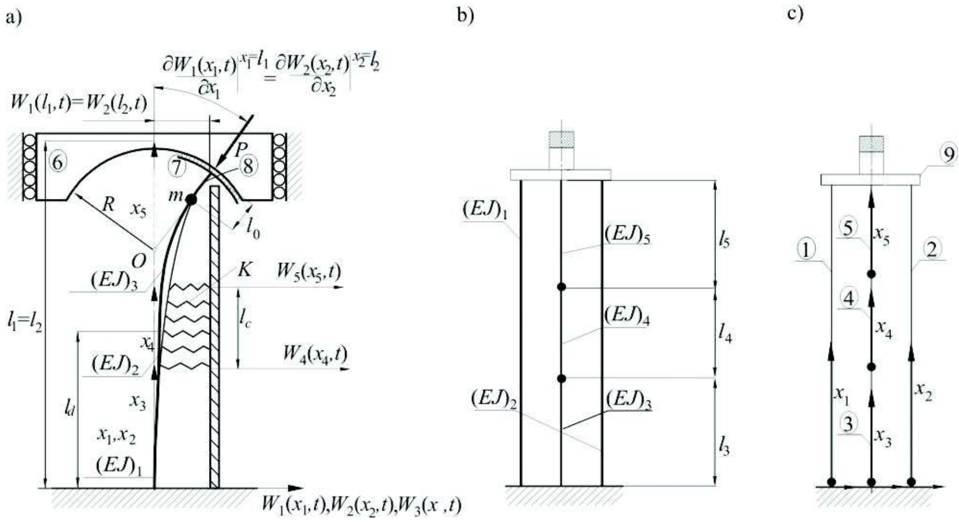

Figure 1. a) The physical model of geometrically nonlinear column (**NW**) locally resting on Winkler elastic foundation,
b, c) The physical model of rods of the geometrically nonlinear column

Taking into account the elastic foundation, the following parameters describing a location and a size of the elastic base relative to the total length of the analysed structure were determined:

$$l_c^* = \frac{l_4}{l_1} = \frac{l_c}{l_1},\; l_d^* = \frac{l_3 + \frac{l_4}{2}}{l_1} = \frac{l_d}{l_1} \tag{10}$$

Assuming that the sum of the total flexural stiffness of the geometrically nonlinear system (**NW**) is constant:

$$\sum_{n=1}^{3} \left(EJ\right)_n = idem\,, \tag{11}$$

the asymmetry of flexural stiffness coefficient μ was defined in the form:

$$\mu = \frac{\left(EJ\right)_3}{(EJ)_1 + \left(EJ\right)_2} \tag{12}$$

Taking into consideration the physical model of the column, a components of kinetic energy and potential energy were specified. The kinetic energy T is a sum of the kinetic energy of particular rods and the kinetic energy of concentrated mass m :

$$T = \frac{1}{2}\sum_{i=1}^{5} (\rho A)_i \int_0^{l_i} \left(\frac{\partial W_i\,(x_i\,,t)}{\partial t} \right)^2 dx_i + \frac{1}{2}\, m \left(\frac{\partial W_1\,(l_1\,,t)}{\partial t} \right)^2 \tag{13}$$

The components of the potential energy V refer to the energy of bending elasticity, the potential energy resulting from external load and the energy of elastic foundation:

$$V = \frac{1}{2}\sum_{i=1}^{5}(EJ)_i \int_0^{l_i}\left[\frac{\partial^2 W_i(x_i,t)}{\partial x_i^{\,2}}\right]^2 dx_i + \frac{1}{2}\sum_{i=1}^{5}(EA)_i \int_0^{l_i}\left[\frac{1}{2}\left(\frac{\partial W_i(x_i,t)}{\partial x_i}\right)^2 + \frac{\partial U_i(x_i,t)}{\partial x_i}\right]^2 dx_i + \\ + PU_1(l_1,t) + \frac{1}{2}P(R-l_0)\left[\frac{\partial W_1(x_1,t)}{\partial x_1}\right]^2 + \frac{1}{2}K\int_0^{l_4}\left(W_4(x_4)\right) dx_4 \tag{14}$$

3. Problem Formulation, Differential Equations of Motion, Boundary Conditions

The issue of the stability and free vibrations of the geometrically nonlinear column was formulated using the Hamilton's principle [6]:

$$\delta\int_{t_1}^{t_2}\left(T-V\right)dt = 0 \tag{15}$$

where: δ – variation operator.

