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CONTROLLABILITY OF DEGENERATE AND SINGULAR PARABOLIC PROBLEMS: THE DOUBLE STRONG CASE WITH NEUMANN BOUNDARY CONDITIONS

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Abstract. We prove a null controllability result for a parabolic problem with Neumann boundary conditions. We consider non smooth coefficients in presence of a strongly singular potential and a strongly degenerate coefficient, both vanishing at an interior point. This paper concludes the study of the Neumann case.

Keywords: strongly singular/degenerate equations, non smooth coefficients, null controllability.

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1. INTRODUCTION

This paper deals with null controllability issues for a class of degenerate and singular parabolic Neumann problems with interior degeneracy and singularity, whose prototype is

$$\begin{cases} u_t - Au - \dfrac{\lambda}{|x-x_0|^{K_b}} = f\chi_\omega & (t,x) \in Q_T := (0,T) \times (0,1),\\ u_x(t,0) = u_x(t,1) = 0 & t \in (0,T),\\ u(0,x) = u_0(x) \in L^2(0,1) & x \in (0,1), \end{cases}$$

where

$$Au := \left(|x-x_0|^{K_a} u_x\right)_x \quad \text{or} \quad Au := |x-x_0|^{K_a} u_{xx}.$$

Here $x_0 \in (0,1)$, the control function f is located in an open set ω compactly contained in $(0,1)$ and λ is a real parameter.

Actually, we shall consider more general operators of the form

$$u_t - (a(x)u_x)_x - \frac{\lambda}{b(x)} u,$$

where $a(x) \sim |x-x_0|^{K_a}$ and $b(x) \sim |x-x_0|^{K_b}$ are possibly *non-smooth* functions (for more comments see [18, 19] or [20]).

Related problems have been studied for example in [1–8, 13, 15–17, 21] and in [10] (for the nonlinear case), when $\lambda = 0$. If $\lambda \neq 0$, the first null controllability result was proved in [29] for the *non degenerate* heat operator with singular potential

$$u_t - u_{xx} - \lambda \frac{1}{x^{K_b}} u, \quad (t,x) \in Q_T, \tag{1.1}$$

under Dirichlet boundary conditions: Carleman estimates (and consequently null controllability properties) are established when $\lambda \leq 1/4$. On the contrary, if $\lambda > 1/4$, in [9] it is proved that the null controllability for (1.1) fails.

As far as we know, [28] is the first paper where a degenerate diffusion coefficient is coupled with a singular potential, precisely considering

$$u_t - (x^{K_a} u_x)_x - \lambda \frac{1}{x^{K_b}} u, \quad (t,x) \in Q_T.$$

Under suitable conditions on λ and assuming $K_a + K_b \leq 2$, but excluding $K_a = K_b = 1$, the author establishes Carleman estimates, and so null controllability results. These results were extended in [12] and in [11] to operators of the form

$$u_t - (a(x) u_x)_x - \lambda \frac{1}{x^{K_b}} u, \quad (t,x) \in Q_T,$$

where $a(x) \sim x^{K_a}$.

We notice that in the previously cited papers the degeneracy and the singularity occur at the boundary of the domain. However, it seems natural to consider degeneracy occurring at an interior point of the space domain. This fact originates some complications because the boundary conditions do not play any role in controlling the loss of ellipticity or the singularity in the equation. For these reasons, a related research has started focusing on *interior* degenerate coefficients, possibly non smooth: for instance, see [3, 4, 15, 16, 21] and [27] when $\lambda = 0$ and [14, 18–20] and [22] when $\lambda \neq 0$, and the references therein.

In particular, problems strictly related to the one studied in this paper are considered in [14, 18, 19] and in [20], to which we refer for any further comment and for the general setting. First of all, let us recall the following possibilities for the degenerate function a, or similarly, for the singular potential b:

(a) $a \in W^{1,1}(0,1)$ is said to be *weakly degenerate*, (WD) for short, if there exists $x_0 \in (0,1)$ such that $a(x_0) = 0$, $a > 0$ on $[0,1] \setminus \{x_0\}$ and there exists $K_a \in (0,1)$ such that $(x-x_0)a' \leq K_a a$ a.e. in $[0,1]$;
(b) $a \in W^{1,\infty}(0,1)$ is said to be *strongly degenerate*, (SD) for short, if there exists $x_0 \in (0,1)$ such that $a(x_0) = 0$, $a > 0$ on $[0,1] \setminus \{x_0\}$ and there exists $K_a \in [1,2)$ such that $(x-x_0)a' \leq K_a a$ a.e. in $[0,1]$.

Standard examples are $a(x) = |x-x_0|^{K_a}$ with $0 < K_a < 2$. The restriction $K_a < 2$ is related to controllability and existence issues ([16] and [24]) and to certain characterizations of the domains of the operators which permit some integrations by parts ([7] and [21]). For this reasons, from now on, we will only consider coefficients $K_a, K_b < 2$.

As already said, related problems have been studied in [14, 18, 19] and in [20]. In [14] the problem in non divergence form was considered under Dirichlet or Neumann boundary conditions; moreover, if a and b were both (SD), the well posedness and the null controllability were proved only in the case $K_a = K_b = 1$. In [18] the problem in divergence form was considered only under Dirichlet boundary conditions and, if a and b were both (SD), only the well posedness was proved, provided that $\lambda < 0$; indeed, when $\lambda > 0$ and small, the controllability was proved for $K_a + K_b \leq 2$, excluding the case $K_i = 1$, as a consequence of Carleman and observability inequalities: these estimates were obtained by the Hardy–Poincaré type inequality with interior degeneracy

$$\int_0^1 \frac{u^2}{b} dx \leq C \int_0^1 a(u')^2 dx,$$

which follows by the inequality

$$\frac{(1-\alpha)^2}{4} \int_0^1 \frac{u^2}{|x-x_0|^{2-\alpha}} dx \leq \int_0^1 |x-x_0|^{\alpha} (u')^2 dx \tag{1.2}$$

valid for every $\alpha \in \mathbb{R}$ and for every $u \in H$. Here

$$u \in H := \left\{ u \ \in W_0^{1,1}(0,1) \ : \ \sqrt{|x-x_0|^{\alpha}} u' \in L^2(0,1), \frac{u}{\sqrt{|x-x_0|^{2-\alpha}}} \in L^2(0,1) \right\}.$$

It is clear that inequality (1.2) fails to be interesting precisely for $\alpha = 1$, in agreement with the celebrated characterization of Muckenhoupt [25]. Thus, if both a and b are (SD), in order to obtain the controllability result, one cannot follow the approach used in [18]. For this reason, in [20], we proceeded in a completely different way, proving the null controllability also when $K_a = K_b = 1$, *only by using cut–off functions.* This technique was applied also in the non divergence case, thus generalizing the result given in [14].

