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OSCILLATION CRITERIA FOR THIRD ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS WITH DAMPING

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Abstract. This note is concerned with the oscillation of third order nonlinear delay differential equations of the form

$$\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)' + p(t)y'(t) + q(t)f(y(g(t))) = 0. \tag{$*$}$$

In the papers [A.Tiryaki, M.F. Aktas, *Oscillation criteria of a certain class of third order nonlinear delay differential equations with damping*, J. Math. Anal. Appl. 325 (2007), 54–68] and [M.F. Aktas, A. Tiryaki, A. Zafer, *Oscillation criteria for third order nonlinear functional differential equations*, Applied Math. Letters 23 (2010), 756–762], the authors established some sufficient conditions which insure that any solution of equation ($*$) oscillates or converges to zero, provided that the second order equation

$$\left(r_2(t)z'(t)\right)' + \left(p(t)/r_1(t)\right)z(t) = 0 \tag{$**$}$$

is nonoscillatory. Here, we shall improve and unify the results given in the above mentioned papers and present some new sufficient conditions which insure that any solution of equation ($*$) oscillates if equation ($**$) is nonoscillatory. We also establish results for the oscillation of equation ($*$) when equation ($**$) is oscillatory.

Keywords: oscillation, third order, delay differential equation.

Mathematics Subject Classification: 34C10, 39A10

1. INTRODUCTION

In this note, we consider a nonlinear third order functional differential equations of the form

$$\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)' + p(t)y'(t) + q(t)f(y(g(t))) = 0, \tag{1.1}$$

where $q \in C(I,R)$, $r_2, p \in C(I,R)$, $r_1 \in C^2(I,R)$, $I = [t_0,\infty) \subset R$, $t_0 \geq 0$, $r_1(t) > 0$, $r_2(t) > 0$, $p(t) \geq 0$, $q(t) > 0$, $g \in C^1(I,R)$ satisfies $0 < g(t) \leq t$, $g'(t) \geq 0$ and $g(t) \to \infty$ as $t \to \infty$ and $f \in C(R,R)$ satisfies $f(u)/u \geq K > 0$ for $u \neq 0$.

A function $y(t)$ is called the solution of equation (1.1) if $y(t) \in C[t_y,\infty)$, $r_1(t)y'(t) \in C^1[t_y,\infty)$ and $r_2(t)\left(r_1(t)y'(t)\right)' \in C^1[t_y,\infty)$ and $y(t)$ satisfies equation (1.1) on $[t_y,\infty)$ for every $t \geq t_y \geq t_0$.

We restrict our attention to those solutions of equation (1.1) which exist on I and satisfy the condition $\sup\{|y(t)| : t_1 \leq t < \infty\} > 0$ for $t_1 \in [t_0,\infty)$. Such a solution is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all of its solutions are oscillatory.

Determining oscillation criteria for particular second order differential equations has received a great deal of attention in the last few years. Compared to second order differential equations, the study of oscillation and asymptotic behavior of third order differential equations has received considerably less attention in the literature. In the ordinary case for some recent results on third order equations the reader can refer to Cecchi and Marini [3, 4], Parhi and Das [10, 11], Parhi and Padhi [12], Skerlik [13], Tiryaki and Yaman [14], Aktas and Tiryaki [1]. It is interesting to note that there are third order delay differential equations which have only oscillatory solutions or have both oscillatory and nonoscillatory solutions. For example,

$$y'''(t) + 2y'(t) + y(t - \pi/2) = 0$$

admit an oscillatory solution $y_1(t) = \sin t$ and a nonoscillatory solution $y_2(t) = e^{\lambda t}$, where $\lambda < 0$ is a root of the characteristic equation of this equation, namely

$$\lambda^3 + 2\lambda + e^{-\lambda\pi/2} = 0.$$

On the other hand, all solutions of

$$y'''(t) + y(t - \tau) = 0, \quad \tau > 0,$$

are oscillatory if and only if $\tau e > 3$ (see [9]). But the corresponding ordinary differential equation

$$y'''(t) + y(t) = 0,$$

admits a nonoscillatory solution $y_1(t) = e^{-t}$ and oscillatory solutions $y_2(t) = e^{t/2}\sin\left(\sqrt{3}/2t\right)$ and $y_3(t) = e^{t/2}\cos\left(\sqrt{3}/2t\right)$.

