Control and Cybernetics

vol. **44** (2015) No. 1

Record-based inference and associated cost analysis for the Weibull distribution*

by

Mahdi Doostparast

Department of Statistics, School of Mathematical Sciences,
Ferdowsi University of Mashhad, P. O. Box 91775-1159, Mashhad, Iran
doostparast@math.um.ac.ir & doustparast@um.ac.ir

Abstract: In statistical process control, record schemes are used to reduce the total time on test for the inspection inquiry. In these schemes, units are examined sequentially and successive minimum values are recorded. On the basis of record data, Samaniego and Whitaker (1986) obtained the maximum likelihood (ML) estimate of the mean for an exponential distribution. Since the two-parameter Weibull model, as an extension of the exponential distribution, has a wide range of application, Hoinkes and Padgett (1994) derived the record-based ML estimators for the parameters of interest in this model. This paper shows that the ML estimates of the Weibull parameters do not always exist for the basis of records. Thus, a new scheme is proposed, in which the ML estimates of the parameters always exist. An analytic cost-based comparison between the usual and the new scheme is also carried out. Finally, some concluding remarks and open problems are formulated.

Keywords: cost analysis; lifetime models; likelihood function; record data; total time on test; Weibull model

1. Introduction

Record data arise in a wide variety of practical situations. Examples include industrial stress testing, meteorological analysis, sporting and athletic events, and oil and mining surveys. According to Samaniego and Whitaker (1986), such data may be represented by $(\mathbf{r},\mathbf{k}) := (r_1, k_1, \cdots, r_m, k_m)$, where r_i is the i-th record value, meaning new minimum (or maximum), and k_i is the number of trials following the observation of r_i that are needed to obtain a new record value (or to exhaust the available observations). There are two sampling schemes for generating such a record-breaking data, called *inverse* and *random* sampling schemes. In the former, items are presented sequentially and sampling is terminated when the m-th minimum is observed, while in the latter a random

*Submitted: January 2014; Accepted: February 2015

sample, say $Y_1, \cdots, Y_n$, is examined sequentially and successive minimum values are recorded. This paper deals with the random sampling scheme. For the inversely random scheme, the reader is referred to Gulati and Padgett (1994).

The Weibull distribution function (DF), denoted by $W(\alpha, \sigma)$, is

$$F(x;\alpha,\sigma) = 1 - \exp\left\{-\left(\frac{x}{\sigma}\right)^{\alpha}\right\}, \quad x \geq 0, \quad \alpha > 0, \quad \sigma > 0, \tag{1}$$

and hence its density is

$$f(x;\alpha,\sigma) = \frac{\alpha x^{\alpha-1}}{\sigma^{\alpha}} \exp\left\{-\left(\frac{x}{\sigma}\right)^{\alpha}\right\}, \quad x \geq 0, \quad \alpha > 0, \quad \sigma > 0. \tag{2}$$

The scale parameter σ is called the *characteristic life*, because it is always the 63.2-th percentile. It determines the spread and has the same units as failure times, for example hours, months, cycles, and so forth. Parameter α is a unitless pure number and determines the shape of the distribution. For $\alpha = 1$, the Weibull distribution is the exponential distribution, denoted by $Exp(\sigma)$. The Weibull distribution appears very frequently in practical problems when we observe data representing minimal values; see, e.g., Lawless (2003).

In the sequel, the following notations are used:

US:	usual scheme for record sampling
NS:	new proposed scheme for record sampling
TC:	total cost for associated record sampling
ETC:	expectation of total cost
TTT:	total time on test
$\ln x$:	the natural logarithm of x
$(\mathbf{r}, \mathbf{k})$:	$(r_1, k_1, \cdots, r_m, k_m)$
$(\mathbf{R}, \mathbf{K})$:	$(R_1, K_1, \cdots, R_m, K_m)$
$N^{(n)}$:	number of records in a random sample size n under US
$N_{\star}^{(n)}$:	number of records in a random sample size n under NS
$\hat{\theta}$:	maximum likelihood estimator of the unknown parameter θ
$I(A)$:	1 if the event A occurs and 0 otherwise
$X \sim W(\alpha, \sigma)$:	X has a Weibull model with shape and scale parameters α and σ, respectively
$\Gamma(r)$:	$\int_0^{+\infty} x^{r-1} e^{-x} dx$, the complete gamma function
$B(a, b)$:	$\int_0^1 x^{a-1}(1-x)^{b-1} dx$, the complete beta function
$\Psi(x)$:	$\partial \ln \Gamma(x) / \partial x$.
$a(n)$:	$E\left(\sum_{i=1}^{N^{(n)}} K_i R_i\right)$,
$b(n)$:	$E\left(\left\{\sum_{i=1}^{N^{(n)}} K_i R_i\right\} I\left(N^{(n-1)} > 1\right)\right)$.

