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EIGEN VALUE APPROACH IN MICROPOLAR ELASTIC MEDIUM *WITH VOIDS*

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The eigen value approach, following the Laplace and Hankel transformation has been employed to find a general solution of the field equations in a micropolar elastic medium with voids for an axisymmetric problem. An infinite space with the mechanical source has been applied to illustrate the utility of the approach. The integral transformations has been inverted by using a numerical inversion technique to get the result in physical domain. The results in the form of normal displacement, volume fraction, normal force stress, tangential force stress and tangential couple stress components have been obtained numerically and illustrated graphically.

Key words: eigen value, voids, Laplace transform, Hankel transform, concentrated force, Romberg's integration.

1. Introduction

The classical theory of elasticity successfully explains the behavior of construction materials (various sorts of steel, aluminum, concrete) provided the stresses do not exceed the elastic limit and no stress concentration occurs. The linear theory of elastic materials with voids is one of the generalizations of the classical theory of elasticity. This theory has practical utility to investigate various type of geological, biological and synthetic porous materials for which the elastic theory is inadequate. It is concerned with elastic materials consisting of a distribution of small pores (voids), in which the void's volume is included among the kinematic variables, and in the limiting case when the volume tends to zero this theory reduces to the classical theory of elasticity. The process of voids is known to affect the estimations of the physical-mechanical properties of the composite and also to weaken the bond as these pores (voids) get spread over a wide area.

It is commonly accepted that the mechanical behavior of the granular masses is strongly affected by their microstructure, namely the relative arrangement of the voids and particles, i.e., the granular fabric. Therefore parameters which characterize the granular mass are of paramount importance in the fundamental description of overall macroscopic stresses and deformation measures. The study of deformations in granular materials is important in many areas of science and technology such as powder metallurgy and earth quake engineering. In recent years, dynamic compaction of powders has been used to manufacture advance composites. A granular medium is composed of a large number of distinct particles as well as some heterogeneous inclusions. Voids may be filled with gas or liquid at the boundaries of discontinuity and a mismatch on the incident wave will produce both transmission and reflection waves of different modes. Wave propagation phenomenon in such media not only depends on the microstructure but on the existence of inclusions and voids.

A non-linear theory of elastic materials with voids was developed by Nunziato ans Cowin (1979). Later Cowin and Nunziato (1983) developed a theory of linear elastic materials with voids. Lewis and Isaak (1982) discussed the voids of minimum stress concentration. Later Nunziato and Cowin (1983) developed a

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theory of linear elastic materials with voids, for a mathematical study of mechanical behavior of porous solids. Pouget and Maugin (1983) established non-homogeneous elastoacoutic equations for piezoelectric powders and discussed the continuum approach to elastoacoutic echoes in piezoelectric powders. Puri and Cowin (1985) studied the behavior of plane harmonic waves in linear elastic materials with voids. Chandersekharahaiah (1986) studied plane waves in the rotating elastic solids with voids. The problem of complete solution in the theory of isotropic elastic materials with voids was discussed by Chandersekharahaiah (1989). Scarpetta (1995) proved some theorem of uniqueness for linear elastic materials with voids.

The particles of classical elastic materials have only translational degree of freedom and transmission of the load across the differential element of the surface is described by a force vector only, whereas the polycrystalline materials do not confirm this. These materials are fibrous and composite in nature, show size effect and have additional micro-deformational degree of freedom, i.e., they possess a microstructure whose size cannot be neglected in comparisons with the length scale of interest. Various degrees of freedom of a microstructure were considered by different authors, e.g., Cosserat and Cosserat (1909), Eringen and Suhubi (1964) and Mindlin (1964). The force at any point of surface elements of the body of these materials is completely characterized by a stress vector and a couple stress vector at that point. In the classical theory of elasticity, the effect of couple stress is neglected. Eringen (1966) modified his earlier theory and renamed it as the "Linear Theory of Micropolar Elasticity".

Iesan (1985) studied shock waves in micropolar elastic materials with voids. Scarpetta (1990) worked on the fundamental solution in micropolar elasticity with voids. Marin (1996) showed the existence and uniqueness of solution for boundary value problems in elasticity of micropolar materials with voids. Marin (1996) discussed generalized solutions in elasticity of micropolar bodies with voids. Marin (1998) derived a temporally evolutionary equation in a micropolar elastic body with voids.

2. Formulation and solution of the problem

We consider a homogeneous, isotropic micropolar elastic solid with voids. We take a cylindrical polar co-ordinates system (r, θ, z) where the z-axis is pointing into the medium. Following Eringen(1968) and Iesan (1985) field equations and constitutive relations in the micropolar elastic solid with voids in the absence of body loads are given by

$$\beta^*\nabla q+\left(\lambda+2\mu+K\right)\nabla(\nabla\cdot\boldsymbol{u})-(\mu+K)\times\nabla\times\nabla\times\boldsymbol{u}+K\nabla\times\varphi=\rho\frac{\partial^2\boldsymbol{u}}{\partial t^2}, \tag{2.1}$$

$$(\alpha+\beta+\gamma)\nabla(\nabla.\varphi)-\gamma\nabla\times\nabla\times\varphi+K\nabla\times\varphi-2K\varphi=\rho j\frac{\partial^2\varphi}{\partial t^2}, \tag{2.2}$$

$$\alpha^*\nabla^2 q^*-\zeta^* q^*-\omega^*\frac{\partial q^*}{\partial t}-\beta^*\nabla\cdot\boldsymbol{u}=\rho K^*\frac{\partial^2 q^*}{\partial t^2}, \tag{2.3}$$

$$t_{ij}=\left(\beta^* q^*+\lambda u_{rr}\right)\delta_{ij}+\mu\left(u_{i,j}+u_{j,i}\right)+K\left(u_{i,j}-\varepsilon_{ijr}\varphi_r\right), \tag{2.4}$$

$$m_{ij}=\alpha\varphi_{rr}\delta_{ji}+\beta\varphi_{j,i}+\gamma\varphi_{j,i}. \tag{2.5}$$

Since we are considering a two-dimensional axisymmetric problem, so we assume the components of the displacement vector $\boldsymbol{u}$ and microrotation vector φ are of the form

$$\boldsymbol{u}=\left(u_r,0,u_z\right)\qquad \varphi=\left(0,\varphi_\theta,0\right), \tag{2.6}$$

Here due to symmetry about the z-axis all the quantities are independent of θ.

