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Descriptor fractional discrete-time linear system with two different fractional orders and its solution

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Abstract. Factional Discrete-time linear systems with fractional different orders are addressed. The Weierstrass-Kronecker decomposition theorem of the regular pencil is extended to the descriptor fractional discrete-time linear system with different fractional orders. Using the extension, method for finding the solution of the state equation is derived. Effectiveness of the method is demonstrated on a numerical example.

Key words: fractional, different order, descriptor, solution.

1. Introduction

Descriptor (singular) linear systems have been considered in many papers and books [1–9]. The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19^{th} century [10, 11] and another one was proposed in 20^{th} century by Caputo [12]. This idea has been used by engineers for modeling different processes [13–15]. Mathematical fundamentals of fractional calculus are given in the monographs [10–12, 16]. Solutions of the state equations of descriptor fractional discrete-time linear systems with regular pencils have been given in [7, 17] and for continuous-time in [5, 6]. Reduction and decomposition of singular fractional discrete-time linear systems has been considered in [18]. Application of the Drazin inverse method to analysis of descriptor fractional discrete-time and continuous-time linear systems have been given in [19, 20]. The positive fractional linear systems has been investigated in [21, 22]. The positive linear systems with different fractional orders have been addressed in [23, 24]. Stability of fractional continuous-time linear systems consisting of n subsystem with different fractional orders has been given in [25]. Reachability and minimum energy control problem for systems with two different fractional orders has been considered in [26]. Solution of the state equation of descriptor fractional continuous-time linear systems with two different fractional orders has been introduced in [27]. Comparison of three different methods for finding the solution of the descriptor fractional discrete-time linear system has been given in [28].

In this paper the solution to descriptor fractional discrete-time linear systems with two different fractional order is derived.

The paper is organized as follows. In Sec. 2 basic information on the fractional discrete-time linear systems with different fractional orders is recalled. Descriptor fractional discrete-time linear systems with different fractional orders are addressed in Sec. 3, where the Weierstrass-Kronecker decomposition is given. Main idea of the paper is presented in Sec. 4, where the solution to descriptor fractional discrete-time linear systems with different fractional orders is derived and illustrated by numerical example. Concluding remarks are given in Sec. 5.

The following notation is used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n \times m$ real matrices, Z_+ – the set of nonnegative integers, I_n – the $n \times n$ identity matrix, A^T – the transpose matrix A.

2. Fractional different orders discrete-time linear systems

Consider the fractional discrete-time linear system with two different fractional orders α and β of the form

$$\begin{aligned}\Delta^\alpha x_1(k+1) &= A_{11}x_1(k) + A_{12}x_2(k) + B_1u(k),\\ \Delta^\beta x_2(k+1) &= A_{21}x_1(k) + A_{22}x_2(k) + B_2u(k),\end{aligned} \tag{1}$$

where $k \in Z_+$, $x_1(k) \in \Re^{n_1}$ and $x_2(k) \in \Re^{n_2}$ are the state vectors, $u(k) \in \Re^m$ is the input vector and $A_{ij} \in \Re^{n_i\times n_j}$, $B_i \in \Re^{n_i\times m}$; i, j = 1,2, $n = n_1 + n_2$.

The fractional difference of α (β) order is defined by [22]

$$\begin{gathered}\Delta^\alpha x(k) = \sum_{j=0}^{k}(-1)^j\begin{pmatrix}\alpha\\ j\end{pmatrix}x(k-j) = \sum_{j=0}^{k}c_\alpha(j)x(k-j),\\ c_\alpha(j) = (-1)^j\begin{pmatrix}\alpha\\ j\end{pmatrix} = (-1)^j\frac{\alpha(\alpha-1)...(\alpha-j+1)}{j!},\\ c_\alpha(0) = 1, \quad j = 1,2,...\end{gathered} \tag{2}$$

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Using (2) we can write Eq. (1) in the matrix form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} - \sum_{j=2}^{k+1} \begin{bmatrix} c_\alpha(j) I_{n_1} & 0 \\ 0 & c_\beta(j) I_{n_2} \end{bmatrix} \begin{bmatrix} x_1(k-j+1) \\ x_2(k-j+1) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k), \tag{3}$$

where $A_{1\alpha} = A_{11} + I_1\alpha$, $A_{2\beta} = A_{22} + I_2\beta$.