The rest of the paper is organized as follows: in Section 2, the problem of estimating unknown parameters based on record-breaking data coming from the

Weibull distribution is reviewed, and it will be shown that the ML estimates do not always exist. In Section 3, a new scheme (NS) is proposed and ML estimates under NS are investigated. In Section 4, the two schemes are compared for estimating the parameters based on a criterion involving a cost function. Section 5 contains a discussion.

2. Record-based Weibull analysis

Assume that the sequence $\{R_1, K_1, \cdots, R_m, K_m\}$ originates from a random sample $Y_1, \cdots, Y_n$ with the common absolutely continuous DF $F(.;\theta)$, where $\theta \in \Theta$ is the parameter vector and Θ stands for the parameter space. Based on record data $(\mathbf{r},\mathbf{k})$, the likelihood function (LF) reads (Samaniego and Whitaker, 1986)

$$L(\theta; \mathbf{r},\mathbf{k}) \equiv \prod_{i=1}^{m} f(r_i;\theta) \{1 - F(r_i;\theta)\}^{k_i - 1} I_{(-\infty, r_{i-1})}(r_i), \tag{3}$$

where $f(.;\theta)$ is the density of the DF $F(.;\theta)$. For the Weibull model, by substituting Equations (1) and (2) into Equation (3), the corresponding LF with $\theta = (\alpha, \sigma)$ is obtained as

$$L(\alpha, \sigma; \mathbf{r}, \mathbf{k}) \quad \equiv \quad \frac{\alpha^m}{\sigma^{m\alpha}} \left\{ \prod_{i=1}^{m} r_i \right\}^{\alpha - 1} \exp\left\{ - \left(\frac{1}{\sigma^\alpha} \sum_{i=1}^{m} k_i r_i^\alpha \right) \right\}. \tag{4}$$

From (4), Hoinkes and Padgett (1994) obtained the ML estimates of the unknown parameters on the basis of record data $(\mathbf{r},\mathbf{k})$ arising from a random sample by solving the following non-linear equations

$$\sigma = \left\{ \frac{1}{m} \sum_{i=1}^{m} k_i r_i^\alpha \right\}^{1/\alpha}, \tag{5}$$

and

$$\frac{m}{\alpha} + \sum_{i=1}^{m} \ln(r_i) - \frac{m}{\sum_{i=1}^{m} k_i r_i^\alpha} \sum_{i=1}^{m} k_i r_i^\alpha \ln(r_i) = 0, \tag{6}$$

where $N^{(n)} = m$ is the number of records among $Y_1, \cdots, Y_n$. One can easily check that Equation (6) is similar to those of Pike (1966) and conclude that Equation (6) has a unique solution. But, according to Nelson (1985), there must be at least two uncensored observations to calculate the estimates for the Weibull parameters from Equations (5) and (6). In what follows, it is proved that the ML estimate of Weibull parameters does not exist in the case when the first observation is the smallest of the n.

Remark 1 *Statistical inference on the basis of upper record values coming from the Weibull distribution has been considered in literature; see, e.g., Soliman et al. (2006).*

In the case of $N^{(n)} = 1$, the LF (4) is simplified as

$$\begin{aligned} L_s(\alpha,\sigma;\mathbf{r},\mathbf{k}) &\equiv f(r_1;\alpha,\sigma)(1-F(r_1;\alpha,\sigma))^{n-1} \\ &= \frac{\alpha r_1^{\alpha-1}}{\sigma^\alpha}\exp\left\{-n\left(\frac{r_1}{\sigma}\right)^\alpha\right\}, \end{aligned} \tag{7}$$

with the corresponding log-likelihood function (LLF)

$$l_s(\alpha,\sigma;\mathbf{r},\mathbf{k}) = \ln\left(L_s(\alpha,\sigma;\mathbf{r},\mathbf{k})\right) = \ln\alpha + (\alpha-1)\ln r_1 - \alpha\ln\sigma - n\left(\frac{r_1}{\sigma}\right)^\alpha. \tag{8}$$

For a given α, the LLF (8) is maximized by solving the equation

$$\partial l_s(\alpha,\sigma)/\partial\sigma = 0.$$

So,

$$\frac{\alpha}{\sigma}\left(-1+\frac{nr_1^\alpha}{\sigma^\alpha}\right) = 0, \tag{9}$$

which yields

$$\sigma = r_1 n^{\frac{1}{\alpha}}. \tag{10}$$

Note that

$$\frac{\partial^2}{\partial\sigma^2} l_s(\alpha,\sigma;\mathbf{r},\mathbf{k}) = \frac{\alpha}{\sigma^2} - \frac{n\alpha(\alpha+1)r_1^\alpha}{\sigma^{\alpha+2}}. \tag{11}$$