Using Eqs (2.6) the set of Eqs (2.1)-(2.3) reduces to

$$\beta^* \frac{\partial q^*}{\partial r} + (\lambda+\mu)\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_r}{\partial r \partial z}\right) - K\frac{\partial \varphi_\theta}{\partial z} + (K+\mu)\left(\nabla^2 u_r - \frac{u_r}{r^2}\right) = \rho\frac{\partial^2 u_r}{\partial t^2}, \quad (2.7)$$

$$\beta^* \frac{\partial q^*}{\partial r} + (\lambda+\mu)\left(\frac{\partial^2 u_z}{\partial z^2} + \frac{1}{r}\frac{\partial u_r}{\partial z} + \frac{\partial^2 u_r}{\partial r \partial z}\right) - K\frac{\partial \varphi_\theta}{\partial z} + (K+\mu)\nabla^2 u_z + \frac{K}{r}\frac{\partial(r\varphi_\theta)}{\partial r} = \rho\frac{\partial^2 u_z}{\partial t^2}, \quad (2.8)$$

$$\gamma\left(\nabla^2\varphi_\theta - \frac{\varphi_\theta}{r^2}\right) + K\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) - 2K\varphi_\theta = \rho j\frac{\partial^2 \varphi_\theta}{\partial t^2}, \quad (2.9)$$

$$\alpha^*\nabla^2 q^* - \zeta^* q^* - \omega^*\frac{\partial q^*}{\partial t} - \beta^*\left(\frac{\partial u_z}{\partial z} + \frac{\partial u_r}{\partial r}\right) = \rho K^*\frac{\partial^2 q^*}{\partial t^2} \quad (2.10)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \quad (2.11)$$

Introducing dimensionless quantities

$$r' = \frac{\omega_1}{c_1}r, \qquad z' = \frac{\omega_1}{c_1}z, \qquad u'_r = \frac{\omega_1}{c_1}u_r, \qquad u'_z = \frac{\omega_1}{c_1}u_z, \qquad \varphi'_\theta = \frac{\omega_1^2}{c_1^2}\varphi_\theta, \qquad t' = \omega_1 t,$$

$$q^{*\prime} = \frac{\omega_1^2}{c_1^2}q^*, \qquad t'_{zz} = \frac{t_{zz}}{\mu}, \qquad t'_{zr} = \frac{t_{zr}}{\mu}, \qquad m'_{z\theta} = \frac{\omega_1}{c_1}m_{z\theta}$$

where $$c_1^2 = \frac{(\lambda + 2\mu + K)}{\rho}, \qquad \omega^{*\prime} = \frac{K}{\rho j}. \quad (2.12)$$

After suppressing the primes for convenience Eqs (2.7)-(2.10) reduce to

$$s_0\frac{\partial q^*}{\partial r} + \left(\frac{\partial^2 u_r}{\partial r^2} + \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_r}{\partial r \partial z}\right) - s_1\left(\nabla^2 u_r - \frac{u_r}{r^2}\right) - s_2\frac{\partial \varphi_\theta}{\partial z} = s_3\frac{\partial^2 u_r}{\partial t^2}, \quad (2.13)$$

$$s_0\frac{\partial q^*}{\partial z} + \left(\frac{\partial^2 u_r}{\partial z^2} + \frac{1}{r}\frac{\partial u_r}{\partial z} + \frac{\partial^2 u_r}{\partial r \partial z}\right) - s_1\nabla^2 u_z - \frac{s_2}{r}\frac{\partial(r\varphi_\theta)}{\partial r} = s_3\frac{\partial^2 u_z}{\partial t^2}, \quad (2.14)$$

$$\left(\frac{\partial^2 \varphi_\theta}{\partial r^2} + \frac{\partial \varphi_\theta}{\partial r} - \frac{\varphi_\theta}{r^2} + \frac{\partial^2 \varphi_\theta}{\partial r \partial z}\right) - s_4\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) - s_5\varphi_\theta = s_6\frac{\partial^2 \varphi_\theta}{\partial t^2}, \quad (2.15)$$

$$\nabla^2 q^* - s_7 q^* - s_8\frac{\partial q^*}{\partial t} - s_9\left(\frac{\partial u_z}{\partial z} + \frac{\partial u_r}{\partial r}\right) = s_{10}\frac{\partial^2 q^*}{\partial t^2} \quad (2.16)$$

where

$$s_0 = \frac{\beta^* c_1^2}{(\lambda+\mu) j\omega_1^2}, \qquad s_1 = \frac{(K+\mu)}{(\lambda+\mu)}, \qquad s_2 = \frac{K c_1^2}{(\lambda+\mu) j\omega_1^2}, \qquad s_3 = \frac{\rho c_1^2}{(\lambda+\mu)},$$

$$s_4 = \frac{Kj}{\gamma}, \qquad s_5 = \frac{2Kc_1^2}{\gamma\omega_1^2}, \qquad s_7 = \frac{\zeta^* c_1^2}{\alpha^*\omega_1^2}, \qquad s_8 = \frac{\omega^* c_1^2}{\alpha^*\omega_1}, \tag{2.17}$$

$$s_9 = \frac{\beta^* j}{\alpha^*}, \qquad s_{10} = \frac{K^*\rho c_1^2}{\alpha^*}.$$

Applying the Laplace and Hankel transforms defined by

$$\overline{f}(r,z,p) = L\{f(r,z,t)\} = \int_0^\infty f(r,z,t)\exp(-pt)dt\ , \tag{2.18}$$

$$\tilde{f}(\xi,z,p) = H\left\{\overline{f}(x,z,p)\right\} = \int_0^\infty r\overline{f}(x,z,p)\, J_n(\xi\, r)dr\ , \tag{2.19}$$

on Eqs (2.13)-(2.16), we obtain

$$\frac{d^2\tilde{u}_r}{dz^2} = \frac{\left(\xi^2 + s_1\xi^2 + s_1 p^2\right)}{s_1}\tilde{u}_r + \frac{\xi}{s_1}\frac{d\tilde{u}_z}{dz} + \frac{s_2}{s_1}\frac{d\tilde{\varphi}_\theta}{dz} + \frac{\xi s_0}{s_1}\tilde{q}^*, \tag{2.20}$$

$$\frac{d^2\tilde{u}_z}{dz^2} = \frac{\left(s_1\xi^2 + s_3 p^2\right)}{1+s_1}\tilde{u}_z - \frac{\xi}{1+s_1}\frac{d\tilde{u}_r}{dz} - \frac{s_0}{1+s_1}\frac{d\tilde{q}^*}{dz} - \frac{\xi s_2}{1+s_1}\tilde{\varphi}_\theta, \tag{2.21}$$

$$\frac{d^2\tilde{\varphi}_\theta}{dz^2} = -s_4\frac{d\tilde{u}_r}{dz} - \xi s_4\tilde{u}_z + \left(\xi^2 + s_5 + s_6 p^2\right)\tilde{\varphi}_\theta, \tag{2.22}$$

$$\frac{d^2\tilde{q}^*}{dz^2} = s_9\frac{d\tilde{u}_r}{dz} - \xi s_9\tilde{u}_r + \left(\xi^2 + s_7 + s_8 p + s_{10} p^2\right)\tilde{q}^*\ . \tag{2.23}$$

The system of Eqs (2.20)-(2.23) can be written as

$$\frac{dW(\xi,z,p)}{dz} = A(\xi,p)W(\xi,z,p) \tag{2.24}$$

where

$$W = \begin{bmatrix} U \\ DU \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & I \\ A_2 & A_1 \end{bmatrix}, \qquad U = \begin{bmatrix} \tilde{u}_r \\ \tilde{u}_z \\ \tilde{\varphi}_\theta \\ \tilde{q}^* \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & a_{12} & a_{13} & 0 \\ a_{21} & 0 & 0 & a_{24} \\ a_{31} & 0 & 0 & 0 \\ 0 & a_{24} & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} b_{11} & 0 & 0 & b_{14} \\ 0 & b_{22} & b_{23} & 0 \\ 0 & b_{32} & b_{33} & 0 \\ b_{41} & 0 & 0 & b_{44} \end{bmatrix},$$

$$a_{12} = \frac{\xi}{s_1}, \qquad a_{13} = \frac{s_2}{s_1}, \qquad a_{21} = -\frac{\xi}{1+s_1}, \qquad a_{24} = \frac{-s_0}{1+s_1}, \qquad D = \frac{d}{dz},$$

$$a_{31} = -s_4, \qquad a_{42} = s_9, \qquad b_{11} = \frac{\left(\xi^2 + s_1\xi^2 + s_1 p^2\right)}{s_1}, \qquad b_{14} = \frac{s_0\xi^2}{s_1}, \tag{2.25}$$

$$b_{22} = \frac{\left(s_1\xi^2 + s_3 p^2\right)}{1+s_1}, \qquad b_{23} = \frac{-s_2\xi^2}{1+s_1}, \qquad b_{32} = -s_4\xi^2, \qquad b_{33} = \left(\xi^2 + s_5 + s_6 p^2\right),$$

$$b_{44} = \left(\xi^2 + s_7 + s_8 p + s_{10} p^2\right), \qquad b_{41} = -s_9\xi$$

where O and I are null and unit matrices of order 4.