The degenerate/singular problem in divergence form with Neumann boundary conditions appeared in [19]. In this case we couldn't use (1.2) due to the lack of Dirichlet conditions, and in the (SSD) case (i.e. both a and b are (SD)) we proved only the well posedness, provided that $\lambda < 0$. Again the null controllability was not considered in the (SSD) case, and this is what we are going to face here. Hence, this paper completes the previous works, concluding the description of the evolution systems

$$\begin{cases} u_t - (a(x)u_x)_x - \dfrac{\lambda}{b(x)} u = f(t,x)\chi_\omega(x), & (t,x) \in Q_T, \\ u_x(t,0) = u_x(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1), \end{cases} \tag{1.3}$$

and

$$\begin{cases} u_t - a(x)u_{xx} - \dfrac{\lambda}{b(x)} u = f(t,x)\chi_\omega(x), & (t,x) \in Q_T, \\ u_x(t,0) = u_x(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1). \end{cases} \tag{1.4}$$

when $K_a, K_b \geq 1$. In particular, we aim at showing null controllability results for (1.3) and (1.4), that is: for every $u_0 \in L^2(0,1)$ there exists $f \in L^2(Q_T)$ such that the related solution u satisfies $u(T,x) = 0$ for every $x \in [0,1]$ and $\|f\|^2_{L^2(Q_T)} \leq C\|u_0\|^2_{L^2(0,1)}$ for some universal positive constant C (for (1.4) replacing L^2 with $L^2_{\frac{1}{a}}$).

A final comment on the notation: by C we shall denote universal positive constants, which are allowed to vary from line to line.

We remark that the divergence form case will be treated in full details, while for the non divergence form case we will be more sketchy, since many calculations are analogous to the former case.

2. WELL POSEDNESS

As just remarked, we focus in (1.3). Let us start introducing the functional setting from [18]. First of all, define the weighted Hilbert spaces

$$H^1_a(0,1) := \left\{ u \in W^{1,1}(0,1) \,:\, \sqrt{a}u' \in L^2(0,1) \right\}$$

and

$$\mathcal{H} := H^1_{a,b}(0,1) := \left\{ u \in H^1_a(0,1) \,:\, \frac{u}{\sqrt{b}} \in L^2(0,1) \right\},$$

endowed with the inner products

$$\langle u, v\rangle_{H^1_a(0,1)} := \int_0^1 au'v'dx + \int_0^1 uv\,dx,$$

and

$$\langle u, v\rangle_{H^1_{a,b}(0,1)} = \int_0^1 au'v'dx + \int_0^1 uv\,dx + \int_0^1 \frac{uv}{b}dx,$$

respectively.

Finally, introduce the Hilbert space

$$H^2_{a,b} := \left\{ u \in H^1_a(0,1) \,:\, au' \in H^1(0,1), u'(0) = u'(1) = 0 \text{ and } Au \in L^2(0,1) \right\},$$

where

$$Au := (au')' + \frac{\lambda}{b}u \quad \text{with} \quad D(A) = H^2_{a,b}(0,1).$$

We recall the following definition:

Definition 2.1. Let $u_0 \in L^2(0,1)$ and $h \in L^2(0,T;\mathcal{H}^*)$. A function u is said to be a (weak) solution of

$$\begin{cases} u_t - (a(x)u_x)_x - \frac{\lambda}{b(x)}u = h(t,x), & (t,x) \in Q_T, \\ u_x(t,0) = u_x(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1), \end{cases} \tag{2.1}$$

if

$$u \in \mathcal{U} := L^2(0,T;\mathcal{H}) \cap H^1([0,T];\mathcal{H}^*)$$

and it satisfies (2.1) in the sense of $\mathcal{H}^*$-valued distributions.

Notice that any solution belongs to $C([0,T];L^2(0,1))$ by [26, Lemma 11.4].

Our fundamental assumption is the following:

(H) a and b are (SD) and $\lambda < 0$.

For completeness, we show that $D(A)$ is dense in $L^2(\Omega)$. Indeed, if

$$T \in D(A)^\perp = \Big\{T \in L^2(\Omega) \,:\, \int\limits_\Omega Tu\,dx = 0 \text{ for all } u \in D(A)\Big\},$$

take $u \in D(A)$ solution of $-Au + u = T$ (the existence of a unique weak solution $u \in \mathcal{H}$ is guaranteed by the Lax-Milgram Theorem and the equation itself implies that $u \in D(A)$). Then

$$0 = \int\limits_\Omega Tu\,dx = \int\limits_\Omega \Big[-(a(x)u_x)_x\,u - \frac{\lambda}{b(x)}u^2 + u^2\Big]dx \geq \int\limits_\Omega u^2 dx,$$

so that $u = 0$, which implies $T = 0$, and thus $D(A)$ is dense in $L^2(\Omega)$.

As a particular case of [19, Theorem 2.1], we have the following well-posedness result.

Theorem 2.2. *Assume* (H). *If $u_0 \in L^2(0,1)$ and $h \in L^2(0,T;\mathcal{H}^*)$, there exists a unique solution of* (2.1). *Moreover, if $u_0 \in D(A)$, then*

$$h \in L^2(Q_T) \Rightarrow u \in H^1(0,T;L^2(0,1)),$$
$$h \in W^{1,1}(0,T;L^2(0,1)) \Rightarrow u \in C^1(0,T;L^2(0,1)) \cap C([0,T];D(A)).$$

3. THE CONTROLLABILITY RESULT

In order to study the controllability property, on the control set ω we assume one of the following hypothesis:

($\mathcal{O}$) First item either

$$\omega = (\alpha,\beta) \subset (0,1) \text{ is such that } x_0 \in \omega, \tag{3.1}$$

or

$$\omega = \omega_1 \cup \omega_2, \tag{3.2}$$

where

$$\omega_i = (\alpha_i,\beta_i) \subset (0,1),\ i = 1,2, \text{ and } \beta_1 < x_0 < \alpha_2.$$

The main result of this paper is the following.

Theorem 3.1. *Assume* (H) *and* ($\mathcal{O}$). *Then, given* $u_0 \in L^2(0,1)$, *there exists* $f \in L^2(Q_T)$ *such that the solution* u *of* (1.3) *satisfies*

$$u(T,x) = 0 \ \ \textit{for every} \ \ x \in [0,1].$$

Moreover,

$$\int\limits_{Q_T} f^2 dxdt \le C \int\limits_0^1 u_0^2 dx \tag{3.3}$$

for some universal positive constant C.

The proof of the previous theorem is based on the next result, that will be proved in the Appendix.