Substituting Equation (10) into Equation (11) gives

$$\begin{aligned} \frac{\partial^2}{\partial\sigma^2} l_s(\alpha,\sigma;\mathbf{r},\mathbf{k})\Big|_{\sigma=r_1 n^{\frac{1}{\alpha}}} &= \left(\frac{\alpha}{\sigma^2} - \frac{n\alpha(\alpha+1)r_1^\alpha}{\sigma^{\alpha+2}}\right)\Big|_{\sigma=r_1 n^{\frac{1}{\alpha}}} \\ &= \left(\frac{\alpha}{\sigma^2} - \frac{\alpha(\alpha+1)}{\sigma^2}\right)\Big|_{\sigma=r_1 n^{\frac{1}{\alpha}}} \\ &= -\left(\frac{\alpha}{\sigma}\right)^2\Big|_{\sigma=r_1 n^{\frac{1}{\alpha}}} \\ &= -\left(\frac{\alpha}{r_1 n^{\frac{1}{\alpha}}}\right)^2 \\ &< 0. \end{aligned} \tag{12}$$

This implies that $\sup_{\sigma>0} l_s(\alpha,\sigma;\mathbf{r},\mathbf{k}) = l_s(\alpha,\sigma = r_1 n^{\frac{1}{\alpha}};\mathbf{r},\mathbf{k})$ for all $\alpha > 0$. So, the ML estimate of α occurs in the region $\{(\alpha,\sigma) : \sigma = r_1 n^{\frac{1}{\alpha}}\}$. Thus, Equation (8) yields

$$\begin{aligned} l_s(\alpha,\sigma = r_1 n^{\frac{1}{\alpha}};\mathbf{r},\mathbf{k}) &= \alpha + (\alpha-1)\ln r_1 - \alpha\ln\left(r_1 n^{\frac{1}{\alpha}}\right) - n\left(\frac{r_1}{r_1 n^{\frac{1}{\alpha}}}\right)^\alpha \\ &= \alpha - \ln r_1 - \ln n - 1, \end{aligned}$$

which is increasing with respect to α, that is, the ML estimate of α is $+\infty$. This unrealistic estimate merely indicates that the true α is very large. But a $W(\alpha, \sigma)$-distribution tends to a degenerate DF as $\alpha \to +\infty$ at point $x = \sigma$ because

$$\lim_{\alpha \to +\infty} F(x; \alpha, \sigma) = \begin{cases} 1 & \text{if } x \geq \sigma, \\ 0 & \text{if } x < \sigma. \end{cases}$$

So we have the following proposition.

PROPOSITION 1 *On the basis of record data arising from a random sampling scheme, when* $N^{(n)} = 1$*, the ML estimate of* α *does not exist.*

One may suggest to use the method of moments (MM) to estimate α and σ. With this in mind, the k-th central moment readily follows from Equations (1) and (2) as

$$\mu'_k := E(X^k) = \sigma^k \Gamma\left(\frac{k}{\alpha} + 1\right). \tag{13}$$

Since $R_1 = Y_1$, we have

$$E(R_1) = \mu'_1 = \sigma \Gamma\left(\frac{1}{\alpha} + 1\right), \quad \text{and} \quad E(R_1^2) = \mu'_2 = \sigma^2 \Gamma\left(\frac{2}{\alpha} + 1\right).$$

Thus, the MM estimates of the unknown parameters are obtained by solving the equations

$$R_1 = \sigma \Gamma\left(\frac{1}{\alpha} + 1\right), \quad \text{and} \quad R_1^2 = \sigma^2 \Gamma\left(\frac{2}{\alpha} + 1\right).$$

So, $\left\{\Gamma\left(\frac{1}{\alpha} + 1\right)\right\}^2 = \Gamma\left(\frac{2}{\alpha} + 1\right)$. Hence, the MM estimate of α is obtained by solving the equation

$$B\left(\frac{1}{\alpha}, \frac{1}{\alpha}\right) = 2\alpha. \tag{14}$$

Letting $g(\alpha) = B\left(\frac{1}{\alpha}, \frac{1}{\alpha}\right) - 2\alpha$, one can verify that

$$\frac{d}{d\alpha} g(\alpha) = \frac{2}{x^2}\left(-\Psi\left(x^{-1}\right) + \Psi\left(2\, x^{-1}\right)\right) B\left(x^{-1}, x^{-1}\right) - 2.$$

It is easy to see that the global minimum point of the function $g(\alpha)$ is 1 and $\lim_{\alpha \to +\infty} g(\alpha) = 0$. Graphs of $g(\alpha)$ and $\frac{d}{d\alpha} g(\alpha)$ are given in Fig. 1. This shows that $\hat{\alpha} = +\infty$. Again, this unrealistic estimate indicates that the true α is very large. Hence, the MM estimate of σ is given by