Using the formulas (13) and (14), after computation of the variation of the kinetic and potential energies, the following equations were obtained:

– the differential equations of transverse displacements

$$(EJ)_j \frac{\partial^4 W_j(x_j,t)}{\partial x_j^4} + S_j(t)\frac{\partial^2 W_j(x_j,t)}{\partial x_j^2} + (\rho A)_j \frac{\partial^2 W_j(x_j,t)}{\partial t^2} = 0,\ j = 1,2,3,5 \tag{16}$$

$$(EJ)_4 \frac{\partial^4 W_4(x_4,t)}{\partial x_4^4} + S_4(t)\frac{\partial^2 W_4(x_4,t)}{\partial x_4^2} + (\rho A)_4 \frac{\partial^2 W_4(x_4,t)}{\partial t^2} + KW_4(x_4,t) = 0 \tag{17}$$

where longitudinal forces in external rods and particular members of the internal rod are defined as:

$$S_i(t) = -(EA)_i\left[\frac{\partial U_i(x_i,t)}{\partial x_i} + \frac{1}{2}\left(\frac{\partial W_i(x_i,t)}{\partial x_i}\right)^2\right],\quad i = 1..5 \tag{18}$$

– the equations on motion in the longitudinal direction:

$$\frac{\partial}{\partial x_i}\left[\frac{\partial U_i(x_i,t)}{\partial x_i} + \frac{1}{2}\left(\frac{\partial W_i(x_i,t)}{\partial x_i}\right)^2\right] = 0,\ i = 1..5 \tag{19}$$

Double integration of the formulas (19) over an adequate ranges and taking account of the relationships (18) allowed the determination of the formulas describing the longitudinal displacements of the each rod of the system:

$$U_i(x_i,t) - U_i(0,t) = -\frac{S_i(t)}{(EA)_i}x_i - \frac{1}{2}\int_0^{x_i}\left(\frac{\partial W_i(x_i,t)}{\partial x_i}\right)^2 dx_i \tag{20}$$

Known geometrical boundary conditions of the considered structure are written as follows:

$$W_1(0,t) = W_2(0,t) = W_3(0,t) = U_1(0,t) = U_2(0,t) = U_3(0,t) = 0$$

$$W_1(l_1,t) = W_5(l_5,t), \quad W_2(l_2,t) = W_5(l_5,t),$$

$$U_1(l_1,t) = U_2(l_2,t) = U_5(l_5,t),$$

$$\left.\frac{\partial W_1(x_1,t)}{\partial x_1}\right|_{x_1=0} = \left.\frac{\partial W_2(x_2,t)}{\partial x_2}\right|_{x_2=0} = \left.\frac{\partial W_3(x_3,t)}{\partial x_3}\right|_{x_3=0} = 0,$$

$$\left.\frac{\partial W_1(x_1,t)}{\partial x_1}\right|_{x_1=l_1} = \left.\frac{\partial W_5(x_5,t)}{\partial x_5}\right|_{x_5=l_5}, \left.\frac{\partial W_2(x_2,t)}{\partial x_2}\right|_{x_2=l_2} = \left.\frac{\partial W_5(x_5,t)}{\partial x_5}\right|_{x_5=l_5} \tag{21-35}$$

$$W_3(l_3,t) = W_4(0,t), \quad W_4(l_4,t) = W_5(0,t),$$

$$U_3(l_3,t) = U_4(0,t), \quad U_4(l_4,t) = U_5(0,t),$$

$$\left.\frac{\partial W_3(x_3,t)}{\partial x_3}\right|_{x_3=l_3} = \left.\frac{\partial W_4(x_4,t)}{\partial x_4}\right|_{x_4=0}, \quad \left.\frac{\partial W_4(x_4,t)}{\partial x_4}\right|_{x_4=l_4} = \left.\frac{\partial W_5(x_5,t)}{\partial x_5}\right|_{x=0},$$

$$W_1(l_1,t) = (R-l_0)\left.\frac{\partial W_1(x_1,t)}{\partial x_1}\right|_{x_1=l_1}$$

Substituting of conditions (21-35) into the equation (15) allowed to determine remaining natural boundary conditions that are expressed by the following formulas:

$$(EJ)_1\left.\frac{\partial^3 W_1(x_1,t)}{\partial x_1^3}\right|_{x=l_1} + (EJ)_2\left.\frac{\partial^3 W_2(x_2,t)}{\partial x_x^3}\right|_{x=l_2} + (EJ)_5\left.\frac{\partial^3 W_5(x_5,t)}{\partial x_5^3}\right|_{x=l_5} +$$

$$-\frac{1}{R-l_0}\left[(EJ)_1\left.\frac{\partial^2 W(x_1,t)_1}{\partial x_1^2}\right|_{x=l_1} + (EJ)_5\left.\frac{\partial^2 W_5(x_5,t)}{\partial x_5^2}\right|_{x=l_5}\right] - m\frac{\partial^2 W_1(x_1,t)}{\partial t^2} = 0$$