To solve Eq.(2.24), we take

$$W(\xi, z, p) = W(\xi, p)e^{qz}. \tag{2.26}$$

For some parameter q, we obtain

$$A(\xi, p)W(\xi, z, p) = qW(\xi, z, p), \tag{2.27}$$

which leads to the eigen value problem. The characteristic equation corresponding to the matrix A is given by

$$\det\left(A - qI\right) = 0. \tag{2.28}$$

Expanding (2.28) gives

$$q^8 - \sigma_1 q^6 + \sigma_2 q^4 - \sigma_3 q^2 + \sigma_4 = 0 \tag{2.29}$$

where

$$\sigma_1 = b_{11} + b_{22} + b_{33} + b_{44} + a_{24}a_{42} + a_{12}a_{21} + a_{13}a_{31}, \tag{2.30}$$

$$\begin{aligned} \sigma_2 = {} & b_{11}b_{22} + b_{22}b_{33} + b_{11}b_{33} + b_{11}b_{44} + b_{22}b_{44} + b_{33}b_{44} - b_{41}a_{24}a_{12} \\ & + b_{11}a_{24}a_{42} + b_{33}a_{24}a_{42} + a_{24}a_{42}a_{13}a_{31} - b_{32}b_{23} + a_{12}a_{21}b_{33} + a_{12}a_{21}b_{44} \\ & - a_{12}a_{31}b_{23} - a_{31}a_{21}b_{32} + a_{13}a_{31}b_{44} - b_{14}b_{41} - b_{14}a_{21}a_{42}, \end{aligned} \tag{2.31}$$

$$\begin{aligned} \sigma_3 = {} & b_{11}b_{22}b_{33} + b_{22}b_{33}b_{44} + b_{11}b_{33}b_{44} + b_{11}b_{22}b_{44} - b_{11}b_{23}b_{32} - b_{44}b_{23}b_{32} + \\ & a_{42}a_{42}b_{11}b_{33} + a_{12}a_{24}b_{41}b_{32} + a_{31}a_{42}b_{41}b_{23} - a_{31}a_{42}b_{14}b_{23} + a_{12}a_{21}b_{33}b_{44} \\ & - a_{13}a_{21}b_{32}b_{45} - a_{12}a_{31}b_{23}b_{44} - a_{12}a_{24}b_{41}b_{33} - a_{42}a_{21}b_{14}b_{33} + a_{13}a_{31}b_{22}b_{44} \\ & - b_{14}b_{41}b_{22} - b_{14}b_{41}b_{33}, \end{aligned} \tag{2.32}$$

$$\sigma_4 = b_{11}b_{22}b_{33}b_{44} - b_{11}b_{44}b_{23}b_{32} - b_{14}b_{41}b_{22}b_{33} + b_{14}b_{41}b_{23}b_{32}. \tag{2.33}$$

The eigen values of the matrix A are characteristic roots of Eq.(2.29).The eigen vectors $X(\xi, p)$ corresponding to the eigen value q_s can be determined by solving the homogenous equations

$$[A - qI]\, X(\xi, p) = 0. \tag{2.34}$$

The set of eigen vectors $X_s(\xi, p)$ $s=1, 2, 3... 8$ may be defined as

$$X_s(\xi, p) = \begin{bmatrix} X_{s1}(\xi, p) \\ X_{s2}(\xi, p) \end{bmatrix} \tag{2.35}$$

where

$$X_{s1} = \begin{bmatrix} a_s q_s \\ b_s \\ -\xi \\ c_s \end{bmatrix}, \qquad X_{s1} = \begin{bmatrix} a_s q_s^2 \\ b_s q_s \\ -\xi q_s \\ c_s q_s \end{bmatrix}, \qquad q = q_s; \quad s = 1, 2, 3, 4 \tag{2.36}$$

$$X_{l1} = \begin{bmatrix} a_s q_s \\ b_s \\ -\xi \\ c_s \end{bmatrix}, \qquad X_{l2} = \begin{bmatrix} a_s q_s^2 \\ b_s q_s \\ -\xi q_s \\ c_s q_s \end{bmatrix}, \qquad l = s + 4, \quad q = -q_s; \quad s = 1, 2, 3, 4 \tag{2.37}$$

$$\begin{aligned} a_s = \frac{\xi}{\Delta_s}\Big[&\left(\xi^2 + s_7 + s_8 p + s_{10} p^2\right)\left\{\left(\xi^2 + s_5 + s_6 p^2\right) + s_2 s_4 - q_s^2\right\} + \\ &- s_2 s_4 q_s^2 - \left(\xi^2 + s_5 + s_6 p^2 - q_s^2\right)\left(s_0 s_9 - q_s^2\right)\Big], \end{aligned} \tag{2.38}$$

$$\begin{aligned} b_s = \frac{-1}{\Delta_s}\Big[&\left(\xi^2 + s_7 + s_8 p + s_{10} p^2 - q_s^2\right)\left\{\left(\xi^2 + s_1\xi^2 + s_3 p^2 - s_1 q_s^2\right)\left(\xi^2 + s_5 + s_6 p^2 - q_s^2\right) + \right. \\ &\left. + s_2 s_4 q_s^2\right\} - s_0 s_9 \xi^2\left(\xi^2 + s_5 + s_6 p^2 - q_s^2\right)\Big], \end{aligned} \tag{2.39}$$

$$c_s = \frac{s_9\left(\xi a_s + b_s\right)}{\left\{q_s^2 - \left(\xi^2 + s_7 + s_8 p + s_{10} p^2\right)\right\}}, \tag{2.40}$$

$$\Delta_s = s_4\left[s_0 s_9\left(q_s^2 - \xi^2\right) - \left(\xi^2 + s_7 + s_8 p + s_{10} p^2 - q_s^2\right)\left\{q_s^2 - \left(\xi^2 + s_1\xi^2 + s_3 p^2 - s_1 q_s^2\right)\right\}\right],$$

$$s = 1, 2, 3, 4. \tag{2.41}$$

Thus the solution of Eq.(2.24) as given by Sharma and Chand (1992) is

$$W(\xi,z,p)=\sum_{s=1}^{4}\left[E_s X_s \exp(q_s z)+E_{s+4}X_{s+4}\exp(-q_s z)\right] \tag{2.42}$$

where $E_1, E_2, E_3, E_4, E_5, E_6, E_7$ and E_8 are eight arbitrary constants. Equation (2.42) represents the solution of the general problem for the axisymmetric case of a micropolar elastic medium with voids and gives displacement, microrotation and volume fraction components in the transformed domain.