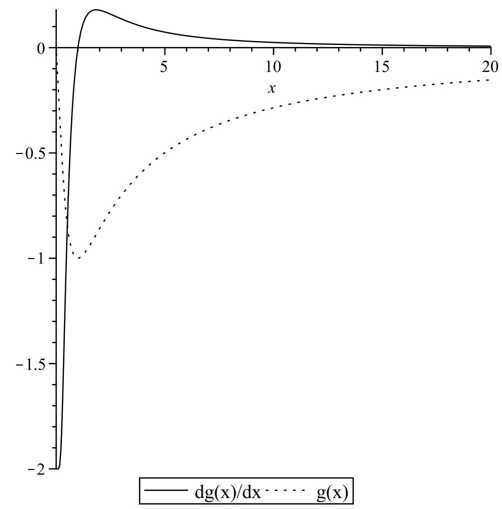

Figure 1. Graph of $g(\alpha)$ and $dg(\alpha)/d\alpha$

$$\lim_{\alpha\to+\infty}\frac{R_1}{\Gamma\left(\frac{1}{\alpha}+1\right)}=R_1. \tag{15}$$

Finally, one may suggest the following estimates of α and σ on the basis of record data arising from a random sample, namely

$$\tilde{\alpha}=\begin{cases}\hat{\alpha} & \text{if } N^{(n)}>1\\ +\infty & \text{if } N^{(n)}=1\end{cases} \tag{16}$$

and

$$\tilde{\sigma}_1=\begin{cases}\left\{\frac{1}{N^{(n)}}\sum_{i=1}^{N^{(n)}}K_iR_i^{\hat{\alpha}}\right\}^{1/\hat{\alpha}} & \text{if } N^{(n)}>1\\ R_1 & \text{if } N^{(n)}=1\end{cases} \tag{17}$$

where $\hat{\alpha}$ is obtained from Equation (6). We leave the problem of obtaining reasonable estimates as an open problem. It should be noticed that we did not use the information on $\{N^{(n)}=1\}$ to estimate α and σ given in Equations (16) and (17), respectively. One may use this information to obtain better estimates.

3. A suggested scheme

As mentioned earlier, one can not use Equations (5) and (6) in order to obtain ML estimate of α and σ on the basis of record data in the case when the first observation is the smallest one of the n; i.e. $N^{(n)}=1$. The probability of this situation is n^{-1}, which vanishes as n goes to infinity. Therefore, for small sample sizes, one needs to propose a new scheme for estimating parameters of the Weibull distribution.

Suppose that we are planning to examine the random sample $Y_1, \cdots, Y_n$ sequentially. For $Y_1, \cdots, Y_{n-1}$, we collect record data as usual, that is, the random sample $Y_1, \cdots, Y_{n-1}$ is examined sequentially and only values smaller than all previous ones are recorded. In the case $N^{(n-1)} = 1$, the last item, i.e. Y_n, is realized without censoring, otherwise, the usual record-based scheme is used for Y_n; that is, Y_n is recorded in the case $Y_n < \min\{Y_1, \cdots, Y_{n-1}\}$. Hence, the available data is

$$\text{data} = \begin{cases} R_1, K_1, \cdots, R_{N^{(n)}}, K_{N^{(n)}}, & \text{if } N^{(n-1)} > 1, \\ \\ R_1, n-2 \text{ observation larger than } R_1, Y_n, & \text{if } N^{(n-1)} = 1. \end{cases}$$

In this setting, the associated LF reads

$$L(\alpha, \sigma; \text{data}) = \begin{cases} \frac{\alpha^m}{\sigma^{m\alpha}} \left\{\prod_{i=1}^m r_i\right\}^{\alpha-1} \exp\left\{-\frac{1}{\sigma^\alpha} \sum_{i=1}^m k_i r_i^\alpha\right\}, & \text{if } N^{(n-1)} > 1, \\ \\ \frac{\alpha^2 (r_1 y_n)^{\alpha-1} \exp\{-n(r_1/\sigma)^\alpha - (y_n/\sigma)^\alpha\}}{\sigma^{2\alpha}}, & \text{if } N^{(n-1)} = 1. \end{cases}$$

In the case of $N^{(n-1)} = 1$, letting $r'_1 := r_1$, $r'_2 := y_n$, $k'_1 := n$, $k'_2 := 1$ and $m = 2$, Equation (4) is obtained on the basis of (r'_1, k'_1, r'_2, k'_2). Hence, similar numerical methods to the ones described in Section 2 may be used to obtain the ML estimates of the unknown parameters.

Remark 2 *One may want to examine the properties of the estimators under one record and see how these properties compare to the properties of the estimators when one has more than one record. We leave this problem as an open problem while in the next section we compare the two schemes for estimating the parameters based on a criterion involving a cost function.*

4. Cost analysis

Which scheme does dominate the other one? The answer depends on the costs. To this end, let C_t and C_u denote the costs of time and unit, respectively. It then readily follows that the total costs associated with US and NS schemes, denoted by TC_{US} and TC_{NS}, respectively, are given by

$$TC_{US} = C_t T_{US} + C_u N^{(n)}, \tag{18}$$

and

$$TC_{NS} = C_t T_{NS} + C_u N_\star^{(n)}, \tag{19}$$

where $N_\star^{(n)}$ is the total number of uncensored observations under NS scheme, i.e.