Theorem 3.2. *Take* $A < B$ *in* $\mathbb{R}$, $a \in W^{1,\infty}(A,B)$, $b \in C([A,B])$ *are such that* $a(x) \ge a_0 > 0$ *and* $b(x) \ge b_0 > 0$ *for all* $x \in [A,B]$. *Assume* $\omega \subset\subset (A,B)$ *is an interval. Then, given* $u_0 \in L^2(A,B)$, *there exists* $f \in L^2((0,T)\times(A,B))$ *such that the solution* u *of*

$$\begin{cases} u_t - (a(x)u_x)_x - \dfrac{\lambda}{b(x)}u = f(t,x)\chi_\omega(x), & (t,x) \in (0,T)\times(A,B), \\ u(t,A) = u(t,B) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (A,B), \end{cases} \tag{3.4}$$

satisfies $u(T,x) = 0$ *for every* $x \in [A,B]$. *Moreover,*

$$\int\limits_0^T \int\limits_A^B f^2 dxdt \le C \int\limits_A^B u_0^2 dx$$

for some universal positive constant C.

Clearly, if a and b are strictly positive then the spaces $H^1_a(0,1)$ and $H^1_{a,b}(0,1)$ coincide with $H^1(0,1)$, while $H^2_{a,b}(0,1)$ coincides with $H^2(0,1)$.

Proof of Theorem 3.1. Let v be the solution of (1.3) with right-hand-side $h\chi_\omega$ and introduce

$$\tilde a(x) := \begin{cases} a(-x), & x \in [-1,0], \\ a(x), & x \in [0,1], \\ a(2-x), & x \in [1,2], \end{cases} \qquad \tilde b(x) := \begin{cases} b(-x), & x \in [-1,0], \\ b(x), & x \in [0,1], \\ b(2-x), & x \in [1,2], \end{cases}$$

$$\tilde h(t,x) := \begin{cases} h(t,-x), & x \in [-1,0], \\ h(t,x), & x \in [0,1], \\ h(t,2-x), & x \in [1,2], \end{cases} \qquad \tilde u_0(x) := \begin{cases} u_0(-x), & x \in [-1,0, \\ u_0(x), & x \in [0,1], \\ u_0(2-x), & x \in [1,2]. \end{cases}$$

Now, assume (3.1) and consider $0 < r' < \tilde{r} < r$ with $(x_0 - r, x_0 + r) \subset \omega$ and take three cut-off functions $\phi_i \in C^\infty([-1,2])$, $|\phi_i| \le 1$, $i = 0, 1, 2$, with

$$\phi_1(x) := \begin{cases} 0, & x \in [-1, -(x_0 - \tilde{r})] \cup [x_0 - \tilde{r}, 2], \\ 1, & x \in [-(x_0 - r), x_0 - r], \end{cases}$$

$$\phi_2(x) := \begin{cases} 0, & x \in [-1, x_0 + \tilde{r}] \cup [2 - (x_0 + \tilde{r}), 2], \\ 1, & x \in [x_0 + r, 2 - (x_0 + r)], \end{cases}$$

and $\phi_0 = 1 - \phi_1 - \phi_2$. Finally, define

$$W(t,x) := \begin{cases} v(t, -x), & x \in [-1, 0], \\ v(t,x), & x \in [0,1], \\ v(t, 2-x), & x \in [1,2], \end{cases} \tag{3.5}$$

and

$$\tilde{\omega} := (-\beta, -\alpha) \cup (\alpha, \beta) \cup (2 - \beta, 2 - \alpha),$$

so that W satisfies

$$\begin{cases} W_t - (\tilde{a} W_x)_x - \lambda \dfrac{W}{\tilde{b}} = \tilde{h} \chi_{\tilde{\omega}}, & (t,x) \in (0,T) \times (-1,2), \\ W_x(t,-1) = W_x(t,2) = 0, & t \in (0,T), \\ W(0,x) = \tilde{u}_0(x), & x \in (-1,2). \end{cases} \tag{3.6}$$

Hence, $v_1 := \phi_1 W$ and $v_2 := \phi_2 W$ satisfy the nondegenerate problems

$$\begin{cases} v_{1,t} - (\tilde{a} v_{1,x})_x - \dfrac{\lambda}{\tilde{b}} v_1 = \bar{h}_1 \chi_{\tilde{\omega}}, & (t,x) \in (0,T) \times (-(x_0 - r'), x_0 - r'), \\ v_1(t, -(x_0 - r')) = v_1(t, x_0 - r') = 0, & t \in (0,T), \\ v_{1,x}(t, -(x_0 - r')) = v_{1,x}(t, x_0 - r') = 0, & t \in (0,T), \\ v_1(0,x) = \phi_1(x) \tilde{u}_0(x), & x \in (-(x_0 - r'), x_0 - r') \end{cases}$$

and

$$\begin{cases} v_{2,t} - (\tilde{a} v_{2,x})_x - \dfrac{\lambda}{\tilde{b}} v_2 = \bar{h}_2 \chi_{\tilde{\omega}}, & (t,x) \in (0,T) \times (x_0 + r', 2 - (x_0 + r')), \\ v_2(t, x_0 + r') = v_2(t, 2 - (x_0 + r')) = 0, & t \in (0,T), \\ v_{2,x}(t, x_0 + r') = v_{2,x}(t, 2 - (x_0 + r')) = 0, & t \in (0,T), \\ v_2(0,x) = \phi_2(x) \tilde{u}_0(x), & x \in (x_0 + r', 2 - (x_0 + r')), \end{cases}$$

with $\bar{h}_i := \phi_i \tilde{h} - (\tilde{a} (\phi_i)_x W)_x - \tilde{a} (\phi_i)_x W_x$, $i = 1, 2$.

Then, by Theorem 3.2, there exist two control functions $h_1 \in L^2((0,T) \times (-(x_0 - r'), x_0 - r'))$ and $h_2 \in L^2((0,T) \times (x_0 + r', 2 - (x_0 + r')))$, such that $v_1(T,x) = 0$ for all $x \in (-(x_0 - r'), x_0 - r')$ and $v_2(T,x) = 0$ for all $x \in (x_0 + r', 2 - (x_0 + r'))$ with

$$\int_0^T \int_{-(x_0 - r')}^{x_0 - r'} h_1^2 \, dx dt \le C \int_0^T \int_{-(x_0 - r')}^{x_0 - r'} \tilde{u}_0^2 \, dx dt$$

and

$$\int_0^T \int_{x_0+r'}^{2-(x_0+r')} h_2^2 dxdt \leq C \int_0^T \int_{x_0+r'}^{2-(x_0+r')} \tilde{u}_0^2 dxdt$$

for some constant C. In particular $h_1 \in L^2((0,T) \times (0, x_0 - r'))$, $h_2 \in L^2((0,T) \times (x_0 + r', 1))$, $v_1(T,x) = 0$ for all $x \in (0, x_0 - r')$, $v_2(T,x) = 0$ for all $x \in (x_0 + r', 1)$ with

$$\int_0^T \int_0^{x_0-r'} h_1^2 dxdt \leq C \int_0^T \int_{-(x_0-r')}^{x_0-r'} \tilde{u}_0^2 dxdt \leq C \int_0^T \int_0^1 u_0^2 dxdt \tag{3.7}$$

and

$$\int_0^T \int_{x_0+r'}^1 h_2^2 dxdt \leq C \int_0^T \int_{x_0+r'}^{2-(x_0+r')} \tilde{u}_0^2 dxdt \leq C \int_0^T \int_0^1 u_0^2 dxdt \tag{3.8}$$

for some constant C.