In the literature there are some papers and books, for example Agarwal et al. [2], Grace and Lalli [5], Parhi and Das [10, 11], Parhi and Padhi [12], Skerlik [13], and Tiryaki and Yaman [14], which deal with the oscillatory and asymptotic behavior of solutions of functional differential equations. In [1, 15], the authors used a generalized Riccati transformation and an integral averaging technique for establishing some sufficient conditions which insure that any solution of equation (1.1) oscillates or converges to zero. The purpose of this note is to improve and unify the results in [1, 15] and present some new sufficient conditions which insure that any solution of equation (1.1) oscillates when equation $(**)$ is nonoscillatory, or oscillatory.

We also apply our results to the equations of the form

$$a_3(t)y'''(t) + a_2(t)y''(t) + a_1(t)y'(t) + q^*(t)f(x(g(t))) = 0, \tag{1.2}$$

where $a_i(t)$, $i = 1, 2, 3$, and $q^*(t)$ are positive continuous functions on $[t_0, \infty)$, g and f are as in equation (1.1).

2. MAIN RESULTS

For the sake of brevity, we define

$$L_0 y(t) = y(t), \quad L_i y(t) = r_i(t)\left(L_{i-1}y(t)\right)', \ i = 1, 2, \quad \text{and} \quad L_3 y(t) = \left(L_2 y(t)\right)'$$

for $t \in [t_0, \infty)$. So equation (1.1) can be written as

$$L_3 y(t) + p(t)y'(t) + q(t)f(y(g(t))) = 0.$$

Remark 2.1. If y is a solution of (1.1), then $z = -y$ is a solution of the equation

$$L_3 z(t) + p(t)z'(t) + q(t)f^*(z(g(t))) = 0,$$

where $f^*(z) = -f(-z)$ and $zf^*(z) > 0$ for $z \neq 0$. Thus, concerning nonoscillatory solutions of (1.1) we can restrict our attention only to solutions which are positive for all large t.

Define the functions

$$R_1(t, t_1) = \int_{t_1}^{t} \frac{ds}{r_1(s)} \quad \text{and} \quad R_2(t, t_1) = \int_{t_1}^{t} \frac{ds}{r_2(s)}$$

for $t_0 \le t_1 \le t < \infty$. We assume that

$$R_1(t, t_0) \to \infty \quad \text{as} \quad t \to \infty, \tag{2.1}$$

and

$$R_2(t, t_0) \to \infty \quad \text{as} \quad t \to \infty. \tag{2.2}$$

In this section we state and prove the following lemmas which we will use in the proof of our main results.

Lemma 2.2 ([15]). *Suppose that*

$$\left(r_2(t)z'(t)\right)' + \left(p(t)/r_1(t)\right)z(t) = 0 \tag{2.3}$$

is nonoscillatory. If y is a nonoscillatory solution of (1.1) on $[t_1, \infty)$, $t_1 \ge t_0$, then there exists a $t_2 \in [t_1, \infty)$ such that $y(t)L_1(y(t)) > 0$ or $y(t)L_1(y(t)) < 0$ for $t \ge t_2$.

In the following two lemmas, we consider the second order delay differential equation

$$(r_2(t)x'(t))' = Q(t)x(h(t)), \tag{2.4}$$

where $r_2(t)$ is as in equation (1.1), $Q \in C(I, R^+)$, and $h \in C^1(I, R)$ such that $h(t) \leq t$, $h'(t) \geq 0$ for $t \geq t_0$ and $h(t) \to \infty$ as $t \to \infty$.

Lemma 2.3. *If*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} Q(s)R_2(h(t), h(s))ds > 1, \tag{2.5}$$

then all bounded solutions of equation (2.4) *are oscillatory.*

Proof. Let $x(t)$ be a bounded nonoscillatory solution of equation (2.4), say $x(t) > 0$ and $x(h(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then there exists a $t_2 \geq t_1$ such that

$$x(t) > 0, \quad x'(t) < 0 \quad \text{and} \quad (r_2(t)x'(t))' \geq 0 \quad \text{for} \quad t \geq t_2. \tag{2.6}$$

Otherwise, $x'(t) > 0$ for $t \geq t_1$ and so, there exist a constant $c^* > 0$ and a $t_1^* \geq t_1$ such that

$$r_2(t)x'(t) \geq c^* \quad \text{for} \quad t \geq t_1^*.$$

Integrating this inequality from t_1^* to t and using condition (2.2), we see that $x(t) \to \infty$ as $t \to \infty$, which contradicts the fact that $x(t)$ is bounded on $[t_1, \infty)$.