$$N_\star^{(n)} = \begin{cases} 2 & \text{if } N^{(n-1)} = 1, \\ N^{(n)} & \text{if } N^{(n-1)} > 1. \end{cases}$$

Also, it can be shown that the total time on test (TTT) under US and NS is given by

$$T_{US} = \sum_{i=1}^{N^{(n)}} K_i R_i, \tag{20}$$

and

$$T_{NS} = \{(n-1)R_1 + Y_n\} I\left(N^{(n-1)} = 1\right) + \left\{\sum_{i=1}^{N^{(n)}} K_i R_i\right\} I\left(N^{(n-1)} > 1\right), \tag{21}$$

respectively. Thus, the expectations of the total costs (ETCs) under different schemes are

$$ETC_{US}(n) = C_t E\left(T_{US}\right) + C_u E\left(N^{(n)}\right), \tag{22}$$

and

$$ETC_{NS}(n) = C_t E\left(T_{NS}\right) + C_u E\left(N_\star^{(n)}\right). \tag{23}$$

To calculate $ETC_{US}(n)$ and $ETC_{NS}(n)$ in Equations (22) and (23), respectively, one needs to obtain $E\left(N^{(n)}\right)$, $E\left(N_\star^{(n)}\right)$, $E\left(T_{US}\right)$ and $E\left(T_{NS}\right)$. In what follows, these quantities are derived.

From Glick (1978), we know that the distribution of $N^{(n)}$ does not depend upon the parent distribution and

$$E(N^{(n)}) = 1 + \frac{1}{2} + \cdots + \frac{1}{n}. \tag{24}$$

For $E\left(N_\star^{(n)}\right)$, the following lemma is helpful.

Lemma 1 *Suppose that $N_\star^{(n)}$ is the total number of uncensored items under the NS scheme. Then*

$$N_\star^{(n)} = N^{(n)} + (1 - I_n) I(N^{(n-1)} = 1), \tag{25}$$

where

$$I_n = I(Y_n < \min\{Y_1, \cdots, Y_{n-1}\}).$$

Proof By definition, we have

$$
\begin{aligned}
N_\star^{(n)} &= 2I(N^{(n-1)}=1)+N^{(n)}I(N^{(n-1)}>1)\\
&= I(N^{(n-1)}=1)+I(N^{(n-1)}=1)+N^{(n)}I(N^{(n-1)}>1)\\
&= I(N^{(n-1)}=1)+N^{(n-1)}I(N^{(n-1)}=1)+(N^{(n-1)}+I_n)I(N^{(n-1)}>1)\\
&= I(N^{(n-1)}=1)+N^{(n-1)}I(N^{(n-1)}\geq 1)+I_nI(N^{(n-1)}>1)\\
&= I(N^{(n-1)}=1)+N^{(n-1)}+I_nI(N^{(n-1)}>1)\\
&= I(N^{(n-1)}=1)+N^{(n-1)}+I_n(1-I(N^{(n-1)}=1))\\
&= I(N^{(n-1)}=1)+N^{(n-1)}+I_n-I_nI(N^{(n-1)}=1)\\
&= I(N^{(n-1)}=1)+N^{(n)}-I_nI(N^{(n-1)}=1)\\
&= N^{(n)}+(1-I_n)I(N^{(n-1)}=1),
\end{aligned}
$$

which establishes the required result. □

It is clear that, for all n,

$$P\left(N^{(n)}=1\right)=\frac{(n-1)!}{n!}=\frac{1}{n}. \tag{26}$$

Equations (24)-(26) and the fact that $N^{(n-1)}$ and I_n are independent random variables (Glick, 1978) imply an expression for the expected number of complete observations resulting from the modified scheme in the form of

$$E\left(N_\star^{(n)}\right)=E\left(N^{(n)}\right)+\frac{1}{n}. \tag{27}$$

Let $Y_1,\cdots,Y_n$ be independent and identically distributed random variables coming from the $W(\alpha,\sigma)$-distribution. From Equation (21), the expectation of TTT under NS is

$$
\begin{aligned}
E(T_{NS}) &= E\left(\{(n-1)R_1+Y_n\}I\left(N^{(n-1)}=1\right)+\left\{\sum_{i=1}^{N^{(n)}}K_iR_i\right\}I\left(N^{(n-1)}>1\right)\right)\\
&= E\left(\{(n-1)R_1+Y_n\}\,I\left(N^{(n-1)}=1\right)\right)\\
&\quad +E\left(\left\{\sum_{i=1}^{N^{(n)}}K_iR_i\right\}I\left(N^{(n-1)}>1\right)\right)\\
&= (n-1)E\left(R_1I\left(N^{(n-1)}=1\right)\right)+E\left(Y_nI\left(N^{(n-1)}=1\right)\right)\\
&\quad +E\left(\left\{\sum_{i=1}^{N^{(n)}}K_iR_i\right\}I\left(N^{(n-1)}>1\right)\right).
\end{aligned} \tag{28}
$$