Note that, the fractional system (3) is equivalent to the 2D standard system with increasing number of delays.

Theorem 1. The solution to the fractional system described by Eq. (1) with initial conditions $x_1(0) = x_{10}$, $x_2(0) = x_{20}$ is given by

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \Phi_k \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \sum_{i=0}^{k-1} \Phi_{k-i-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(i), \quad k \in Z_+, \tag{4}$$

where Φ_k is defined by

$$\Phi_k = \begin{cases} I_{n_1+n_2} & \text{for } k = 0, \\ A\Phi_{k-1} - D_1\Phi_{k-2} - \ldots - D_{k-1}\Phi_0 & \text{for } k = 1, 2, \ldots, i \\ A\Phi_{k-1} - D_1\Phi_{k-2} - \ldots - D_i\Phi_{k-i-1} & \text{for } k = i+1, i+2, \ldots \end{cases} \tag{5a}$$

and

$$A = \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix}, \quad D_k = \begin{bmatrix} c_\alpha(k+1) I_{n_1} & 0 \\ 0 & c_\beta(k+1) I_{n_2} \end{bmatrix}. \tag{5b}$$

Proof is given in [22].

3. Descriptor fractional different orders discrete-time linear systems

Consider the descriptor fractional discrete-time linear system with two different fractional orders

$$\begin{aligned} E_1 \Delta^\alpha x_1(k+1) &= A_{11}x_1(k) + A_{12}x_2(k) + B_1u(k), \\ E_2 \Delta^\beta x_2(k+1) &= A_{21}x_1(k) + A_{22}x_2(k) + B_2u(k), \end{aligned} \tag{6}$$

where $k \in Z_+$, $x_1(k) \in \Re^{n_1}$ and $x_2(k) \in \Re^{n_2}$ are the state vectors, $u(k) \in \Re^m$ is the input vector and $E_i, A_{ij} \in \Re^{n_i \times n_j}$, $B_i \in \Re^{n_i \times m}$; $i, j = 1, 2$.

Similar as in Sec. 2, using (2) we can write Eq. (6) in the matrix form

$$\begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} - \sum_{j=2}^{k+1} \begin{bmatrix} c_\alpha(j) E_1 & 0 \\ 0 & c_\beta(j) E_2 \end{bmatrix} \begin{bmatrix} x_1(k-j+1) \\ x_2(k-j+1) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k), \tag{7}$$

where $A_{1\alpha} = A_{11} + E_1\alpha$, $A_{2\beta} = A_{22} + E_2\beta$ and $E_1 \in \Re^{n_1 \times n_1}$, $E_2 \in \Re^{n_2 \times n_2}$, which represents the descriptor 2D standard system with increasing number of delays.

Finding a solution for the system (7) is very difficult since $\det E_1 = 0$, $\det E_2 = 0$ and the pencil is very complex.

It is much easier to use a standard form

$$E \begin{bmatrix} \Delta^\alpha x_1(k+1) \\ \Delta^\beta x_2(k+1) \end{bmatrix} = A \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + Bu(k), \tag{8a}$$

where

$$E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{8b}$$

It is assumed that

$$\det E = 0 \tag{9a}$$

and

$$\det \left[\begin{bmatrix} E_1z_1 & 0 \\ 0 & E_2z_2 \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right] \neq 0 \tag{9b}$$

for some $z \in C$ (the field of complex numbers). This assumption lead to descriptor system with regular pencil.

Now, using the Weierstrass-Kronecker decomposition theorem of the regular pencil [22, 29] and adopting it to the systems with two different fractional orders, we have the following Lemma.