Now, let u_3 be the solution of

$$\begin{cases} u_t - (a(x)u_x)_x - \dfrac{\lambda}{b(x)} u = 0 & (t,x) \in (0,T) \times (0,1), \\ u_x(t,0) = u_x(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1), \end{cases} \tag{3.9}$$

and denote by u_1 and f_1 (resp. u_2 and f_2) the trivial extensions of v_1 and h_1 (resp. u_2 and h_2) to $[x_0 - r', 1]$ (resp. $[0, x_0 + r']$), so that all functions are defined in the interval $[0,1]$. Finally, take

$$u(t,x) = u_1(t,x) + u_2(t,x) + \frac{T-t}{T} \phi_0(x) u_3(t,x).$$

Then, $u(T,x) = 0$ for all $x \in [0,1]$ and u satisfies problem (1.3) in the domain Q_T with

$$f = f_1 \chi_\omega + f_2 \chi_\omega - \frac{1}{T} \phi_0 u_3 - \phi_0' \frac{T-t}{T} a u_{3,x} - \left(\phi_0' \frac{T-t}{T} a u_3 \right)_x.$$

Since a belongs to $W^{1,\infty}(0,1)$, one has that $f \in L^2(Q_T)$, as required. Moreover, it is easy to see that the support of f is contained in ω.

Now, we prove (3.3) proceeding as in [20]. To this aim, consider the equation satisfied by u_3 and multiply it by u_3. Then, integrating over $(0,1)$, we have

$$\frac{1}{2} \frac{d}{dt} \|u_3(t)\|^2_{L^2(0,1)} + \|\sqrt{a} u_{3,x}(t)\|^2_{L^2(0,1)} - \lambda \left\| \frac{u_3}{\sqrt{b}} \right\|^2_{L^2(0,1)} \leq 0.$$

Using the fact that $\lambda < 0$, we get

$$\frac{d}{dt} \|u_3(t)\|^2_{L^2(0,1)} \leq \frac{d}{dt} \|u_3(t)\|^2_{L^2(0,1)} + 2 \|\sqrt{a} u_{3,x}(t)\|^2_{L^2(0,1)} \leq 0.$$

Thus, the function $t \mapsto \|u_3(t)\|^2_{L^2(0,1)}$ is decreasing. This implies that

$$\|u_3(t)\|^2_{L^2(0,1)} \le \|u_0\|^2_{L^2(0,1)} \quad \text{for all } t \in [0,T]$$

and so

$$\|u_3\|^2_{L^2(Q_T)} \le C\|u_0\|^2_{L^2(0,1)}. \tag{3.10}$$

Now, integrating over $(0,T)$ the inequality

$$\frac{d}{dt}\|u_3(t)\|^2_{L^2(0,1))} + 2\|\sqrt{a}u_{3,x}(t)\|^2_{L^2(0,1)} \le 0,$$

we immediately find

$$\|\sqrt{a}u_{3,x}\|^2_{L^2(Q_T)} \le C\|u_0\|^2_{L^2(Q_T)} \tag{3.11}$$

for some $C > 0$.

Now, let us note that, since $a \in W^{1,\infty}(0,1)$, then

$$\|(au_3)_x\|_{L^2(Q_T)} \le C\left(\|u_3\|_{L^2((Q_T)} + \|\sqrt{a}u_{3,x}\|_{L^2(Q_T)}\right).$$

By using (3.10) and (3.11) in the previous inequality, we get

$$\|(au_3)_x\|_{L^2(Q_T)} \le \|u_0\|^2_{L^2(Q_T)} \tag{3.12}$$

for some $C > 0$.

In conclusion, by (3.10), (3.11), (3.12), from the definition of f and by (3.7) and (3.8), inequality (3.3) follows immediately.

Now, assume (3.2). Take $r > 0$ such that $\beta_1 < x_0 - r$ and $x_0 + r < \alpha_2$. As before, take three cut-off functions $\varphi_i \in C^\infty([-1,2])$, $|\varphi_i| \le 1$, $i = 0,1,2$, with

$$\varphi_1(x) := \begin{cases} 0, & x \in [-1,-\beta_1] \cup [\beta_1, 2], \\ 1, & x \in [-\alpha_1, \alpha_1], \end{cases}$$
$$\varphi_2(x) := \begin{cases} 0, & x \in [-1,\alpha_2] \cup [2-\alpha_2, 2], \\ 1, & x \in [\beta_2, 2-\beta_2], \end{cases}$$

and $\varphi_0 = 1 - \varphi_1 - \varphi_2$. Defining W as in (3.5), we have that W satisfies (3.6) with

$$\tilde{\omega} := (-\beta_2, -\alpha_2) \cup (-\beta_1, -\alpha_1) \cup \omega \cup (2-\beta_2, 2-\alpha_2) \cup (2-\beta_1, 2-\alpha_1).$$