Lemma 2, provided below, gives the exact values of $E\left(R_1I\left(N^{(n-1)}\right)\right)$ and $E\left(Y_nI\left(N^{(n-1)}=1\right)\right)$, while Tables 1 and 2 present simulated values of

$\sigma^{-1}E\Big(\Big\{\sum_{i=1}^{N^{(n)}} K_iR_i\Big\}I\big(N^{(n-1)}>1\big)\Big)$ and $\sigma^{-1}E\left(\left\{\sum_{i=1}^{N^{(n)}} K_iR_i\right\}\right)$ for some selected values of n and α. The simulation study has been conducted 10^4 times using the mathematical package MATLAB version *6.0*.

Lemma 2 *Let $\{R_1, K_1, \cdots, R_m, K_m\}$ be records coming from a random sample $Y_1, \cdots, Y_n$ with the common $W(\alpha,\sigma)$-distribution. Then*

$$E(R_1 I\left(N^{(n-1)}=1\right)) = \sigma(n-1)^{-(1+1/\alpha)}\Gamma\left(\frac{1}{\alpha}+1\right), \tag{29}$$

and

$$E\left(Y_n I\left(N^{(n-1)}=1\right)\right) = \frac{\sigma}{(n-1)}\Gamma\left(\frac{1}{\alpha}+1\right). \tag{30}$$

Proof By definition, $R_1 = Y_1$. Thus,

$$\begin{aligned}
E(R_1 I\left(N^{(n-1)}=1\right)) &= \int_0^{+\infty} E\left(R_1 I\left(N^{(n-1)}=1\right)|R_1=x\right) f_{R_1}(x)dx \\
&= \int_0^{+\infty} xE\left(I\left(N^{(n-1)}=1\right)|R_1=x\right) f(x)dx \\
&= \int_0^{+\infty} xP\left(N^{(n-1)}=1|R_1=x\right) f(x)dx \\
&= \int_0^{+\infty} xP\left(Y_1=x, Y_1<\min\{Y_2,\cdots,Y_{n-1}\}|R_1=x\right) f(x)dx \\
&= \int_0^{+\infty} xP\left(x<\min\{Y_2,\cdots,Y_{n-1}\}\right) f(x)dx \\
&= \int_0^{+\infty} x[\bar{F}(x)]^{n-2} f(x)dx,
\end{aligned} \tag{31}$$

where $\bar{F}(x) = 1-F(x)$ is the survival function of the parent population. Substituting Equations (1) and (2) into Equation (31), implies

$$\begin{aligned}
E\left(R_1 I\left(N^{(n-1)}=1\right)\right) &= \int_0^{+\infty} xe^{-(n-1)\left(\frac{x}{\sigma}\right)^\alpha}\frac{\alpha x^{\alpha-1}}{\sigma^\alpha}dx \\
&= \frac{1}{n-1}\int_0^{+\infty} x\frac{(n-1)\alpha x^{\alpha-1}}{\sigma^\alpha}e^{-(n-1)\left(\frac{x}{\sigma}\right)^\alpha}dx \\
&= \frac{1}{n-1}E(U),
\end{aligned}$$

where $U \sim W(\alpha, \sigma(n-1)^{-1/\alpha})$. Equation (13) with $k=1$ concludes

$$\begin{aligned}
E(R_1 I\left(N^{(n-1)}=1\right)) &= \frac{1}{n-1}\sigma(n-1)^{-1/\alpha}\Gamma\left(\frac{1}{\alpha}+1\right) \\
&= \sigma(n-1)^{-(1+1/\alpha)}\Gamma\left(\frac{1}{\alpha}+1\right),
\end{aligned}$$

Table 1. Values of $\sigma^{-1}a(n)$ (upper figures) and $\sigma^{-1}b(n)$ (lower figures) for $n = 5(5)20$ and $\alpha = 0.1, 0.3, 0.5, 0.7, 0.9$ and 1.

	α					
n	0.1	0.3	0.5	0.7	0.9	1
5	1773163.6579	11.3305	2.9091	2.2834	2.2439	2.2746
	800520.6334	8.7172	2.1749	1.6566	1.5955	1.6023
10	1444736.7741	11.5008	3.1307	2.6388	2.8038	2.8954
	1337331.7079	10.3527	2.8458	2.3586	2.4722	2.5414
15	1283326.3400	10.6061	3.1473	2.7577	3.1118	3.3071
	1198194.4427	10.0727	2.9681	2.5872	2.9071	3.0654
20	1645772.2623	10.6459	3.1411	2.9189	3.3271	3.6366
	1471068.8908	10.1423	3.0035	2.7841	3.1586	3.4473
50	1325350.7229	10.4230	3.3048	3.2061	3.9906	4.4744
	1280857.5155	10.1978	3.2599	3.1607	3.9327	4.3969
100	1865200.6225	10.9559	3.2288	3.3781	4.4025	5.1181
	1653056.0640	10.6648	3.2101	3.3522	4.3723	5.0828