3. Application

Case-I Mechanical normal point source

We consider an infinite micropolar elastic space with voids in which a concentrated force $F=-F_0\dfrac{\delta(r)\delta(t)}{2\pi r}$ where F_0 is the magnitude of the force, acting in the direction of the *z*-axis at the origin of the cylindrical polar co-ordinate system as shown in Fig.1. The boundary condition for present problem on the plane $z=0$ are

$$u_r(r,0^+,t)-u_r(r,0^-,t)=0, \qquad u_z(r,0^+,t)-u_z(r,0^-,t)=0, \tag{3.1}$$

$$\varphi_\theta(r,0^+,t)-\varphi_\theta(r,0^-,t)=0, \qquad q^*(r,0^+,t)-q^*(r,0^-,t)=0, \tag{3.2}$$

$$t_{zr}\left(r,0^+,t\right)-t_{zr}\left(r,0^-,t\right)=0, \qquad t_{zz}\left(r,0^+,t\right)-t_{zz}\left(r,0^-,t\right)=-F_0\frac{\delta(r)\delta(t)}{2\pi r}, \tag{3.3}$$

$$m_{z\theta}\left(r,0^+,t\right)-m_{z\theta}\left(r,0^-,t\right)=0, \qquad \frac{\partial q^*\left(r,0^+,t\right)}{\partial z}-\frac{\partial q^*\left(r,0^-,t\right)}{\partial z}=0. \tag{3.4}$$

Making use of Eqs (2.6) and (2.12) on Eqs (2.4)-(2.5) and $F_0'=\dfrac{F_0}{K}$, we get the stresses in the non-dimensional form with primes. After suppressing the primes and applying the Laplace and Hankel transforms defined by Eqs (2.18) and (2.19) on the resulting equations and from Eqs (3.1)-(3.4) we get the transformed components of displacement, microrotation, volume fraction, tangential force stress, normal force stress and tangential couple stress for $z>0$ which are given by

$$\tilde{u}_r(\xi,z,p)=\left\{a_1q_1E_5e^{-q_1z}+a_2q_2E_6e^{-q_2z}+a_3q_3E_7e^{-q_3z}+a_4q_4E_8e^{-q_4z}\right\}, \tag{3.5}$$

$$\tilde{u}_z(\xi,z,p)=b_1E_5e^{-q_1z}+b_2E_6e^{-q_2z}+b_3E_7e^{-q_3z}+b_4E_8e^{-q_4z}, \tag{3.6}$$

$$\tilde{\varphi}_\theta(\xi,z,p)=-\xi\left\{E_5e^{-q_1z}+E_6e^{-q_2z}+E_7e^{-q_3z}+E_8e^{-q_4z}\right\}, \tag{3.7}$$

$$\tilde{q}^*(\xi,z,p)=c_1E_5e^{-q_1z}+c_2E_6e^{-q_2z}+c_3E_7e^{-q_3z}+c_4E_8e^{-q_4z}, \tag{3.8}$$

$$\tilde{t}_{zr}(\xi,z,p)=\left(s_{11}a_1q_1^{\,2}-\xi\, b_1+\xi\, s_{12}\right)E_5e^{-q_1z}+\left(s_{11}a_2q_2^{\,2}-\xi b_2+\xi s_{12}\right)E_6e^{-q_2z}+$$
$$+\left(s_{11}a_3q_3^{\,2}-\xi\, b_3+\xi\, s_{12}\right)E_7e^{-q_3z}+\left(s_{11}a_4q_4^{\,2}-\xi b_4+\xi s_{12}\right)E_8e^{-q_4z}, \tag{3.9}$$

$$\tilde{t}_{zz}(\xi,z,p)=\Big[\left(s_{13}c_1-s_{14}\xi a_1q_1+s_{15}b_1q_1\right)E_5e^{-q_1z}+\left(s_{13}c_2-s_{14}\xi a_2q_2+s_{15}b_2q_2\right)E_6e^{-q_2z}+$$
$$+\left(s_{13}c_3-s_{14}\xi a_3q_3+s_{15}b_3q_3\right)E_7e^{-q_3z}+\left(s_{13}c_4-s_{14}\xi a_4q_4+s_{15}b_4q_4\right)E_8e^{-q_4z}\Big], \tag{3.10}$$

$$\tilde{m}_{z\theta}(\xi,z,p)=\xi\left\{q_1E_1e^{-q_1z}+q_2E_6e^{-q_2z}+q_3E_7e^{-q_3z}+q_4E_8e^{-q_4z}\right\}, \tag{3.11}$$

for $z<0$, the above expression gets modified as

$$\tilde{u}_r(\xi,z,p)=\left\{a_1q_1E_1e^{-q_1z}+a_2q_2E_2e^{-q_2z}+a_3q_3E_3e^{-q_3z}+a_4q_4E_4e^{-q_4z}\right\}. \tag{3.12}$$

Making use of the transformed displacement, microrotation, volume fraction and stress components given by Eqs (3.5)-(3.12) in the boundary conditions, we obtain eight linear relations between $E_1, E_2, E_3, E_4, E_5, E_6, E_7$ and E_8, which on solving give

$$E_1=E_5=\frac{F_0}{4\pi q_1\Delta_1}\left[c_2(a_3-a_4)+c_3(a_4-a_2)+c_4(a_2-a_3)\right], \tag{3.13}$$

$$E_2=E_6=\frac{F_0}{4\pi q_2\Delta_1}\left[c_1(a_4-a_3)+c_3(a_1-a_4)+c_4(a_3-a_1)\right], \tag{3.14}$$

$$E_3=E_7=\frac{F_0}{4\pi q_3\Delta_1}\left[c_1(a_2-a_4)+c_2(a_4-a_1)+c_4(a_1-a_2)\right], \tag{3.15}$$

$$E_4=E_8=\frac{F_0}{4\pi q_4\Delta_1}\left[c_1(a_3-a_2)+c_2(a_1-a_3)+c_3(a_2-a_1)\right] \tag{3.16}$$

where

$$\Delta_1=s_{15}\Big[c_1\left\{(a_2b_3-a_3b_2)+(a_3b_4-a_4b_3)+(a_4b_2-a_2b_4)\right\}+$$
$$+c_2\left\{(a_3b_1-a_1b_3)+(a_1b_4-a_4b_1)+(a_4b_3-a_3b_4)\right\}+$$
$$+c_3\left\{(a_1b_2-a_2b_1)+(a_4b_1-a_1b_4)+(a_2b_4-a_4b_2)\right\}+$$
$$+c_4\left\{(a_2b_1-a_1b_2)+(a_1b_3-a_3b_1)+(a_3b_2-a_2b_3)\right\}\Big]. \tag{3.17}$$

Thus the functions $\tilde{u}_1, \tilde{u}_3, \tilde{\varphi}_2, \tilde{T}, \tilde{t}_{31}, \tilde{t}_{33}$ and $\tilde{m}_{32}$ have been determined in the transform domain and thy enable us to find the displacements, microrotation, temperature field and stresses.