Setting $v_4 := \varphi_1 W$ and $v_5 := \varphi_2 W$, one has that v_4 and v_5 satisfy the nondegenerate problems

$$\begin{cases} v_{4,t} - (\tilde{a}v_{4,x})_x - \dfrac{\lambda}{\tilde{b}}v_4 = \bar{h}_4\chi_{(-\beta_1,-\alpha_1)\cup(\alpha_1,\beta_1)}, & (t,x) \in (0,T) \times (-(x_0-r), x_0-r), \\ v_4(t,-(x_0-r)) = v_4(t,x_0-r) = 0, & t \in (0,T), \\ v_{4,x}(t,-(x_0-r)) = v_{4,x}(t,x_0-r) = 0, & t \in (0,T), \\ v_4(0,x) = \varphi_1(x)\tilde{u}_0(x), & x \in (-(x_0-r), x_0-r) \end{cases}$$

and

$$\begin{cases} v_{5,t}-(\tilde{a}v_{5,x})_x-\dfrac{\lambda}{\tilde{b}}v_5=\bar{h}_5\chi_{(\alpha_2,\beta_2)\cup(2-\beta_2,2-\alpha_2)}, & (t,x)\in(0,T)\times(x_0+r,2-(x_0+r)),\\ v_5(t,x_0+r)=v_5(t,2-(x_0+r))=0, & t\in(0,T),\\ v_{5,x}(t,x_0+r)=v_{5,x}(t,2-(x_0+r))=0, & t\in(0,T),\\ v_5(0,x)=\varphi_2(x)\tilde{u}_0(x), & x\in(x_0+r,2-(x_0+r)) \end{cases}$$

with $\bar{h}_i := \varphi_{i-3}\tilde{h}-(\tilde{a}(\varphi_{i-3})_xW)_x-\tilde{a}(\varphi_{i-3})_xW_x$, $i=4,5$. Again, by Theorem 3.2, there exist two control functions $h_4\in L^2((0,T)\times(-(x_0-r),x_0-r))$ and $h_5\in L^2((0,T)\times(x_0+r,2-(x_0+r)))$, such that $v_4(T,x)=0$ for all $x\in(-(x_0-r),x_0-r)$ and $v_5(T,x)=0$ for all $x\in(x_0+r,2-(x_0+r))$ with

$$\int_0^T\int_{-(x_0-r)}^{x_0-r} h_4^2\,dxdt \le C\int_0^T\int_{-(x_0-r)}^{x_0-r}\tilde{u}_0^2\,dxdt$$

and

$$\int_0^T\int_{x_0+r}^{2-(x_0+r)} h_5^2\,dxdt \le C\int_0^T\int_{x_0+r}^{2-(x_0+r)}\tilde{u}_0^2\,dxdt$$

for some constant C. In particular $h_4\in L^2((0,T)\times(0,x_0-r))$, $h_5\in L^2((0,T)\times(x_0+r,1))$, $v_4(T,x)=0$ for all $x\in(0,x_0-r)$, $v_5(T,x)=0$ for all $x\in(x_0+r,1)$ with

$$\int_0^T\int_0^{x_0-r} h_4^2\,dxdt \le C\int_0^T\int_{-(x_0-r)}^{x_0-r}\tilde{u}_0^2\,dxdt \le C\int_0^T\int_0^1 u_0^2\,dxdt$$

and

$$\int_0^T\int_{x_0+r}^{1} h_5^2\,dxdt \le C\int_0^T\int_{x_0+r}^{2-(x_0+r)}\tilde{u}_0^2\,dxdt \le C\int_0^T\int_0^1 u_0^2\,dxdt$$

for some constant C.

As before, let u_4 and f_4, u_5 and f_5 be the trivial extensions of v_4 and h_4, v_5 and h_5 in $[x_0 - r, 1]$ and $[0, x_0 + r]$, respectively, considered in the interval $[0, 1]$. Finally, set

$$u(t,x) = u_4(t,x) + u_5(t,x) + \frac{T-t}{T}\varphi_0(x)u_3(t,x),$$

where u_3 is the solution of (3.9). As before, $u(T,x) = 0$ for all $x \in [0,1]$ and u satisfies problem (1.3) in the domain Q_T with

$$f = f_4\chi_{(\alpha_1,\beta_1)} + f_5\chi_{(\alpha_2,\beta_2)} - \frac{1}{T}\varphi_0 u_3 - \varphi_0' \frac{T-t}{T} a u_{3,x} - \left(\varphi_0' \frac{T-t}{T} a u_3\right)_x.$$

Again $f \in L^2(Q_T)$, as required and the support of f is contained in ω. In order to conclude we have to prove (3.3) for the control function f, but such an estimate can be obtained as above, thus concluding the proof. □

Remark 3.3. We strongly remark that if a is (WD), the previous approach *does not work*. Indeed, the function f found in the previous proof *is not* in $L^2\big(Q_T\big)$, since a is only of class $W^{1,1}(0,1)$.

Remark 3.4. If a is (SD) and b is (WD) the technique above, and so the controllability result, still works provided that there exists a solution of (1.3), for example if $\lambda < 1/C^*$ and $K_a + K_b \leq 2$ (see [19, Theorem 3.2]) (observe that in this case Theorem 3.2 still holds true). Thus, we re–obtain the controllability result in [19]. However, we observe that in [19] in order to prove the controllability result we required the following additional assumptions:

1. if $K_a > \frac{4}{3}$, then there exists a constant $\theta \in (0, K_a]$ such that

$$x \mapsto \frac{a(x)}{|x-x_0|^\theta} \quad \begin{cases} \text{is non increasing on the left of } x = x_0, \\ \text{is non decreasing on the right of } x = x_0; \end{cases} \tag{3.13}$$

2. if $K_a > \frac{3}{2}$ the function in (3.13) is bounded below away from 0 and there exists a constant $\Sigma > 0$ such that

$$|a'(x)| \leq \Sigma |x - x_0|^{2\theta-3} \text{ for a.e. } x \in [0,1];$$

3. if $\lambda < 0$ we assume that $(x - x_0)b'(x) \geq 0$ in $[0,1]$.

Hence, Theorem 3.1 improves the result of [19].

3.1. THE NON DIVERGENCE CASE

The null controllability for the problem in *non divergence* form (1.4) was studied in [14] (see also [17, Hypotheses 4.2 and 5.2]) requiring additional assumptions: for example, $(x - x_0)b'(x) \geq 0$ in $[0,1]$ when $\lambda < 0$, as recalled in Remark 3.4. However, using the technique used in the proof of Theorem 3.1, in order to prove the global controllability result, one has to require only the conditions for the existence theorem (see [14, Hypothesis 3.1]) and for the analogous of Theorem 3.2 in the non divergence case. Indeed, proceeding as in the proof of Theorem 3.1 but with problems written in non divergence form, the control function f for (1.4) is given by

$$f = f_1\chi_\omega + f_2\chi_\omega - \frac{1}{T}\phi_0 u_3 - \phi_0' \frac{T-t}{T} a u_{3,x} - a\frac{T-t}{T}\left(\phi_0' u_3\right)_x,$$

if ω satisfies (3.1) or

$$f = f_4\chi_{(\alpha_1,\beta_1)} + f_5\chi_{(\alpha_2,\beta_2)} - \frac{1}{T}\varphi_0 u_3 - \varphi_0' \frac{T-t}{T} a u_{3,x} - a\frac{T-t}{T}\left(\varphi_0' u_3\right)_x,$$

if ω satisfies (3.2). In every case f belongs to the $L^2_{\frac{1}{a}}(Q_T)$ as required (for the definition of the space see, e.g., [14]). Hence, the next theorem holds.