which is the desired result in Equation (29). Since $I\left(N^{(n-1)}=1\right)$ is a function of $Y_1, \cdots, Y_{n-1}$, the random variables Y_n and $I\left(N^{(n-1)}=1\right)$ are independent. Thus, Equations (13) and (26) yield

$$\begin{aligned} E\left(Y_n I\left(N^{(n-1)}=1\right)\right) &= E(Y_n) E\left(I\left(N^{(n-1)}=1\right)\right) \\ &= \sigma\Gamma\left(\frac{1}{\alpha}+1\right) P\left(N^{(n-1)}=1\right) \\ &= \frac{\sigma}{(n-1)}\Gamma\left(\frac{1}{\alpha}+1\right), \end{aligned}$$

and the proof of Equation (30) is completed. □

From Equations (20), (21) and (28) and Lemma 2, the following proposition is derived:

Proposition 2 *Expectations of TTTs under US and NS schemes are*

$$E(T_{US}) = a(n), \tag{32}$$

$$E(T_{NS}) = \sigma(n-1)^{-(1/\alpha)}\Gamma\left(\frac{1}{\alpha}+1\right) + \frac{\sigma}{(n-1)}\Gamma\left(\frac{1}{\alpha}+1\right) + b(n), \tag{33}$$

Table 2. Values of $\sigma^{-1}a(n)$ (upper figures) and $\sigma^{-1}b(n)$ (lower figures) for $n = 5(5)20$ and $\alpha = 1.5, 2, 5, 7, 10$ and 15.

			α			
n	1.5	2	5	7	10	15
5	2.5671	2.8789	3.8110	4.0827	4.3251	4.5328
	1.7301	1.8984	2.3728	2.5034	2.6371	2.7233
10	3.7142	4.4043	6.8450	7.5668	8.1988	8.7411
	3.2088	3.7297	5.6023	6.1377	6.6702	7.0901
15	4.5582	5.6951	9.5699	10.8230	11.8786	12.8145
	4.1598	5.1654	8.4461	9.5178	10.3703	11.1971
20	5.2007	6.7375	12.1833	13.9196	15.4488	16.7911
	4.8859	6.3025	11.0533	12.6806	13.9961	15.1754
50	7.7985	11.2630	25.8260	30.8771	35.4840	39.6962
	7.6342	11.0039	25.0102	29.8400	34.2171	38.3013
100	10.3482	16.3389	45.1733	56.1652	66.4266	75.9091
	10.2558	16.1647	44.4356	55.1937	65.3455	74.6648

where

$$a(n) = E\left(\left\{\sum_{i=1}^{N^{(n)}} K_i R_i\right\}\right)$$

and

$$b(n) = E\left(\left\{\sum_{i=1}^{N^{(n)}} K_i R_i\right\} I\left(N^{(n-1)} > 1\right)\right).$$

From Equations (22), (23), (24), (27), (29), (30) and Proposition 2, the expectations of the total cost under US and NS schemes are derived as

$$\begin{aligned} ETC_{US} &= C_t E(T_{US}) + C_u E(N^{(n)}) \\ &= C_t a(n) + C_u \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right), \end{aligned}$$

and

$$\begin{aligned} ETC_{NS} &= C_t E(T_{NS}) + C_u E(N_\star^{(n)}) \\ &= C_t \left\{ \sigma(n-1)^{-(1/\alpha)} \Gamma\left(\frac{1}{\alpha}+1\right) + \frac{\sigma}{(n-1)} \Gamma\left(\frac{1}{\alpha}+1\right) + b(n) \right\} \\ &\quad + C_u \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n}\right), \end{aligned}$$

respectively. The difference between the costs of the two schemes is

$$\begin{aligned} D &:= ETC_{NS} - ETC_{US} \\ &= C_t \left\{ \sigma(n-1)^{-1/\alpha} \Gamma\left(\frac{1}{\alpha}+1\right) \right. \\ &\quad \left. + \frac{\sigma}{(n-1)} \Gamma\left(\frac{1}{\alpha}+1\right) + b(n) - a(n) \right\} + \frac{C_u}{n} \\ &= C_t \left\{ \sigma\Gamma\left(\frac{1}{\alpha}+1\right) \left[(n-1)^{-1/\alpha} + \frac{1}{(n-1)} \right] + b(n) - a(n) \right\} + \frac{C_u}{n} \\ &= C_t(b(n) - a(n) - h(\alpha, \sigma, n)) + \frac{C_u}{n}, \qquad (34) \end{aligned}$$

where

$$h(\alpha, \sigma, n) = \sigma\Gamma\left(\frac{1}{\alpha}+1\right) \left[(n-1)^{-1/\alpha} + \frac{1}{(n-1)} \right].$$