$$\left.\frac{\partial^2 W_3(x_3,t)}{\partial x_3^2}\right|_{x_2=l_3} = \left.\frac{\partial^2 W_4(x_4,t)}{\partial x_4^2}\right|_{x_3=0}, \quad \left.\frac{\partial^2 W_4(x_4,t)}{\partial x_4^2}\right|_{x_4=0} = \left.\frac{\partial^2 W_5(x_5,t)}{\partial x_5^2}\right|_{x_5=0}, \tag{32-37}$$

$$\left.\frac{\partial^3 W_3(x_3,t)}{\partial x_3^3}\right|_{x_3=l_3} = \left.\frac{\partial^3 W_4(x_4,t)}{\partial x_4^3}\right|_{x_4=0}, \quad \left.\frac{\partial^3 W_4(x_4,t)}{\partial x_4^3}\right|_{x_4=0} = \left.\frac{\partial^3 W_5(x_5,t)}{\partial x_5^3}\right|_{x_5=0}$$

$$\sum_{n=1}^{3} S_n - P = 0$$

4. Results of Numerical Computations

Taking into account the solution of the boundary problem, the numerical calculations were carried out relating to the free vibrations of the considered system.

An exemplary scope of change in the value of the first frequency of vibrations (parameter Ω^*) as a function of external load (parameter λ_c^*) for given parameters of Winkler elastic foundation (K^*, l_c^*, l_d^*) and parameter of the loading head R^* was presented in Figure 2.

$$\lambda_c^* = \frac{Pl_1^2}{\sum_{n=1}^{3}(EJ)_n}, \quad \Omega^* = \frac{\sum_{n=1}^{3}(\rho A)_n l_n^4 \omega^2}{\sum_{n=1}^{3}(EJ)_n}, \quad K^* = \frac{Kl_1^{\,4}}{\sum_{n=1}^{3}(EJ)_n}, \quad R^* = \frac{R - l_0}{l_1}, \tag{38a-d}$$

Depending on the external load, the curve representing eigenvalues may be positively or negatively inclined to the axis of ordinates. In order to compare the results, besides the trajectories of curves relating to the frequencies of vibrations of **NW** system, the scope of changes in frequencies for **L** and **N** columns were presented too. The geometrically nonlinear column **N** is built of three rods (two external and one internal rods). The physical model of geometrically nonlinear column is identical as the physical model of **NW** column (without considering the Winkler elastic foundation). The linear column **L** consists of two external rods ①,② of the column **N-** the flexural stiffness of these elements is the same as the flexural stiffness of external rods of the nonlinear column **N** for given value of the asymmetry of flexural stiffness coefficient μ. The linear system is treated in this paper as a comparative system.

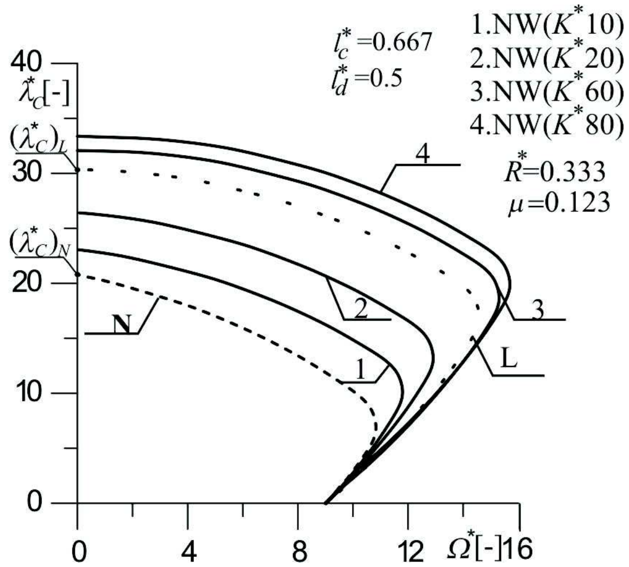

Figure 2. The curves representing the first frequency of free vibrations of the **NW**, **N** and **L** columns **-** the changes in frequencies of nonlinear and linear systems

The value of the bifurcational load parameter λ_c^* for each of the courses of curves of the free vibrations occurs while $\Omega^* = 0$. On the basis of the obtained results it has been proved that there are such a parameters of Winkler elastic foundation, for which the system can "exit" from the range of local loss of stability ($\lambda_c^* > \lambda_L^*$, compare curves 3, 4 in Figure 2.). The detailed results of studies were presented during the 27th Symposium on vibrations in physical systems.

5. Conclusions

The analysis of the free vibrations of the geometrically nonlinear column **NW** locally resting on the Winkler elastic foundation subjected to the follower force directed towards the positive pole was the subject of this paper. The Winkler elastic foundation, its length and location effect on the value of frequency of free vibrations of the examined system. Consideration of the Winkler elastic foundation in the physical model of the column causes an increase in the value of frequency of vibrations of the system. The value of bifurcational load is rising with the increase in elastic base stiffness. The elastic foundation of sufficiently large stiffness causes an "exit" from the range of the local loss of stability.

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