Now, for $v \geq u \geq t_2$ we have

$$\begin{aligned} x(u) - x(v) &= -\int_u^v x'(s)ds = -\int_u^v (r_2(s))^{-1} (r_2(s)x'(s))\, ds \\ &\geq \left(\int_u^v (r_2(s))^{-1}ds\right) (-r_2(v)x'(v)) = R_2(v,u)\,(-r_2(v)x'(v))\,. \end{aligned} \tag{2.7}$$

For $t \geq s \geq t_2$, setting $u = h(s)$ and $v = h(t)$ in (2.7), we get

$$x(h(s)) \geq R_2(h(t), h(s))\,(-r_2(h(t))x'(h(t)))\,. \tag{2.8}$$

Integrating equation (2.4) from $h(t) \geq t_2$ to t, we have

$$-r_2(h(t))x'(h(t)) \geq r_2(t)x'(t) - r_2(h(t))x'(h(t)) = \int_{h(t)}^{t} Q(s)x(h(s))ds. \tag{2.9}$$

Using (2.8) in (2.9), we have

$$-r_2(h(t))x'(h(t)) \geq \left(\int_{h(t)}^{t} Q(s)R_2(h(t), h(s))ds\right) (-r_2(h(t))x'(h(t)))$$

or

$$1 \geq \int_{h(t)}^{t} Q(s)R_2(h(t), h(s))ds. \tag{2.10}$$

We take the lim sup as $t \to \infty$ of both sides of inequality (2.10), we have a contradiction to condition (2.5) and this completes the proof of the lemma. □

Lemma 2.4. *If*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \left(r_2^{-1}(u) \int_{u}^{t} Q(s)ds \right) du > 1, \tag{2.11}$$

then all bounded solutions of equation (2.4) *are oscillatory.*

Proof. Let $x(t)$ be a bounded nonoscillatory solution of equation (2.4), say $x(t) > 0$ and $x(h(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. As in Lemma 2.3, we obtain (2.6). Integrating equation (2.4) from u to t, we have

$$r_2(t)x'(t) - r_2(u)x'(u) = \int_{u}^{t} Q(s)x(h(s))ds$$

or

$$-x'(u) \geq \left(\left(r_2^{-1}(u)\right) \int_{u}^{t} Q(s)ds \right) x(h(t)).$$

Integrating this inequality from $h(t)$ to t, we get

$$x(h(t)) \geq \left[\int_{h(t)}^{t} \left(\left(r_2^{-1}(u)\right) \int_{u}^{t} Q(s)ds \right) du \right] x(h(t))$$

or

$$1 \geq \left[\int_{h(t)}^{t} \left(\left(r_2^{-1}(u)\right) \int_{u}^{t} Q(s)ds \right) du \right].$$

The rest of the proof is similar to that of Lemma 2.3 and hence is omitted. This completes the proof. □

Now, we are ready to establish the main results of this note.

Theorem 2.5. *Let conditions* (2.1), (2.2) *hold and equation* (2.3) *is nonoscillatory. If there exist two functions* ρ *and* $h \in C^1(I, R)$ *such that* $g(t) \leq h(t) < t$, $h'(t) \geq 0$ *and* $\rho(t) > 0$ *for* $t \geq t_0$ *such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[K\rho(s)q(s) - \frac{r_1(g(s))\left(\rho'(s)r_1(s) - \rho(s)p(s)R_2(g(s), t_1)\right)^2}{4\rho(s)R_2(g(s), t_1)g'(s)r_1^2(s)} \right] ds = \infty, \tag{2.12}$$

for all large t and condition (2.5) *or* (2.11) *holds with*

$$Q(t) = [Kq(t)R_1(h(t), g(t)) - (p(t)/r_1(t))] \geq 0 \quad \textit{for} \quad t \geq t_1,$$

then equation (1.1) *is oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution of (1.1) on $[t_1, \infty)$, $t \geq t_1$. Without loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. It follows from Lemma 2.2 that $L_1 y(t) < 0$ or $L_1 y(t) > 0$ for $t \geq t_1$. If $L_1 y(t) > 0$ for $t \geq t_1$, then one can easily see that $L_2 y(t) > 0$ for $t \geq t_1$. Otherwise, $L_2 y(t) < 0$ for $t \geq t_1$ as so there exist a constant $c^* < 0$ and a $t_1^* \geq t_1$ such that