Theorem 3.5. *Assume $a \in W^{1,\infty}(0,1)$ and ($\mathcal{O}$). Then, given $u_0 \in L^2_{\frac{1}{a}}(0,1)$, there exists $f \in L^2_{\frac{1}{a}}(Q_T)$ such that the solution u of* (1.4) (*if there exists*) *satisfies*

$$u(T,x) = 0 \quad \textit{for every} \quad x \in [0,1].$$

Moreover,

$$\int\limits_{Q_T} \frac{f^2}{a} dxdt \le C \int\limits_0^1 \frac{u_0^2}{a} dx,$$

for some universal positive constant C.

We remark that the previous theorem generalizes the result given in [14] in the sense that here we prove the controllability result under weaker assumptions. This is due to the fact that in [14] the controllability result is proved via Carleman estimates and observability inequality, while here we use only cut off functions. Of course, since here we do not make any assumption guaranteeing the well posedness, we can refer to [14, Hypothesis 3.1] for some sufficient conditions.

Observe that the proof of Theorem 3.5 is based on the analogous of Theorem 3.2 for the problem in non divergence form, whose proof is similar to that of Theorem 3.2 and is based on the nondegenerate nonsingular Carleman estimate in non divergence form proved in [14, Proposition 4.1]).

4. APPENDIX: PROOF OF THEOREM 3.2

In this section we will prove, for the reader's convenience, Theorem 3.2. The proof is standard: it is based on the equivalence between the null controllability for the non degenerate problem (3.4) and the observability inequality (4.7) below for the associated adjoint problem

$$\begin{cases} v_t + (a(x)v_x)_x + \dfrac{\lambda}{b(x)} v = 0, & (t,x) \in (0,T) \times (A,B), \\ v(t,A) = v(t,B) = 0, & t \in (0,T), \\ v(T,x) = v_T(x) \in L^2(A,B). \end{cases} \tag{4.1}$$

In order to prove the observability inequality for the solution v of (4.1), the Carleman estimate proved in [18, Proposition 4.8] is crucial:

Proposition 4.1 (Nondegenerate nonsingular Carleman estimate)**.** *Let z be the solution of*

$$\begin{cases} z_t + (az_x)_x + \lambda \dfrac{z}{b} = h \in L^2\big((0,T) \times (A,B)\big), \\ z(t,A) = z(t,B) = 0, \ t \in (0,T), \end{cases}$$

where $b \in C\big([A,B]\big)$ is such that $b \geq b_0 > 0$ in $[A,B]$ and $a \in W^{1,\infty}(A,B)$ is such that $a \geq a_0 > 0$ in (A,B). Then, for all $\lambda \in \mathbb{R}$, there exist three positive constants C, r and s_0 such that for any $s > s_0$

$$\int_0^T \int_A^B \left(s\Theta(z_x)^2 + s^3\Theta^3 z^2\right) e^{2s\Phi} dxdt \leq C \left(\int_0^T \int_A^B h^2 e^{2s\Phi} dxdt - (B.T.) \right),$$

where

$$(B.T.) = sr \int_0^T \left[ae^{2s\Phi}\Theta e^{r\zeta}(z_x)^2\right]_{x=A}^{x=B} dt.$$

Here the function Φ is defined as $\Phi(t,x) := \Theta(t)\rho(x)$, where $\Theta(t) := \dfrac{1}{[t(T-t)]^4}$,

$$\rho(x) := e^{r\zeta(x)} - \mathfrak{c}$$

and

$$\zeta(x) = \mathfrak{d} \int_x^B \frac{1}{a(t)} dt.$$

Here $\mathfrak{d} = \|a'\|_{L^\infty(A,B)}$ and $\mathfrak{c} > 0$ is chosen in such a way that $\max_{[A,B]} \rho < 0$.

For completeness, we recall that the previous result is in the lines of the Carleman estimates in the Lipschitz case proved in [23].

Moreover, we will also need the following Caccioppoli–type inequality.

Lemma 4.2 (Caccioppoli's inequality)**.** *Assume that the Hypotheses of Proposition 4.1 hold. Let ω' and ω be two open subintervals of (A,B) such that $\omega' \subset\subset \omega \subset\subset (A,B)$. Then, there exist two positive constants C and s_0 such that every solution v of (4.1) satisfies*

$$\int_0^T \int_{\omega'} (v_x)^2 e^{2s\Phi} dxdt \ \leq \ C \int_0^T \int_{\omega} v^2 dxdt,$$

for all $s \geq s_0$.

The proof of the previous inequality is similar to the one in [18], but actually easier since in this case the problem is non degenerate, so we omit it.

Thanks to the previous estimates we can prove a Carleman estimate for the solutions of (4.1).

Lemma 4.3. *Assume that a and b are as in the previous Proposition and let $\omega \subset\subset (A,B)$ be an open interval. Then there exist two positive constants C and s_0 such that every solution v of (4.1) satisfies, for all $s \geq s_0$,*

$$\int_0^T \int_A^B \left(s\Theta(v_x)^2 + s^3\Theta^3 v^2\right) e^{2s\Phi} dxdt \leq C \int_0^T \int_{\omega} v^2 dxdt.$$

Proof. Let $\omega = (\alpha, \beta)$ and consider a smooth cut-off function $\xi : [A,B] \to [0,1]$, such that

$$\begin{cases} \xi(x) = 1, & x \in (\alpha_2, \beta_1), \\ \xi(x) = 0, & x \in (A, \alpha_1) \cup (\beta_2, B), \end{cases}$$

where $\alpha < \alpha_1 < \alpha_2 < \beta_1 < \beta_2 < \beta$. We define $w := \xi v$, where v is the solution of (4.1). Then w satisfies

$$\begin{cases} w_t + (a(x)w_x)_x + \dfrac{\lambda}{b(x)} w = f(x), & (t,x) \in (0,T) \times (A,B), \\ w(t,A) = w(t,B) = 0, & t \in (0,T), \end{cases} \tag{4.2}$$

with $f(x) := (a\xi_x v)_x + \xi_x a v_x$. Therefore, applying Proposition 4.1, using that $w_x(t,A) = w_x(t,B) = 0$, we have

$$\int_0^T \int_A^B \left(s\Theta(w_x)^2 + s^3\Theta^3 w^2\right) e^{2s\Phi} dxdt \leq C \int_0^T \int_A^B f^2 e^{2s\Phi} dxdt. \tag{4.3}$$