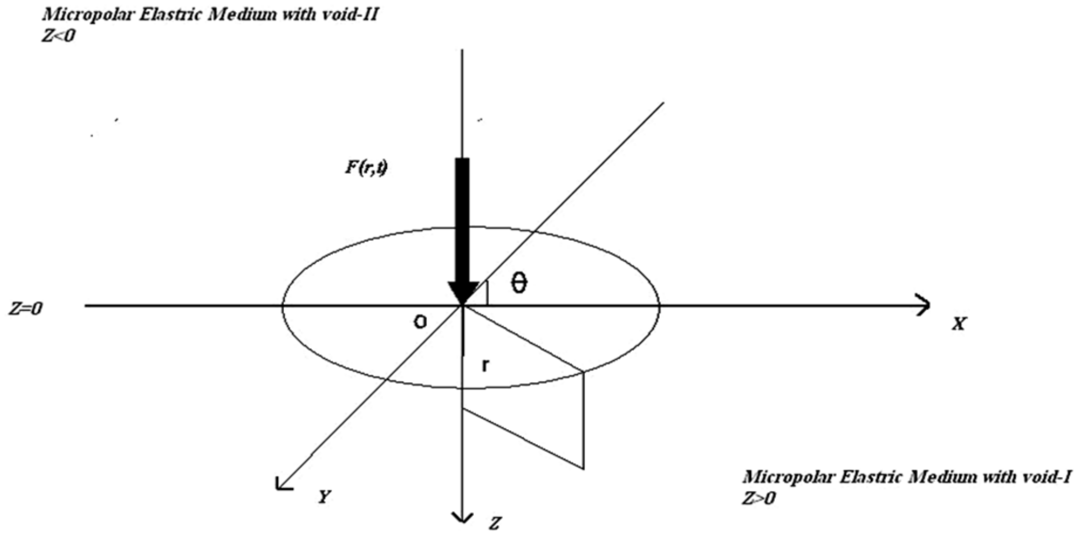

Fig.1. Geometry of the problem.

Case-II Mechanical continuous normal point source

When plane boundary is subjected to continuous normal point force, the boundary conditions at the interface $z=0$ are

$$u_r(r,0^+,t)-u_r(r,0^-,t)=0, \qquad u_z(r,0^+,t)-u_z(r,0^-,t)=0, \tag{3.18}$$

$$\varphi_\theta(r,0^+,t)-\varphi_\theta(r,0^-,t)=0, \qquad q^*(r,0^+,t)-q^*(r,0^-,t)=0, \tag{3.19}$$

$$t_{zr}(r,0^+,t)-t_{zr}(r,0^-,t)=0, \qquad t_{zz}(r,0^+,t)-t_{zz}(r,0^-,t)=-F_0\frac{\delta(r)H(t)}{2\pi r}, \tag{3.20}$$

$$m_{z\theta}(r,0^+,t)-m_{z\theta}(r,0^-,t)=0, \qquad \frac{\partial q^*(r,0^+,t)}{\partial z}-\frac{\partial q^*(r,0^-,t)}{\partial z}=0, \tag{3.21}$$

where F_0 is the magnitude of the force applied and H(t) is the Heaviside distribution.

With the help of these boundary conditions (3.18)-(3.21), the expressions for the components of displacement, force stress, couple stress and volume fraction field are given by Eqs (3.5)–(3.12) after replacing Δ_1 with Δ_1/p.

Particular Case I: Neglecting the influence of the voids i.e., $\left(\alpha^*=\beta^*=\zeta^*=\omega^*=K^*=0\right)$; the expressions for the displacement components, force stresses and couple stress, are obtained in a micropolar elastic medium.

Particular Case II: If the effect of micropolarity is ignored i.e., $\left(K=j=\alpha=\beta=\gamma=0\right)$ expressions for the displacement components, force stresses and volume fraction field are obtained in an elastic medium with voids.

Particular Case III: If the effect of micropolarity and voids is neglected i.e., $\left(K=j=\alpha=\beta=\gamma=0\right)$ and $\left(\alpha^*=\beta^*=\zeta^*=\omega^*=K^*=0\right)$; we obtain expression for the displacement components and force stresses in an elastic medium. Again the resulting expressions matches with those obtained by Achenbach (1973) with the change of notations to be the same as used by the author.

4. Method for the inversion of transforms

The transformed solutions are functions of the form $\tilde{f}(\xi, z, p)$ and to get the function $f(r, z, p)$, first we invert the Hankel transform by using

$$\bar{f}(\xi, z, p) = \int_0^\infty \xi \tilde{f}(\xi, z, p) J_n(\xi r) d\xi . \tag{4.1}$$

The expression (4.1) gives us the Laplace transform $\bar{f}(\xi, z, p)$ of the function $\bar{f}(\xi, z, p)$. Now for the fixed values of r and z the function $\bar{f}(\xi, z, p)$ can be considered as the Laplace transform $\bar{g}(p)$ of some function $g(t)$. Following Honig and Hirdes (1984), the Laplace transformed function $\bar{g}(p)$ can be inverted numerically as given below.

The function $g(t)$ can be obtained from $\bar{g}(p)$ by using the inversion formula

$$g(t) = \frac{1}{2\iota\pi} \int_{c+\iota\infty}^{c+\iota\infty} e^{pt} \bar{g}(p) d\xi \tag{4.2}$$

where C is an arbitrary real number greater than all the real parts of the singularities of $\bar{g}(p)$. The actual procedure to invert the Laplace transform consists of Eq.(4.2) together with the ε -algorithm. The values of C and L are chosen according to the criteria outlined by Honig and Hirdes (1984).

The last step is to calculate the integral in Eq.(4.1). The method for evaluating this integral is described by Press *et al.* (1986). The method involves the use of Romberg's integration with an adaptive step size. It also uses the results from successive refinements of the extended trapezoidal rule followed by extrapolation of the results to the limit when the step size tends to zero.

Numerical results and discussion

Following Eringen (1984), we take the following values of relevant parameters for the case of Magnesium crystal for physical constants as

$$\lambda = 9.4 \times 10^{10} N/m^2, \quad \mu = 4 \times 10^{10} N/m^2, \quad K = 1 \times 10^{10} N/m^2,$$

$$\rho = 1.74 \times 10^3 kg/m^3, \quad \gamma = 0.779 \times 10^{-9} N, \quad j = 0.2 \times 10^{-19} m^2,$$

$$\alpha^\# = 3.688 \times 10^{-9} N, \quad \beta^\# = 1.138494 \times 10^{10} N/m^2, \quad \zeta^\# = 1.1475 \times 10^{10} N/m^2,$$

$$\omega^\# = 0.0787 \times 10^{-1} N \times \sec/m^2, \quad K^\# = 1.1753 \times 10^{-19} m^2 .$$

The computations were carried out for non-dimensional time $t=0.1$ at $z=1$ in the range $0 \le r \le 8$. The variations in non-dimensional normal displacement $U_z \left(= 4\pi u_z / F_0\right)$, non-dimensional volume fraction

$Q\left(=4\pi q^{*}/F_0\right)$, non-dimensional normal stress $T_{zz}\left(=4\pi t_{zz}/F_0\right)$ and non-dimensional tangential couple stress $M_{z\theta}\left(=4\pi m_{z\theta}/F_0\right)$ with non-dimensional distance '*r*' are shown in Figs 1-9. The solid line gives the variation in component for a micropolar elastic solid with voids (MESV) whereas the very small dashe are for a micropolar elastic solid (MES), the line corresponds to the variations in micropolar elastic solid with void (ESV) and large dashed lines are for an elastic solid (ES).

5.1. Mechanical normal point source

The variations in normal displacement, volume fraction field, normal force stress and tangential couple stress with distance *r* for MESV, MES, EVS and ES when a mechanical normal point source is applied are shown in Figs 2, 3, 4,5, respectively.

Figure 2. The variations of normal displacement U_Z with *r* for all four theories (MESV, MES, ESV, ES) are shown in Fig.2 and it is observed that the behavior of U_Z for MES is opposite to MESV, ESV and ES. The values of U_Z decrease sharply as *r* lies between $0 \le r \le 3$ whereas for MES the values of U_Z increase in the same range. The magnitude of U_Z is largest for MESV and smallest for MES close to the point of action of source, as the source is very near to the point of action.