Lemma 1. If (9a) and (9b) hold for the systems with two different fractional orders (8), then there exist nonsingular matrices $P, Q \in \Re^{n \times n}$ such that

$$P \left[\begin{bmatrix} E_1z_1 & 0 \\ 0 & E_2z_2 \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right] Q = \begin{bmatrix} \overline{E}_1z_1 & 0 \\ 0 & \overline{E}_2z_2 \end{bmatrix} - \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix}, \tag{10}$$

where

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \tag{11}$$

and the following decomposition is possible

$$
\begin{aligned}
\overline{E}_1 = P_1 E_1 Q_1 &= \begin{bmatrix} I_{n_1^1} & 0 \\ 0 & N_1 \end{bmatrix}, \\
\overline{E}_2 = P_2 E_2 Q_2 &= \begin{bmatrix} I_{n_2^1} & 0 \\ 0 & N_2 \end{bmatrix}, \\
\overline{A}_{11} = P_1 A_{11} Q_1 &= \begin{bmatrix} \widetilde{A}_{11} & 0 \\ 0 & I_{n_1^2} \end{bmatrix}, \\
\overline{A}_{12} = P_1 A_{12} Q_2 &= \begin{bmatrix} \widetilde{A}_{12}^1 & 0 \\ 0 & \widetilde{A}_{12}^2 \end{bmatrix}, \\
\overline{A}_{21} = P_2 A_{21} Q_1 &= \begin{bmatrix} \widetilde{A}_{21}^1 & 0 \\ 0 & \widetilde{A}_{21}^2 \end{bmatrix}, \\
\overline{A}_{22} = P_2 A_{22} Q_2 &= \begin{bmatrix} \widetilde{A}_{22} & 0 \\ 0 & I_{n_2^2} \end{bmatrix},
\end{aligned}
\tag{12}
$$

where $N_1 \in \Re^{n_1^2 \times n_1^2}$, $N_2 \in \Re^{n_2^2 \times n_2^2}$ are a nilpotent matrices with the index μ_i, $i = 1, 2$ (i.e. $N_i^{\mu_i} = 0$ and $N_i^{\mu_i - 1} \neq 0$), $\widetilde{A}_{11} \in \Re^{n_1^1 \times n_1^1}$, $\widetilde{A}_{22} \in \Re^{n_2^1 \times n_2^1}$, $\widetilde{A}_{21}^1 \in \Re^{n_2^1 \times n_1^1}$, $\widetilde{A}_{21}^2 \in \Re^{n_2^2 \times n_1^2}$, $\widetilde{A}_{12}^1 \in \Re^{n_1^1 \times n_2^1}$, $A_{12}^2 \in \Re^{n_1^2 \times n_2^2}$ and $\operatorname{rank} E_1 = n_1^1$, $\operatorname{rank} E_2 = n_2^1$, $n_1^1 + n_1^2 = n_1$, $n_2^1 + n_2^2 = n_2$, $n_1 + n_2 = n$.

Computation methods for the matrices P and Q have been given e.g. in [8, 9, 22].

Using Lemma 1 and solution presented in [6, 7], the solution $x(k)$ to Eq. (6) with given initial conditions $x(0)$ and an input vector $u(k)$ for $k \in Z_+$ is derived in the next section.

4. Solution of the state equation

Premultiplying the state Eq. (8a) by the matrix $P \in \Re^{n \times n}$ and introducing new state vector

$$
\begin{gathered}
\begin{bmatrix} \overline{x}_1^1(k) \\ \overline{x}_1^2(k) \\ \overline{x}_2^1(k) \\ \overline{x}_2^2(k) \end{bmatrix} = Q^{-1} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \\
\overline{x}_1^1(k) \in \Re^{n_1^1}, \quad \overline{x}_1^2(k) \in \Re^{n_1^2}, \\
\overline{x}_2^1(k) \in \Re^{n_2^1}, \quad \overline{x}_2^2(k) \in \Re^{n_2^2}
\end{gathered}
\tag{13}
$$

we obtain

$$
\begin{aligned}
& PEQQ^{-1} \begin{bmatrix} \Delta^\alpha x_1(k+1) \\ \Delta^\beta x_2(k+1) \end{bmatrix} \\
&= PAQQ^{-1} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + PBu(k), \quad k \in Z_+.
\end{aligned}
\tag{14}
$$