$$L_1 y(t) \leq \frac{c^*}{r_2(t)} \quad \text{for} \quad t \geq t_1^*.$$

Integrating this inequality from t_1^* to t and using condition (2.2) we see that $L_1 y(t) \to -\infty$ as $t \to \infty$. Thus there exist a constant $c^{**} < 0$ and a $t_1^{**} \geq t_1^*$ such that

$$y'(t) \leq \frac{c^{**}}{r_1(t)} \quad \text{for} \quad t \geq t_1^{**}.$$

Integrating this inequality from t_1^* to t and using condition (2.1) we find that $y(t) \to -\infty$ as $t \to \infty$, which contradicts the fact that $y(t) > 0$ for $t \geq t_1$. Next, we define

$$w(t) = \rho(t) \frac{L_2 y(t)}{y(g(t))} \quad \text{for} \quad t \geq t_1. \tag{2.13}$$

First we claim that

$$L_1 y(t) \geq L_1 y(g(t)) \geq R_2(g(t), t_1) L_2\left(y(g(t))\right) \geq R_2(g(t), t_1) L_2\left(y(t)\right) \quad \text{for} \quad t \geq t_1. \tag{2.14}$$

To this end we have,

$$L_1 y(g(t)) \geq \int_{t_1}^{g(t)} \left(L_1 y(s)'\right) ds = \int_{t_1}^{g(t)} \frac{1}{r_2(s)} L_2 y(s) ds \geq L_2 y(g(t)) R_2(g(t), t_1).$$

Since $L_3 y(t) \leq 0$, we get $L_2 y(g(t)) \geq L_2 y(t)$. This completes the proof of the claim.

By equation (1.1) and (2.14), we have

$$w'(t) \leq -K\rho(t)q(t) - \left[w^2(t) \left(\frac{R_2(g(t), t_1) g'(t)}{r_1(g(t))\rho(t)} \right) - w(t) \frac{\rho'(t)}{\rho(t)} - \rho(t) \frac{R_2(g(t), t_1)}{r_1(t)} \right], \tag{2.15}$$

and hence

$$w'(t) \leq -K\rho(t)q(t) + \frac{r_1(g(t)) \left(\rho'(t) r_1(t) - \rho(t) p(t) R_2(g(t), t_1)\right)^2}{4\rho(t) R_2(g(t), t_1) g'(t) r_1^2(t)}.$$

Integrating this inequality from t_1 to t we have

$$\int_{t_1}^{t} \left[K\rho(s)q(s) - \frac{r_1(g(s))\left(\rho'(s)r_1(s) - \rho(s)p(s)R_2(g(s),t_1)\right)^2}{4\rho(s)R_2(g(s),t_1)g'(s)r_1^2(s)} \right] ds$$
$$\leq w(t_1) - w(t) \leq w(t_1)$$

which contradicts condition (2.2). Next, we let $L_1y(t) < 0$ for $t \geq t_1$ and consider the function $L_2y(t)$. The case $L_2y(t) \leq 0$ cannot hold for all large t, say $t \geq t_2 \geq t_1$, since by integration of inequality

$$y'(t) \leq \frac{L_2y(t_2)}{r_1(t)}, \quad t \geq t_2,$$

we obtain from (2.1) $y(t) < 0$ for all large t, a contradiction.

Let $y(t) > 0, L_1y(t) < 0$ and $L_2y(t) \geq 0$ for all large t, say $t \geq t_3 \geq t_2$. Now, for $v \geq u \geq t_3$, we have

$$y(u) - y(v) = -\int_{u}^{v} \frac{1}{r_1(\tau)}\left(r_1(\tau)y'(\tau)\right)d\tau \geq \left(\int_{u}^{v} \frac{1}{r_1(\tau)}d\tau\right)\left(-L_1y(v)\right)$$
$$= R_1(v,u)\left(-L_1y(v)\right).$$

Setting $u = g(t)$ and $v = h(t)$, we get

$$y(g(t)) \geq R_1(h(t),g(t))\left(-L_1y(h(t))\right) = R_1(h(t),g(t))x(h(t)) \quad \text{for} \quad t \geq t_3,$$

where $x(t) = -L_1y(t) > 0$ for $t \geq t_3$. From equation (1.1) and the fact that x is decreasing and $g(t) \leq h(t) \leq t$ we obtain

$$\left(r_2(t)x'(t)\right)' + \left(p(t)/r_1(t)\right)x(h(t)) \geq Kq(t)R_1(h(t),g(t))x(h(t))$$

or

$$\left(r_2(t)x'(t)\right)' \geq \left(Kq(t)R_1(h(t),g(t)) - \left(p(t)/r_1(t)\right)\right)x(h(t)) \quad \text{for} \quad t \geq t_3.$$