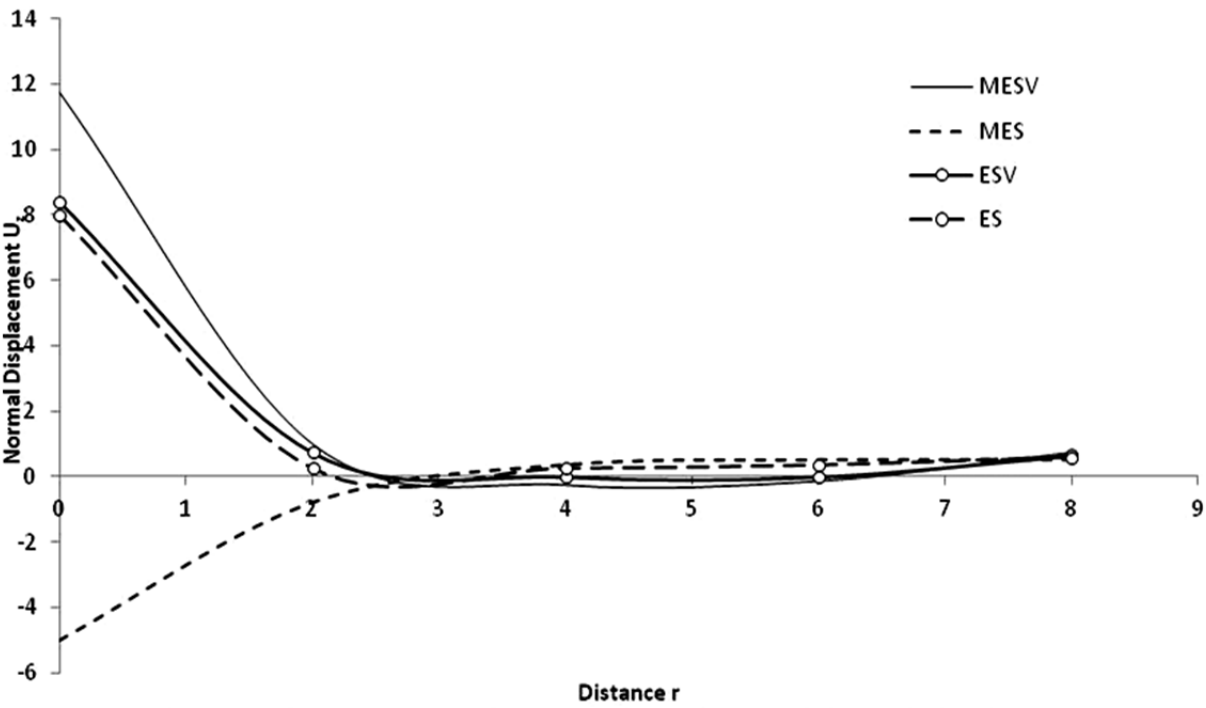

Fig.2. Variations of normal distance U_z $(=4\pi u_z/F_0)$ due to normal point source with distance *r*.

Figure 3 depicts the variations of the volume fraction Q with '*r*' for MESV and ESV. The values of Q decrease sharply in the range $0 \le r \le 2.5$ The value for MESV are greater than those for ESV in the initial range $0 \le r \le 2.5$ and then in the range $7 \le r \le 8$.

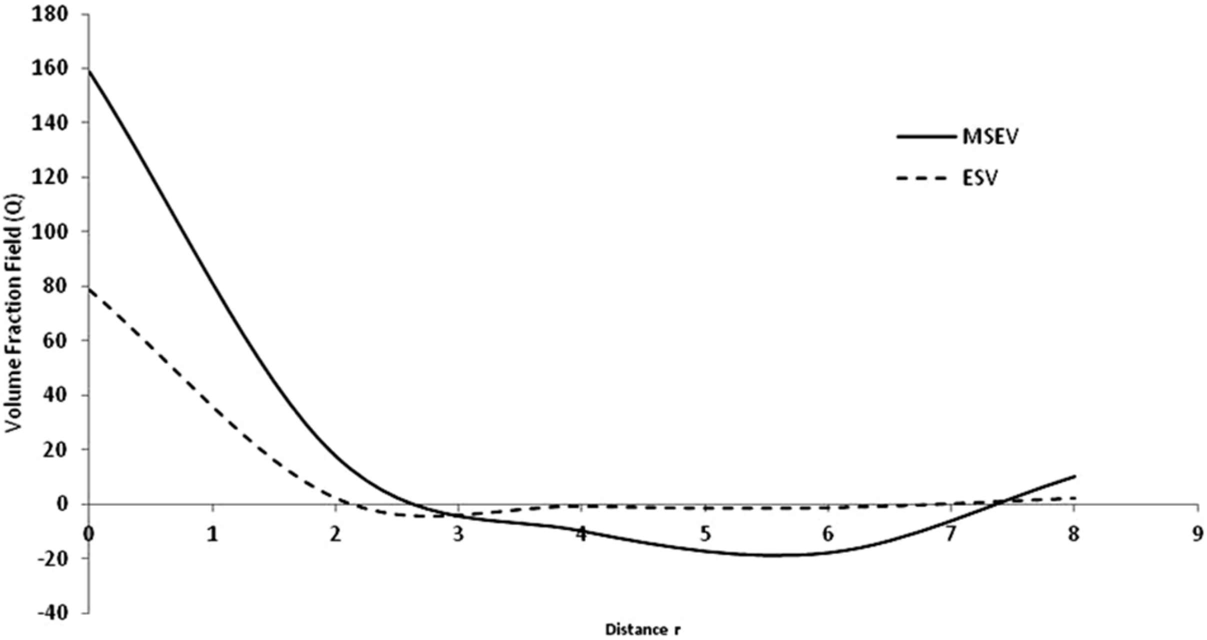

Fig.3. Variations of volume fraction field $Q\,(=4\pi q/F_0)$ due to normal point source with distance r.

The variations of normal force stress T_{ZZ} with r are shown in Fig.4. The value of T_{ZZ} start with a sharp decrease for the cases MESV, MES and ES whereas for the case of ESV it starts with a small increase. The magnitude of values of T_{ZZ} is largest for ES and smallest for MES.

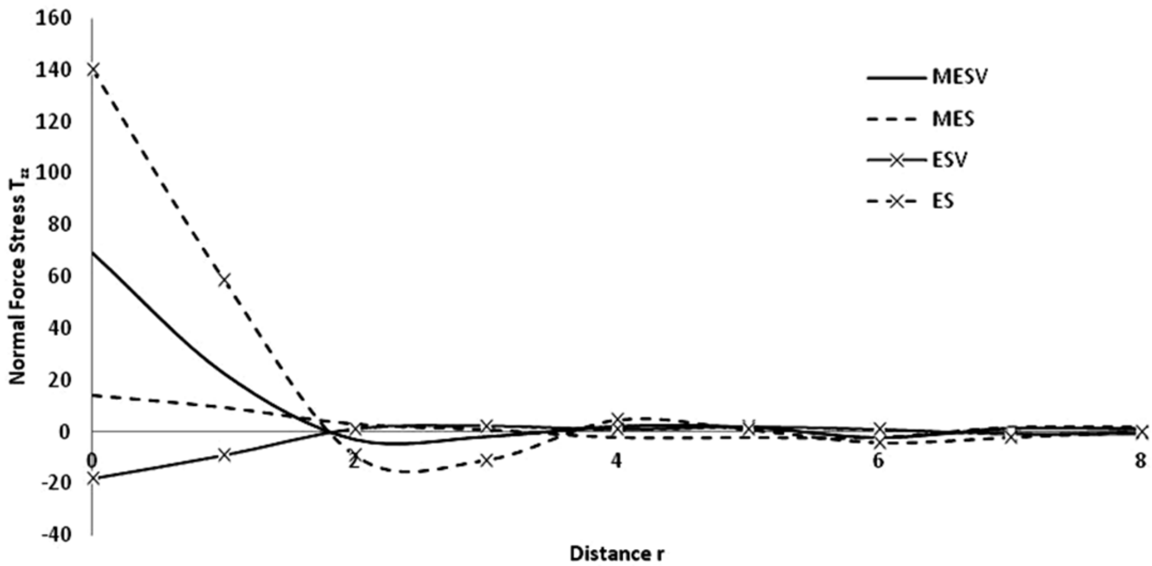

Fig.4. Variations of normal force stress T_{zz} $(=4\pi t_{zz}/F_0)$ due to normal point source with distance r.