Now, substituting (12) and (13) into (14) we have

$$
\begin{aligned}
& \begin{bmatrix} I_{n_1^1} & 0 & 0 & 0 \\ 0 & N_1 & 0 & 0 \\ 0 & 0 & I_{n_2^1} & 0 \\ 0 & 0 & 0 & N_2 \end{bmatrix} \begin{bmatrix} \Delta^\alpha \overline{x}_1^1(k+1) \\ \Delta^\alpha \overline{x}_1^2(k+1) \\ \Delta^\beta \overline{x}_2^1(k+1) \\ \Delta^\beta \overline{x}_2^2(k+1) \end{bmatrix} \\
&= \begin{bmatrix} \widetilde{A}_{11} & 0 & \widetilde{A}_{12}^1 & 0 \\ 0 & I_{n_1^2} & 0 & \widetilde{A}_{12}^2 \\ \widetilde{A}_{21}^1 & 0 & \widetilde{A}_{22} & 0 \\ 0 & \widetilde{A}_{21}^2 & 0 & I_{n_2^2} \end{bmatrix} \begin{bmatrix} \overline{x}_1^1(k) \\ \overline{x}_1^2(k) \\ \overline{x}_2^1(k) \\ \overline{x}_2^2(k) \end{bmatrix} + \begin{bmatrix} \widetilde{B}_1^1 \\ \widetilde{B}_1^2 \\ \widetilde{B}_2^1 \\ \widetilde{B}_2^2 \end{bmatrix} u(k)
\end{aligned}
\tag{15}
$$

for $k \in Z_+$, where

$$
\begin{aligned}
\widetilde{B}_1^1 \in \Re^{n_1^1 \times m}, \qquad \widetilde{B}_1^2 \in \Re^{n_1^2 \times m}, \\
\widetilde{B}_2^1 \in \Re^{n_2^1 \times m}, \qquad \widetilde{B}_2^2 \in \Re^{n_2^2 \times m}.
\end{aligned}
\tag{16}
$$

Lastly, from (8) we can distinguish two subsystems. The standard one

$$
\begin{aligned}
& \begin{bmatrix} \Delta^\alpha \overline{x}_1^1(k+1) \\ \Delta^\beta \overline{x}_2^1(k+1) \end{bmatrix} \\
&= \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12}^1 \\ \widetilde{A}_{21}^1 & \widetilde{A}_{22} \end{bmatrix} \begin{bmatrix} x_1^1(k) \\ x_2^1(k) \end{bmatrix} + \begin{bmatrix} \widetilde{B}_1^1 \\ \widetilde{B}_2^1 \end{bmatrix} u(k)
\end{aligned}
\tag{17}
$$

and nilpotent one

$$
\begin{aligned}
& \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} \Delta^\alpha \overline{x}_1^2(k+1) \\ \Delta^\beta \overline{x}_2^2(k+1) \end{bmatrix} \\
&= \begin{bmatrix} I_{n_1^2} & \widetilde{A}_{12}^2 \\ \widetilde{A}_{21}^2 & I_{n_2^2} \end{bmatrix} \begin{bmatrix} x_1^2(k) \\ x_2^2(k) \end{bmatrix} + \begin{bmatrix} \widetilde{B}_1^2 \\ \widetilde{B}_2^2 \end{bmatrix} u(k).
\end{aligned}
\tag{18}
$$

Using Theorem 1, a solution to the subsystem (17) can be computed by the use of the following formula

$$
\begin{gathered}
\begin{bmatrix} \overline{x}_1^1(k) \\ \overline{x}_2^1(k) \end{bmatrix} = \Phi_k \begin{bmatrix} \overline{x}_1^1(0) \\ \overline{x}_2^1(0) \end{bmatrix} + \sum_{i=0}^{k-1} \Phi_{k-i-1} \begin{bmatrix} \widetilde{B}_1^1 \\ \widetilde{B}_2^1 \end{bmatrix} u(i), \\
k \in Z_+,
\end{gathered}
\tag{19}
$$

where

$$
\Phi_k = \begin{cases} I_{n_1^1 + n_2^1} \ \text{ for } \ k = 0, \\ \widetilde{A}\Phi_{k-1} - D_1 \Phi_{k-2} - \ldots - D_{k-1}\Phi_0 \\ \qquad \text{for } \ k = 1, 2, \ldots, i, \\ \widetilde{A}\Phi_{k-1} - D_1 \Phi_{k-2} - \ldots - D_i \Phi_{k-i-1} \\ \qquad \text{for } \ k = i+1, i+2, \ldots, \end{cases}
\tag{20a}
$$