Then, using the definition of ξ and in particular the fact that ξ_x and ξ_{xx} are supported in $\omega' := (\alpha_1, \beta_2)$, we can write

$$f^2 = ((a\xi_x v)_x + a\xi_x v_x)^2 \leq C(v^2 + (v_x)^2)\chi_{\omega'}.$$

Hence, by (4.3) and Lemma 4.2 we obtain

$$\begin{aligned}\int_0^T\int_{\alpha_2}^{\beta_1}\left(s\Theta(v_x)^2+s^3\Theta^3v^2\right)e^{2s\Phi}dxdt&=\int_0^T\int_{\alpha_2}^{\beta_1}\left(s\Theta(w_x)^2+s^3\Theta^3w^2\right)e^{2s\Phi}dxdt\\&\le\int_0^T\int_A^B\left(s\Theta(w_x)^2+s^3\Theta^3w^2\right)e^{2s\Phi}dxdt\le\ C\int_0^T\int_\omega v^2dxdt.\end{aligned}\tag{4.4}$$

Now, we consider a smooth function $\tau:[A,B]\to[0,1]$ such that

$$\tau(x)=\begin{cases}0 & x\in[A,\alpha_2],\\ 1 & x\in[\beta_1,B].\end{cases}$$

Set $z:=\tau v$, so that z satisfies (4.2), with $f:=(a\tau_xv)_x+a\tau_xv_x$. Since f is supported in $\tilde\omega:=[\alpha_2,\beta_1]$, by Proposition 4.1 and Lemma 4.2 we get

$$\begin{aligned}\int_0^T\int_{\beta_1}^{B}\left(s\Theta(v_x)^2+s^3\Theta^3v^2\right)e^{2s\Phi}dxdt&=\int_0^T\int_{\beta_1}^{B}\left(s\Theta(z_x)^2+s^3\Theta^3z^2\right)e^{2s\Phi}dxdt\\&\le\int_0^T\int_A^B\left(s\Theta(z_x)^2e^{2s\Phi}+s^3\Theta^3z^2e^{2s\Phi}\right)dxd\le C\int_0^T\int_A^B e^{2s\Phi}f^2dxdt\\&\le C\int_0^T\int_{\tilde\omega}\left(v^2+(v_x)^2\right)e^{2s\Phi}dxdt\le C\int_0^T\int_\omega v^2dxdt.\end{aligned}\tag{4.5}$$

Note that the boundary term in Proposition 4.1 is negative, and thus it is neglected. To complete the proof it is sufficient to prove a similar inequality on the interval $[A,\alpha_2]$. Working as just done, the boundary term would have a positive sign, and so we perform a reflection procedure as in [16], introducing the functions

$$W(t,x):=\begin{cases}v(t,x), & x\in[A,B],\\ -v(t,2A-x), & x\in[2A-B,A],\end{cases}$$

$$\tilde a(x):=\begin{cases}a(x), & x\in[A,B],\\ a(2A-x), & x\in[2A-B,A]\end{cases}$$

and

$$\tilde b(x):=\begin{cases}b(x), & x\in[A,B],\\ b(2A-x), & x\in[2A-B,A].\end{cases}$$

Then W satisfies

$$\begin{cases}W_t+(\tilde aW_x)_x+\dfrac{\lambda}{\tilde b(x)}W=0, & (t,x)\in(0,T)\times(2A-B,B),\\ W(t,2A-B)=W(t,B)=0, & t\in(0,T).\end{cases}$$

Now, take a smooth function $\rho : [2A-B, B] \to [0,1]$ such that

$$\rho(x) = \begin{cases} 0, & x \in [2A-B, 2A-\beta_1] \cup [\beta_1, B], \\ 1, & x \in [2A-\alpha_2, \alpha_2] \end{cases}$$

and set $Z = \rho W$. Then Z solves

$$\begin{cases} Z_t + (\tilde{a} Z_x)_x + \dfrac{\lambda}{\tilde{b}(x)} Z = \bar{h}, & (t,x) \in (0,T) \times (2A-B, B), \\ Z(t, 2A-B) = Z(t,B) = 0, & t \in (0,T), \end{cases}$$

with $\bar{h} = (\tilde{a}\rho_x W)_x + \tilde{a}\rho_x W_x$, supported in $[2A-\beta_1, 2A-\alpha_2] \cup [\alpha_2, \beta_1]$.

Set $\tilde{\Phi} := \Theta(t)\tilde{\rho}(x)$, with

$$\tilde{\rho}(x) := e^{r\tilde{\zeta}(x)} - \mathfrak{c}$$

and

$$\tilde{\zeta}(x) = \mathfrak{d} \int_x^B \frac{1}{\tilde{a}(t)} dt \text{ with } x \in [2A-B, B],$$

apply Proposition 4.1 in $[2A-B, B]$ and Lemma 4.2 and find

$$\begin{aligned} &\int_0^T \int_A^{\alpha_2} \left[s\Theta (v_x)^2 e^{2s\Phi} + s^3 \Theta^3 v^2 e^{2s\Phi} \right] dxdt \\ &= \int_0^T \int_A^{\alpha_2} \left[s\Theta (Z_x)^2 e^{2s\Phi} + s^3 \Theta^3 Z^2 e^{2s\Phi} \right] dxdt \\ &= \int_0^T \int_A^{\alpha_2} \left[s\Theta (Z_x)^2 e^{2s\tilde{\Phi}} + s^3 \Theta^3 Z^2 e^{2s\tilde{\Phi}} \right] dxdt \\ &\le \int_0^T \int_{2A-B}^{B} \left[s\Theta (Z_x)^2 e^{2s\tilde{\Phi}} + s^3 \Theta^3 Z^2 e^{2s\tilde{\Phi}} \right] dxdt \le C \int_0^T \int_\omega v^2 dxdt. \end{aligned} \tag{4.6}$$

Thus, by (4.4), (4.5) and (4.6), the thesis follows. □

Using Lemma 4.3 and proceeding as in [18, Lemma 4.9], we will deduce the following observability inequality:

Proposition 4.4. *Take $A < B$ in $\mathbb{R}$ and let $a, b \in W^{1,\infty}(A,B)$ be such that $a(x) \ge a_0 > 0$ and $b(x) \ge b_0 > 0$ for all $x \in [A,B]$. Let $\omega \subset\subset (A,B)$ be an open interval. Then there exists a positive constant C_T such that every solution $v \in C([0,T]; L^2(A,B)) \cap L^2(0,T; H^1(A,B))$ of (4.1) satisfies*

$$\int_A^B v^2(0,x) dx \le C_T \int_0^T \int_\omega v^2(t,x) dxdt. \tag{4.7}$$

Hence Theorem 3.2 follows in a standard way.