Figure 4 depicts the variations of $M_{Z\theta}$ with r. Beginning with a small decrease in the range $0 \le r \le 1.5$, the value of the couple stress for MESV and MES starts to grow with a small variation. The behavior of $M_{Z\theta}$ for both the cases is same whereas their corresponding values are different.

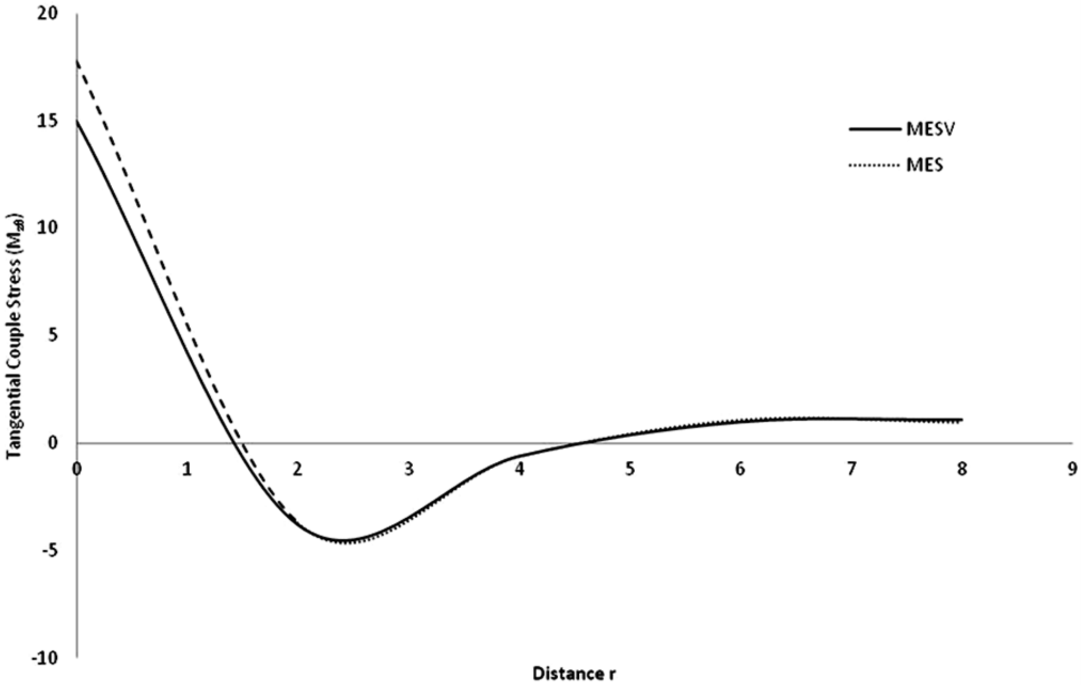

Fig.5. Variations of tangential couple stress $M_{z\theta}$ $(=4\pi m_{z\theta}/F_0)$ due to normal point intance source with distance r.

5.2. Mechanical continuous normal point source

The variations in normal displacement, volume fraction field, normal force stress and tangential couple stress with distance r for MESV, MES ESV and ES when a continuous normal point source is applied are shown in Figs 6, 7, 8, 9, respectively.

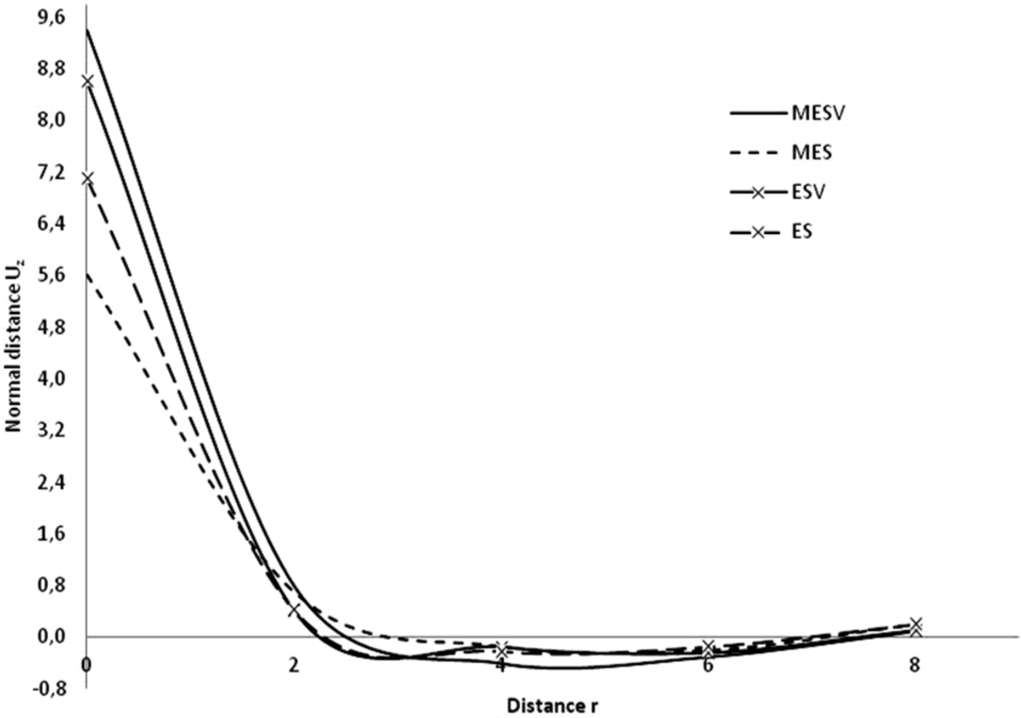

Fig.6. Variations of normal distance U_z $(=4\pi u_z/F_0)$ due to continuous source with distance r.

Figure 6 depicts the variations of normal displacement U_Z with *r*. The behavior of U_Z for all four cases is same. The value of normal displacement start with sharp decrease and the approaches to zero as *r* increases. The values of U_Z are greatest for the case of MESV and smallest for the case of MES in the range $0 \le r \le 3$.

The variations of volume fraction Q with *r* for MESV and ESV are shown in Fig.7. Q decreases sharply in the range $0 \le r \le 3$. The values for ESV are greater for MESV in the range $0 \le r \le 3$.

Figure 8 depicts the variations in normal force stress T_{ZZ} with *r*. Initially, with a sharp increase in the values of the force stress its value approaches to zero. In the range $0 \le r \le 1$ the values are greatest for MES and smallest for MESV.

The variations in $M_{Z\theta}$ with *r* are shown in Fig.9. It is observed that the behavior of couple stress is just opposite to each other in the whole range $0 \le r \le 8$. $M_{Z\theta}$ starts with a rapid decrease in the case of MESV whereas it starts with a sharp increase in the case of MES.