Proceeding exactly as in the proof of Lemma 2.3 and Lemma 2.4, we obtain the desired conclusion completing the proof of the theorem. □

Remark 2.6. From the proof of Theorem 2.5 we obtain

$$w'(t) \leq -K\rho(t)q(t) + \frac{r_1(g(t))\left(P(t,t_1)\right)^2}{4\rho(t)R_2(g(t),t_1)g'(t)r_1^2(t)},$$

where $P(t,t_1) = \rho'(t)r_1(t) - \rho(t)p(t)R_2(g(t),t_1)$. Now, if $P(t,t_1) \geq 0$ for $t \geq t_3$, we have

$$\rho'(t)r_1(t) \geq P(t,t_1) \quad \text{for} \quad t \geq t_3,$$

and hence

$$w'(t) \le -K\rho(t)q(t) + \frac{r_1(g(t))\,(\rho'(t)r_1(t))^2}{4\rho(t)R_2(g(t),t_1)g'(t)r_1^2(t)} \quad \text{for} \quad t \ge t_3.$$

It is easy to see that condition (2.12) can be replaced by

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[K\rho(s)q(s) - \frac{r_1(g(s))\,(\rho'(s)r_1(s))^2}{4\rho(s)R_2(g(s),t_1)g'(s)r_1^2(s)}\right] ds = \infty \tag{2.16}$$

for all large t.

Next, if the function $P(t,t_1) \le 0$ for $t \ge t_3$, we see that condition (2.12) can be replaced by

$$\int_{t_1}^{\infty} \rho(s)q(s)ds = \infty, \tag{2.17}$$

for all large t.

Finally, if $\rho'(t) \le 0$ for $t \ge t_3$, we see from (2.15) that

$$w'(t) \le -K\rho(t)q(t) + \rho(t)\frac{R_2(g(t),t_1)}{r_1(t)},$$

and so, condition (2.12) can replaced by

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[K\rho(s)q(s) - \rho(s)\frac{R_2(g(s),t_1)}{r_1(s)}\right] ds = \infty, \tag{2.18}$$

for all large t. The details are left to the reader.

The following examples are illustrative.

Example 2.7. Consider the equation

$$y'''(t) + e^{-t}y'(t) + (1-e^{-t})y\Big(t - \frac{5\pi}{2}\Big) = 0. \tag{2.19}$$

It is easy to check that all conditions of Theorem 2.5 are satisfied for $h(t) = t - 2\pi$, $K = 1$ and $\rho(t) = 1$ and hence equation (2.19) is oscillatory. One such solution is $y(t) = \sin t$.

Example 2.8. Consider the equation

$$y'''(t) + e^{2-2t}y'(t) + \frac{1}{e}y(t-1)\left(1 + y^2(t-1)\right) = 0. \tag{2.20}$$

Here we take $K = 1$, $\rho(t) = 1$ and $h(t) = t - 1/2$. Now, it is easy to check that all hypotheses of Theorem 2.5 are fulfilled except conditions (2.5) and (2.11). We note that equation (2.20) admits the nonoscillatory solution $y(t) = e^{-t}$.

Next, we present the following comparison result.

Theorem 2.9. *If in Theorem 2.5 the condition* (2.12) *is replaced by the first order delay equation*

$$w'(t) + \left[\frac{p(t)}{r_1(t)} R_2(g(t), t_1) + Kq(t) \left(\int_{t_1}^{t} \frac{R_2(g(s), t_1)}{r_1(s)} ds \right) \right] w(g(t)) = 0, \tag{2.21}$$

is oscillatory, then the conclusion of Theorem 2.5 holds.

Proof. Let $y(t)$ be a nonoscillatory solution of (1.1) on $[t_1, \infty)$, $t \geq t_1$. Without loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. It follows from Lemma 2.2 that $L_1 y(t) < 0$ or $L_1 y(t) > 0$ for $t \geq t_1$. If $L_1 y(t) > 0$ for $t \geq t_1$, then one can easily see that $L_2 y(t) > 0$ for $t \geq t_1$. As in the proof of Theorem 2.5, we obtain (2.14).