Lemma 2 and Equation (34) imply that

$$\begin{aligned} &D = \\ &-C_t[(n-1)E(R_1 I(N^{(n-1)} = 1) + E(Y_n I(N^{(n-1)} = 1))) + h(\alpha, \sigma, n)] + \frac{C_u}{n}, \\ &= -C_t[-h(\alpha, \sigma, n) + h(\alpha, \sigma, n)] + \frac{C_u}{n} \\ &= \frac{C_u}{n} > 0. \qquad (35) \end{aligned}$$

On the basis of records arising from a random sample of size n under the $W(\alpha, \sigma)$-distribution, Equation (35) states that the proposed new scheme is always more costly (on average) than the usual one, while in the preceding section it was proved that the usual scheme with probability $1/n$ does not give the estimate value of the shape parameter α. This shows the additional price for having the estimates of both parameters α and σ.

REMARK 3 *For* $\alpha = 1$, *i.e. in the exponential case, since (Samaniego and Whitaker, 1986)* $\sum_{i=1}^{N^{(n)}} K_i R_i | (N^{(n)} = m)$, *there follows the gamma distribution* *with the shape parameter m and the scale parameter σ. Thus*

$$\begin{aligned} a(n) &= \sum_{m=1}^{n} E\left(\sum_{i=1}^{N^{(n)}} K_i R_i | N^{(n)} = m\right) P\left(N^{(n)} = m\right) \\ &= \sigma \sum_{m=1}^{n} mP\left(N^{(n)} = m\right) \\ &= \sigma E\left(N^{(n)}\right) \\ &= \sigma\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right). \end{aligned}$$

For this case, a comparison between complete data and record has been carried out and some special cases are discussed in more details by Doostparast and Balakrishnan (2010).

5. Discussion

Record schemes are used to reduce the TTT of an experiment when the costs of time and/or units are high. Usually, records are rare, especially when the sample sizes are small (Glick, 1978). In this paper, it is demonstrated that the ML estimates of the parameters of the Weibull distributions do not exist if $N^{(n)} = 1$. To overcome this problem, a new scheme was proposed and the ML estimates of the parameters of interest were derived. Equation (35) indicates that the proposed new scheme is always more costly (on average) than the usual one. It was also proved that the usual scheme with probability $1/n$ does not give the estimate value of the shape parameter α. This shows the additional price for having the estimates of both parameters α and σ. Therefore, the proposed scheme in this paper can be used successfully in small-sample size lifetime testing experiments when the test units or time are prohibitively expensive as in some statistical process controls. Note that Equation (26) implies that $\lim_{n \to +\infty} P\left(N^{(n-1)} = 1\right) = 0$, this leading to the conclusion that US and NS schemes are identical asymptotically. Also by Equation (35), one can see that $\lim_{n \to +\infty} D = 0$, as expected. Various schemes might be defined to overcome the above mentioned problem. For example, one may define the termination time of the experiment as $T^\star := \max\{T_j, n\}$, where T_j is the j-th record time and $j \geq 2$. Investigations in this direction are currently carried out by the author.

Acknowledgements

The author is grateful to anonymous referee and the Associate Editor for their useful suggestions and comments on an earlier version of this paper. This research was supported by a grant from Ferdowsi University of Mashhad (No MS88123DSP).

References

DOOSTPARAST, M. (2009) A note on estimation based on record data. *Metrika* **69**, 69-80.

DOOSTPARAST, M. and BALAKRISHNAN, N. (2010) Optimal sample size for record data and associated cost analysis for exponential distribution. *Journal of Statistical Computation and Simulation* **80**(12), 1389-1401.

GLICK, N. (1978) Breaking record and breaking boards. *American Mathemathical Monthly* **85**, 2-26.

GULATI, S. and PADGETT, W. J. (1994) Smooth nonparametric estimation of the distribution and density functions from record-breaking data. *Communications in Statistics - Theory and Methods* **23**(5), 1259-1274.

HOINKES, L. A. and PADGETT, W. J. (1994) Maximum likelihood estimation from record-breaking data for the Weibull distribution. *Quality and Reliability Engineering International* **10**, 5-13.

LAWLESS, J. F. (2003) *Statistical Models and Methods for Lifetime Data,* second edition. John Wiley & Sons, New York.

NELSON, W. (1985) Weibull analysis of reliability data with few or no failures. *Journal of Quality Technology* **17**(3), 140-146.

PIKE, M. (1966) A suggested method of analysis of a certain class of experiments in carcinogenesis. *Biometrics* **22**, 142-161.

SAMANIEGO, F. J. and WHITAKER, L. R. (1986) On estimating population characteristics from record breaking observations I. Parametric results. *Naval Research Logistics Quarterly* **33**, 531-543.

SOLIMAN, A. A., ABD ELLAHB, A. H. and SULTAN, K. S. (2006) Comparison of estimates using record statistics from Weibull model: Bayesian and non-Bayesian approaches. *Computational Statistics and Data Analysis* **51**, 2065-2077.