Remark 4.5. Lemma 4.3 and Proposition 4.4 still hold true if $\omega = \omega_1 \cup \omega_2$, where $\omega_i = (\lambda_i, \gamma_i) \subset\subset (A, B)$, $i = 1, 2$, $\gamma_1 < \lambda_2$. Indeed, in the proof of Lemma 4.3 under this assumption it is enough to proceed as before taking α_i, β_i, $i = 1, 2$ such that $\lambda_1 < \alpha_1 < \alpha_2 < \gamma_1$ and $\lambda_2 < \beta_1 < \beta_2 < \gamma_2$, ξ as before,

$$\tau(x) = \begin{cases} 0, & x \in [A, \alpha_1], \\ 1, & x \in [\alpha_2, B] \end{cases}$$

and

$$\rho(x) = \begin{cases} 0, & x \in [2A - B, 2A - \beta_2] \cup [\beta_2, B], \\ 1, & x \in [2A - \beta_1, \beta_1]. \end{cases}$$

As a consequence, Theorem 3.2 still holds true if $\omega = \omega_1 \cup \omega_2$.

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REFERENCES

[1] F. Alabau-Boussouira, P. Cannarsa, G Fragnelli, *Carleman estimates for degenerate parabolic operators with applications to null controllability*, J. Evol. Eqs **6** (2006), 161–204.

[2] K. Beauchard, P. Cannarsa, R. Guglielmi, *Null controllability of Grushin-type operators in dimension two*, J. Eur. Math. Soc. (JEMS) **16** (2014), 67–101.

[3] I. Boutaayamou, G. Fragnelli, L. Maniar, *Lipschitz stability for linear cascade parabolic systems with interior degeneracy*, Electron. J. Diff. Equ. **2014** (2014), 1–26.

[4] I. Boutaayamou, G. Fragnelli, L. Maniar, *Carleman estimates for parabolic equations with interior degeneracy and Neumann boundary conditions*, J. Anal. Math. **135** (2018), 1–35.

[5] P. Cannarsa, G. Fragnelli, D. Rocchetti, *Null controllability of degenerate parabolic operators with drift*, Netw. Heterog. Media **2** (2007), 693–713.

[6] P. Cannarsa, G. Fragnelli, D. Rocchetti, *Controllability results for a class of one-dimensional degenerate parabolic problems in nondivergence form*, J. Evol. Equ. **8** (2008), 583–616.

[7] P. Cannarsa, P. Martinez, J. Vancostenoble, *Null controllability of the degenerate heat equations*, Adv. Differential Equations **10** (2005), 153–190.

[8] P. Cannarsa, P. Martinez, J. Vancostenoble, *Carleman estimates for a class of degenerate parabolic operators*, SIAM J. Control Optim. **47** (2008), 1–19.

[9] S. Ervedoza, *Null Controllability for a singular heat equation: Carleman estimates and Hardy inequalities*, Comm. Partial Differential Equations **33** (2008), 1996–2019.

[10] G. Floridia, *Approximate controllability for nonlinear degenerate parabolic problems with bilinear control*, J. Differential Equations **257** (2014), 3382–3422.

[11] M. Fotouhi, L. Salimi, *Null controllability of degenerate/singular parabolic equations*, J. Dyn. Control Syst. **18** (2012), 573–602.

[12] M. Fotouhi, L. Salimi, *Controllability results for a class of one dimensional degenerate/singular parabolic equations*, Commun. Pure Appl. Anal. **12** (2013), 1415–1430.

[13] G. Fragnelli, *Null controllability of degenerate parabolic equations in non divergence form via Carleman estimates*, Discrete Contin. Dyn. Syst. Ser. S **6** (2013), 687–701.

[14] G. Fragnelli, *Interior degenerate/singular parabolic equations in nondivergence form: well-posedness and Carleman estimates*, J. Differential Equations **260** (2016), 1314–1371.

[15] G. Fragnelli, D. Mugnai, *Carleman estimates and observability inequalities for parabolic equations with interior degeneracy*, Advances in Nonlinear Analysis **2** (2013), 339–378.

[16] G. Fragnelli, D. Mugnai, *Carleman estimates, observability inequalities and null controllability for interior degenerate non smooth parabolic equations*, Mem. Amer. Math. Soc. **242** (2016) 1146.

[17] G. Fragnelli, D. Mugnai, *Corrigendum and improvements to "Carleman estimates, observability inequalities and null controllability for interior degenerate non smooth parabolic equations, and its consequences*", to appear in Mem. Amer. Math. Soc.

[18] G. Fragnelli, D. Mugnai, *Carleman estimates for singular parabolic equations with interior degeneracy and non smooth coefficients*, Adv. Nonlinear Anal. **6** (2017), 61–84.

[19] G. Fragnelli, D. Mugnai, *Singular parabolic equations with interior degeneracy and non smooth coefficients: the Neumann case*, to appear in Discrete Contin. Dyn. Syst. Ser. S.

[20] G. Fragnelli, D. Mugnai, *Controllability of strongly degenerate parabolic problems with strongly singular potentials*, Electron. J. Qual. Theory Differ. Equ. **2018**, no. 50 , 1–11.

[21] G. Fragnelli, G. Ruiz Goldstein, J.A. Goldstein, S. Romanelli, *Generators with interior degeneracy on spaces of L^2 type*, Electron. J. Differential Equations **2012** (2012), 1–30.

[22] A. Hajjaj, L. Maniar, J. Salhi, *Carleman estimates and null controllability of degenerate/singular parabolic systems*, Electron. J. Differential Equations **2016** (2016) 292, pp. 1–25.

[23] H. Koch, D. Tataru, *Carleman estimates and unique continuation for second order parabolic equations with nonsmooth coefficients*, Comm. Partial Differential Equations **34** (2009), 305–366.

[24] S. Micu, E. Zuazua, *On the lack of null controllability of the heat equation on the half-line*, Trans. Amer. Math. Soc. **353** (2001), 1635–1659.

[25] B. Muckenhoupt, *Hardy's inequality with weights*, Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, I. Studia Math. **44** (1972), 31–38.

[26] M. Renardy, R.C. Rogers, *An Introduction to Partial Differential Equations*, 2nd ed., Texts Appl. Math. 13, Springer, New York, 2004.

[27] D.D. Repovš, *The Ambrosetti-Prodi problem with degenerate potential and Neumann boundary condition*, arXiv:1802.03194.

[28] J. Vancostenoble, *Improved Hardy-Poincaré inequalities and sharp Carleman estimates for degenerate/singular parabolic problems*, Discrete Contin. Dyn. Syst. Ser. S **4** (2011), 761–790.

[29] J. Vancostenoble, E. Zuazua, *Null controllability for the heat equation with singular inverse-square potentials*, J. Funct. Anal. **254** (2008), 1864–1902.

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