From (2.14) we have

$$r_1(t) y'(t) = L_1 y(t) \geq R_2(g(t), t_1) L_2 \left(y(g(t)) \right) \quad \text{for} \quad t \geq t_1.$$

Dividing this inequality by $r_1(t)$ and integrating from t_1 to t one can easily find

$$y(t) \geq \left(\int_{t_1}^{t} \frac{R_2(g(s), t_1)}{r_1(s)} ds \right) L_2 y(g(t)). \tag{2.22}$$

Using (2.14) and (2.22) in equation (1.1) we have

$$w'(t) + \left(\frac{p(t)}{r_1(t)} \right) R_2(g(t), t_1) w(g(t)) + Kq(t) \left(\int_{t_1}^{t} \frac{R_2(g(s), t_1)}{r_1(s)} ds \right) w(g(t)) \leq 0,$$

where $w(t) = L_2 y(t) > 0$. This inequality has a positive solution and hence by Theorem 2.6.3 in [2], equation (2.21) has a positive solution, which is a contradiction. The proof of the case when $L_1 y(t) < 0$ for $t \geq t_1$ is similar to that of Theorem 2.5 and hence is omitted. This completes the proof. □

The following result is immediate.

Corollary 2.10. *If in Theorem 2.5 the condition* (2.12) *is replaced by*

$$\liminf_{t \to \infty} \int_{g(t)}^{t} \left[\frac{p(u)}{r_1(u)} R_2(g(u), t_1) + Kq(u) \left(\int_{t_1}^{u} \frac{R_2(g(s), t_1)}{r_1(s)} ds \right) \right] du \geq \frac{1}{e}, \tag{2.23}$$

then the conclusion of Theorem 2.5 holds.

Next, if equation (2.1) is oscillatory, we give the following result.

Theorem 2.11. *Let conditions* (2.1), (2.2) *hold and equation* (2.3) *is oscillatory. If there exists a function* $h \in C(I, R)$ *such that* $g(t) \leq h(t) \leq t$ *and* $h'(t) \geq 0$ *for* $t \geq t_0$ *such that* (2.5) *or* (2.11) *holds with* $Q(t)$ *is as in Theorem 2.5, then every solution* y *of equation* (1.1) *either* $y(t)$ *is oscillatory or* $y'(t)$ *is oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution of (1.1) on $[t_1, \infty)$, $t \geq t_1$. Without loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Now, we consider the case $L_1 y(t) < 0$ or $L_1 y(t) > 0$ for $t \geq t_1$. If $L_1 y(t) > 0$ for $t \geq t_1$ holds, then equation (1.1) becomes

$$(r_2(t)x'(t))' + (p(t)/r_1(t))\, x(t) \leq 0 \quad \text{for} \quad t \geq t_2 \geq t_1,$$

where $x(t) = L_1 y(t) > 0$. By [6, Lemma 2.6], equation (2.3) has a positive solution, a contradiction. The proof of the case when $L_1 y(t) < 0$ for $t \geq t_2 \geq t_1$ is similar to that of Theorem 2.5 and hence is omitted. This completes the proof of the theorem. □

Example 2.12. Consider the equation

$$y'''(t) + \frac{1}{2}y'(t) + \frac{1}{2}y\Big(t - \frac{3\pi}{2}\Big) = 0. \tag{2.24}$$

Let $h(t) = t - \pi$. It is easy that to check that all hypotheses of Theorem 2.9 are satisfied and hence every solution y of equation (2.24) is oscillatory or y' is oscillatory. One such solution is $y(t) = \sin t$. We note that none of the results in [3, 8, 10–15] are applicable to equation (2.24).

Finally, we can easily extend Theorems 2.5 and 2.9 to the equation

$$\Big(r_2(t)\,(r_1(t)y'(t))'\Big)' + p(t)y'(h(t)) + q(t)f(y(g(t))) = 0, \tag{2.25}$$

where $h \in C(I, R)$ such that $g(t) \leq h(t) \leq t$ and $h'(t) \geq 0$ for $t \geq t_0$.