$$
\begin{gathered}
\widetilde{A} = \begin{bmatrix} \widetilde{A}_{1\alpha} & \widetilde{A}_{12}^1 \\ \widetilde{A}_{21}^1 & \widetilde{A}_{2\beta} \end{bmatrix}!, \\
D_k = \begin{bmatrix} c_\alpha(k+1) I_{n_1^1} & 0 \\ 0 & c_\beta(k+1) I_{n_2^2} \end{bmatrix}, \\
\widetilde{A}_{1\alpha} = \widetilde{A}_{11} + I_{n_1^1}\alpha, \quad \widetilde{A}_{2\beta} = \widetilde{A}_{22} + I_{n_2^1}\beta.
\end{gathered}
\tag{20b}
$$

To find a solution of the subsystem (18) for $N_1 \neq 0$, $N_2 \neq 0$ nilpotent (e.g. for

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

we have three equations with three unknown elements) we simple start by solving the equation related with the zero row and then continue solving the rest of the equations, see e.g. [6, 7].

If $N_1 = 0$, $N_2 = 0$ then from (18) we have

$$\begin{bmatrix} \overline{x}_1^2(k) \\ \overline{x}_2^2(k) \end{bmatrix} = \begin{bmatrix} [I_{n_1^2} - \widetilde{A}_{12}^2 \widetilde{A}_{21}^2]^{-1} [\widetilde{A}_{12}^2 \widetilde{B}_2^2 - \widetilde{B}_1^2] \\ [I_{n_2^2} - \widetilde{A}_{21}^2 \widetilde{A}_{12}^2]^{-1} [\widetilde{A}_{21}^2 \widetilde{B}_1^2 - \widetilde{B}_2^2] \end{bmatrix} u(k), \quad k \in Z_+. \tag{21}$$

Finally, knowing

$$\begin{bmatrix} \overline{x}_1^1(k) \\ \overline{x}_2^1(k) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \overline{x}_1^2(k) \\ \overline{x}_2^2(k) \end{bmatrix},$$

from (13), we can find the desired solution of the system (6) in the form

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = Q \begin{bmatrix} \overline{x}_1^1(k) \\ \overline{x}_1^2(k) \\ \overline{x}_2^1(k) \\ \overline{x}_2^2(k) \end{bmatrix}, \qquad k \in Z_+. \tag{22}$$

Remark 1. In this study, only the diagonal form of matrix E has been considered. These considerations can be extended to the systems with the matrix E of (8) in general form. It is well-known [29, 30] that by the use of elementary row operations it is always possible to reduce matrix to its diagonal form.

Examples 1. Find the solution of the descriptor fractional linear system (6) with the fractional orders $\alpha = 0.5$, $\beta = 0.6$, matrices

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 4 & 2 \\ 1 & 4 & 1 \end{bmatrix},$$

$$\begin{gathered} A_{11} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 4 & 11 & 6 \\ 2 & 5 & 2 \\ 0 & 0 & -2 \end{bmatrix}, \\ A_{21} = \begin{bmatrix} 3 & 2 & 6 \\ 9 & 2 & 3 \\ 3 & 7 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0.8 & 1.7 & 2.8 \\ 0.4 & 0.8 & 1.4 \\ 2.2 & 4.6 & 2.2 \end{bmatrix}, \end{gathered} \tag{23}$$

$$B_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

and consistent initial conditions $x_1(0) = [\ 1 \quad 2 \quad -1\ ]^T$, $x_2(0) = [\ 0 \quad 2 \quad 1\ ]^T$.