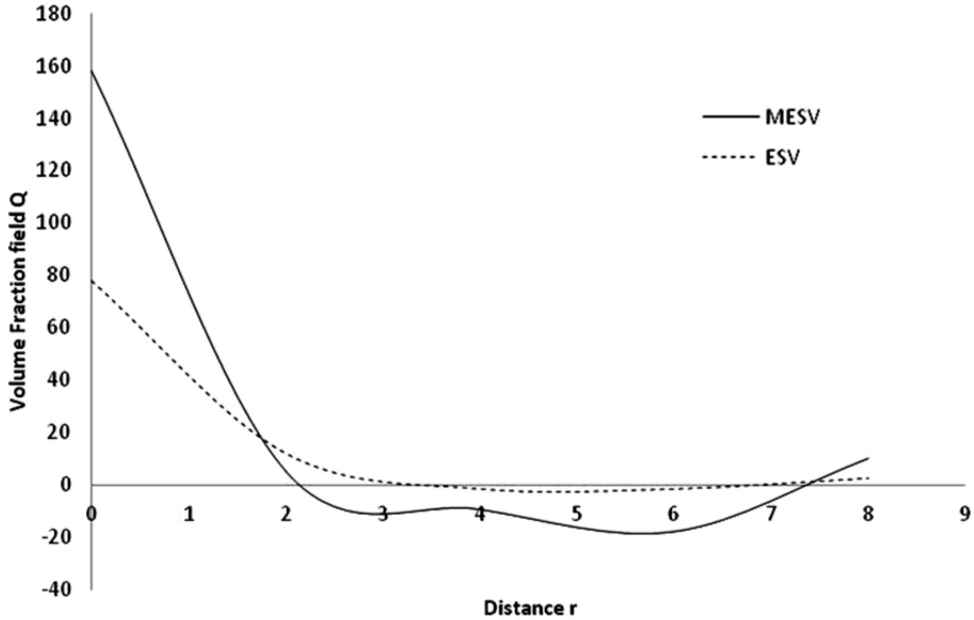

Fig.7. Variations of volume fraction field Q $(=4\pi q/F_0)$ due to continuous source with distance *r*.

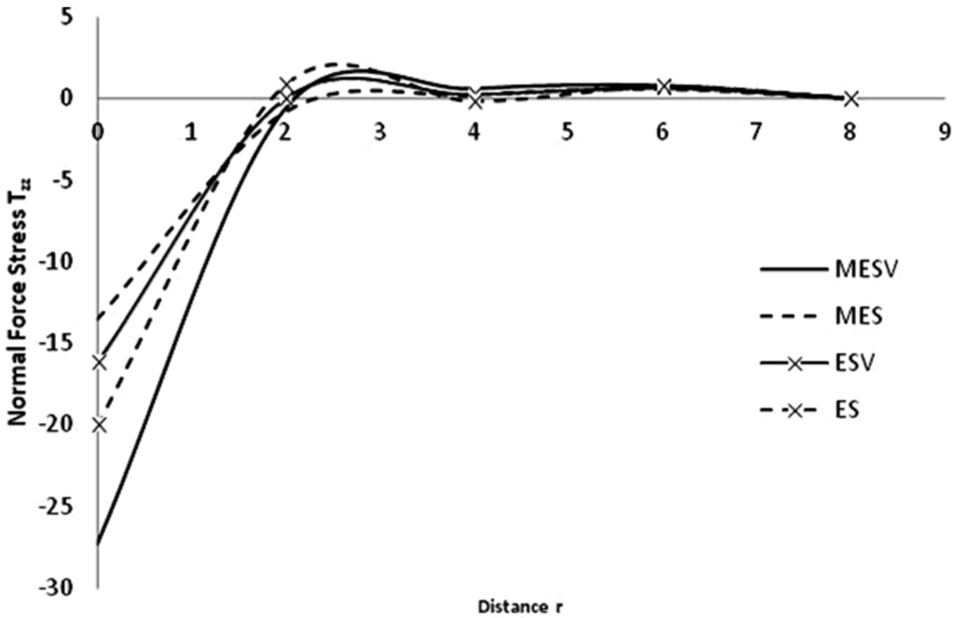

Fig.8. Variations of normal force stress T_{zz} $(=4\pi t_{zz}/F_0)$ due to continuous source with distance *r*.

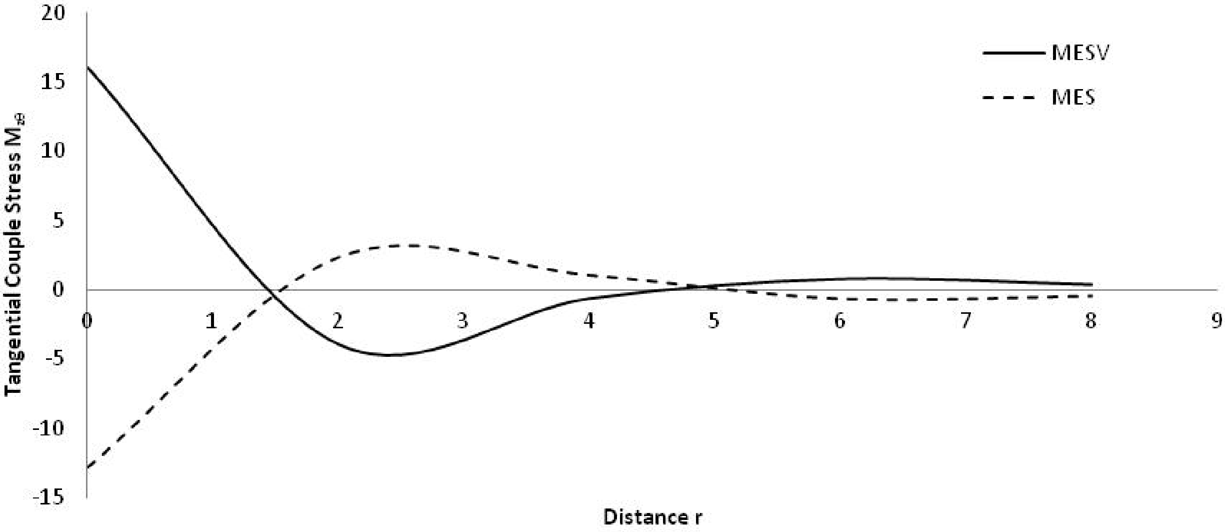

Fig.9. Variations tangential couple stress $M_{z\theta}$ ($=4\pi m_{z\theta}/F_0$) due to continuous source with distance *r*.

Conclusion

A significant effect of void and micropolarity has been observed from the above numerical discussion. The magnitude of variations of normal displacement, normal force stress and tangential couple stress is observed for mechanical normal point source and continuous normal sources. The void effect is appreciable in the sources. It is observed that the components of displacement, force stress, volume fraction field have large values which become smaller and smaller with the increase in the value of the distance '*r*'.

Nomenclature

j – micro-inertia
m_{ij} – couple stress tensor
q^* – volume fraction
t_{ij} – force stress tensor
$\boldsymbol{u}$ – displacement vector
α, β, γ, K – micropolar material constants
Δ – gradient operator
δ_{ij} – Kronecker delta
ε_{ijr} – alternating tensor
λ, μ – Lame's constants
$\lambda^*, \alpha^*, \beta^*, \gamma^*, K^*$ – void constants
ρ – density
ϕ – microrotation vector

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