Theorem 2.13. *Let conditions* (2.1), (2.2) *hold and the equation*

$$(r_2(t)x'(t))' + (p(t)/r_1(h(t)))\, x(h(t)) = 0 \tag{2.26}$$

is oscillatory. If condition (2.5) *or* (2.11) *holds with*

$$Q(t) = [Kq(t)R_1(h(t), g(t)) - (p(t)/r_1(h(t)))] \geq 0 \quad \textit{for} \quad t \geq t_1,$$

then every solution y *of equation* (2.25) *either* $y(t)$ *is oscillatory or* $y'(t)$ *is oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution of (2.25) on $[t_1, \infty)$, $t \geq t_1$. Without loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. As in the proof of Theorem 2.5, we obtain either $L_1 y(t) < 0$ or $L_1 y(t) > 0$ for $t \geq t_1$. If $L_1 y(t) > 0$ for $t \geq t_1$ holds, then equation (2.25) becomes

$$(r_2(t)x'(t))' + (p(t)/r_1(h(t)))\, x(h(t)) \leq 0 \quad \text{for} \quad t \geq t_2 \geq t_1, \tag{2.27}$$

where $x(t) = L_1 y(t) > 0$. By [6, Lemma 2.6], equation (2.26) has a positive solution, a contradiction. The proof of the case when $L_1 y(t) < 0$ for $t \geq t_2 \geq t_1$ is similar to that of Theorem 2.5 and hence is omitted. This completes the proof of the theorem. □

We note that there are many criteria in the literature for the oscillation of second order dynamic equations, and so by applying these results to equation (1.1) and (2.25), we can obtain many oscillation results which are of similar types to these in [1, 15] or else, of different types. The formulations of such results are left to the reader.

The following examples are illustrative.

Example 2.14. Consider the equation

$$y'''(t) + y'(t-\pi) + 2y\Big(t - \frac{3\pi}{2}\Big) = 0. \tag{2.28}$$

It is easy to check that all hypotheses of Theorem 2.11 are satisfied with $h(t) = t - 2\pi$ and hence every solution y of equation (2.28) either $y(t)$ is oscillatory or $y'(t)$ is oscillatory. One such solution is $y(t) = \sin t$. We note that none of the known results appeared in the literature are applicable to this equation because of the delay that appeared in the damping term.

Example 2.15. Consider the equation (2.28) without delays, namely

$$y'''(t) + y'(t) + 2y(t) = 0. \tag{2.29}$$

has a nonoscillatory solution $y(t) = e^{-t}$ and $y'(t) = -e^{-t}$ is also nonoscillatory. Conditions which involved delays in Theorem 2.11 are not fulfilled. The solution set of equation (2.29) is

$$\Big\{e^{-t}, e^{t/2}\cos(\sqrt{7}/2)t, e^{t/2}\sin(\sqrt{7}/2)t\Big\}.$$

We note that the presence of delays in equation (2.29) generate oscillation.

In order to apply results to equation (1.2), we can rewrite equation (1.2) in the form

$$\left(\exp\left(\int_{t_0}^{t} a_2(s)/a_3(s)ds\right)y''(t)\right)' + \exp\left(\int_{t_0}^{t} a_2(s)/a_3(s)ds\right)(a_1(t)/a_3(t))\,y'(t)$$
$$+\exp\left(\int_{t_0}^{t} a_2(s)/a_3(s)ds\right)(a_1(t)/a_3(t))\,(q^*(t)/a_3(t))\,f(y(g(t))) = 0.$$

In this case, our results are applicable to equation (1.2) if we let

$$r_1(t) = 1,$$
$$r_2(t) = \exp\left(\int_{t_0}^{t} a_2(s)/a_3(s)ds\right),$$
$$p(t) = \exp\left(\int_{t_0}^{t} a_2(s)/a_3(s)ds\right)(a_1(t)/a_3(t))$$

and

$$q(t) = \exp\left(\int_{t_0}^{t} a_2(s)/a_3(s)ds\right)(a_1(t)/a_3(t))\,(q^*(t)/a_3(t)).$$

The formulation of the results as a special case of these obtained above are left to the reader.

3. GENERAL REMARKS

1. The results of this paper are presented in a form that is essentially new and of a high degree of generality.
2. It would be of interest to consider equations (1.1) and (2.22) and try to obtain some oscillation criteria if for $p(t) < 0$ and $q(t) < 0$.
3. Finally, we note that the results in [15] are applicable to equation (1.1) if $g(t) \leq t$, while our oscillation results are applicable to equation (1.1) if $g(t) < t$. Thus, as is well known, it is the delay in equation (1.1) that can generate the oscillations.

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