It is easy to check that the matrices (23) satisfy the assumptions (9). In this case the matrices P and Q have the form

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix},$$

$$\begin{gathered} P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad P_2 = \frac{1}{11} \begin{bmatrix} 1 & -2 & 5 \\ -2 & 4 & 1 \\ 4 & 3 & -2 \end{bmatrix}, \\ Q_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{gathered} \tag{24}$$

and

$$PEQ = \begin{bmatrix} I_{n_1^1} & 0 & 0 & 0 \\ 0 & N_1 & 0 & 0 \\ 0 & 0 & I_{n_2^1} & 0 \\ 0 & 0 & 0 & N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Q^{-1} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} \overline{x}_1^1(k) \\ \overline{x}_1^2(k) \\ \overline{x}_2^1(k) \\ \overline{x}_2^2(k) \end{bmatrix} = \begin{bmatrix} \overline{x}_{11}^1(k) \\ \overline{x}_{12}^1(k) \\ \overline{x}_{11}^2(k) \\ \overline{x}_{21}^1(k) \\ \overline{x}_{22}^1(k) \\ \overline{x}_{21}^2(k) \end{bmatrix},$$

$$\begin{aligned} PAQ &= \begin{bmatrix} \widetilde{A}_{11} & 0 & \widetilde{A}_{12}^1 & 0 \\ 0 & I_{n_1^2} & 0 & \widetilde{A}_{12}^2 \\ \widetilde{A}_{21}^1 & 0 & \widetilde{A}_{22} & 0 \\ 0 & \widetilde{A}_{21}^2 & 0 & I_{n_2^2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0.1 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{bmatrix}, \end{aligned} \tag{25}$$

$$PB = \begin{bmatrix} \widetilde{B}_1^1 \\ \widetilde{B}_2^1 \\ \widetilde{B}_1^2 \\ \widetilde{B}_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0.545 \\ -0.091 \\ 0.182 \end{bmatrix},$$

$$n_1^1 = n_2^1 = 2, \qquad n_1^2 = n_2^2 = 1,$$

$$n_1 = n_2 = 3, \qquad n = n_1 + n_2 = 6.$$

Taking under considerations n_1^1, n_2^1, formula (19) has the form

$$\begin{bmatrix} \overline{x}_{11}^1(k) \\ \overline{x}_{12}^1(k) \\ \overline{x}_{21}^1(k) \\ \overline{x}_{22}^1(k) \end{bmatrix} = \Phi_k \begin{bmatrix} \overline{x}_{11}^1(0) \\ \overline{x}_{12}^1(0) \\ \overline{x}_{21}^1(0) \\ \overline{x}_{22}^1(0) \end{bmatrix} + \sum_{i=0}^{k-1} \Phi_{k-i-1} \begin{bmatrix} \widetilde{B}_{11}^1 \\ \widetilde{B}_{12}^1 \\ \widetilde{B}_{21}^1 \\ \widetilde{B}_{22}^1 \end{bmatrix} u(i), \quad k \in Z_+, \tag{26}$$

where Φ_k is defined by (20a) with

$$\widetilde{A} = \begin{bmatrix} \widetilde{A}_{1\alpha} & \widetilde{A}_{12}^1 \\ \widetilde{A}_{21}^1 & \widetilde{A}_{2\beta} \end{bmatrix} = \begin{bmatrix} 1.5 & 0 & 1 & 2 \\ 0 & 0.5 & 3 & 4 \\ 3 & 0 & 0.7 & 1 \\ 1 & 3 & 0 & 0.8 \end{bmatrix},$$

$$D_i = \begin{bmatrix} c_\alpha(i+1)I_2 & 0 \\ 0 & c_\beta(i+1)I_2 \end{bmatrix}, \tag{27}$$

$$\begin{bmatrix} \widetilde{B}_{11}^1 \\ \widetilde{B}_{12}^1 \\ \widetilde{B}_{21}^1 \\ \widetilde{B}_{22}^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.545 \\ -0.051 \end{bmatrix}.$$

To compute (26), first we have to compute matrices Φ_k for $k \in Z_+$, which in this example have the form

$$\Phi_0 = I_4,$$

$$\Phi_1 = \begin{bmatrix} \widetilde{A}_{1\alpha} & \widetilde{A}_{12}^1 \\ \widetilde{A}_{21}^1 & \widetilde{A}_{2\beta} \end{bmatrix} = \begin{bmatrix} 1.5 & 0 & 1 & 2 \\ 0 & 0.5 & 3 & 4 \\ 3 & 0 & 0.7 & 1 \\ 1 & 3 & 0 & 0.8 \end{bmatrix},$$

$$\Phi_2 = \begin{bmatrix} \widetilde{A}_{1\alpha} & \widetilde{A}_{12}^1 \\ \widetilde{A}_{21}^1 & \widetilde{A}_{2\beta} \end{bmatrix}^2 - \begin{bmatrix} \frac{\alpha(\alpha-1)}{2!}I_2 & 0 \\ 0 & \frac{\beta(\beta-1)}{2!}I_2 \end{bmatrix} = \begin{bmatrix} 7.375 & 6 & 2.2 & 5.6 \\ 13 & 12.375 & 3.6 & 8.2 \\ 7.6 & 3 & 3.64 & 7.5 \\ 2.3 & 3.9 & 10 & 14.79 \end{bmatrix}, \tag{28}$$

$$\Phi_3 = \widetilde{A}\Phi_2 - D_1\widetilde{A} - D_2 I_4 = \begin{bmatrix} 23.263 & 19.8 & 27.059 & 45.718 \\ 38.5 & 30.663 & 53.077 & 86.236 \\ 30.177 & 24 & 19.043 & 36.978 \\ 48.359 & 46.677 & 21 & 41.941 \end{bmatrix}, \ \ldots$$

then we can commute vector

$$[\ \overline{x}_{11}^1(k) \quad \overline{x}_{12}^1(k) \quad \overline{x}_{21}^1(k) \quad \overline{x}_{22}^1(k)\ ]^T.$$

Nilpotent subsystem (18) has the form

$$\begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} = \begin{bmatrix} I_{n_1^2} & \widetilde{A}_{12}^2 \\ \widetilde{A}_{21}^2 & I_{n_2^2} \end{bmatrix} \begin{bmatrix} \overline{x}_{11}^2(k) \\ \overline{x}_{21}^2(k) \end{bmatrix} + \begin{bmatrix} \widetilde{B}_{11}^2 \\ \widetilde{B}_{21}^2 \end{bmatrix} u(k), \quad k \in Z_+, \tag{29}$$

where

$$N_1 = N_2 = 0,$$

$$\begin{bmatrix} I_{n_1^2} & \widetilde{A}_{12}^2 \\ \widetilde{A}_{21}^2 & I_{n_2^2} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \quad \begin{bmatrix} \widetilde{B}_{11}^2 \\ \widetilde{B}_{21}^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.182 \end{bmatrix}. \tag{30}$$

In this case, from (29) for (30) we have

$$\begin{bmatrix} \overline{x}_{11}^2(k) \\ \overline{x}_{21}^2(k) \end{bmatrix} = \begin{bmatrix} 0.127 \\ -0.564 \end{bmatrix} u(k), \qquad k \in Z_+. \tag{31}$$

Finally, the desired solution of the descriptor fractional linear system (6) with (23) is given by

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = Q \begin{bmatrix} \overline{x}_1^1(k) \\ \overline{x}_1^2(k) \\ \overline{x}_2^1(k) \\ \overline{x}_2^2(k) \end{bmatrix} = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} \overline{x}_{11}^1(k) \\ \overline{x}_{12}^1(k) \\ \overline{x}_{11}^2(k) \\ \overline{x}_{21}^1(k) \\ \overline{x}_{22}^1(k) \\ \overline{x}_{21}^2(k) \end{bmatrix}, \tag{32}$$

where

$$\begin{bmatrix} \overline{x}_1^1(k) \\ \overline{x}_2^1(k) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \overline{x}_1^2(k) \\ \overline{x}_2^2(k) \end{bmatrix}$$

are determined by (26) and (31), respectively.

5. Concluding remarks

The fractional discrete-time linear systems with two different fractional orders has been analyzed. The Weierstrass-Kronecker decomposition theorem of the regular pencil has been extended to the descriptor fractional discrete-time linear system with two different fractional orders. The method for finding the solution of the state equation has been derived. Effectiveness of the method has been demonstrated on a numerical example. Extension of these considerations on systems consisting of n subsystems with different fractional orders is possible. An open problem is the application of the Drazin inverse to finding the solution of the system with different fractional orders.

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