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Optimal decoupling controllers revisited*

by

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Abstract: The problem of decoupling a linear system by dynamic compensation into multi-input multi-output subsystems is studied by applying proper and stable fractional representations of transfer matrices. A necessary and sufficient condition is given for a decoupling and a stabilizing controller to exist. The set of all controllers that decouple and simultaneously stabilize the system is determined in parametric form. Optimal decoupling controllers are then obtained by an appropriate selection of the parameter.

Keywords: linear systems, fractional representations, decoupling controllers, stabilizing controllers, optimal controllers.

1. Introduction

Decoupling is a way to decompose a complex system into non-interacting subsystems. In fact, certain applications necessitate controlling independently different parts of the system. Even if this is not required, the absence of interaction can significantly simplify the synthesis of the desired control laws.

The decoupling problem has received much attention in the literature. For linear systems, different approaches have been used and control laws of various structure and complexity applied.

The basic form of decoupling into single-input single-output subsystems is often referred to as the diagonal decoupling. This problem was posed by Voznesenskij (1936) and studied by Kavanagh (1957), Strejc (1960), Mejerov (1965),

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and Wolovich (1974). The studies were related to the inversion problem of rational matrices. Attention was paid to the existence of proper rational transfer matrices. The issue of stability, however, was not properly addressed.

A deeper insight was provided by the state-space approach. The pioneering work is due to Morgan (1964), who posed the problem of decoupling by static state feedback. Falb and Wolovich (1967) established a solvability condition while Gilbert (1969) related this condition to state feedback invariants of the system. Descusse and Dion (1982) then interpreted this condition in terms of system's structure at infinity.

The use of restricted static state feedback, namely the static output feedback, in decoupling was studied by Howze and Pearson (1970), Howze (1973), Denham (1973), Hazlerigg and Sinha (1978), Filev (1982b), Descusse and Malabre (1982), and Descusse, Lafay and Kučera (1984). This is a very restricted problem, whose solution is hard to obtain, but it is very useful in applications.

A more general form of decoupling into multi-input multi-output subsystems is referred to as the block decoupling. This problem was introduced by Wonham and Morse (1970) and Basile and Marro (1970). Using a geometric approach, they determined the solvability of the problem by static state feedback in several special cases. An alternative algebraic approach based on the structure algorithm was presented by Silverman and Payne (1971). Relationships between the two approaches were studied by Filev (1982a).

The decoupling by dynamic state feedback was studied via the geometric approach by Morse and Wonham (1970), who obtained a deep insight into the internal structure of the decoupled system. By this time, the problem of decoupling by dynamic state feedback was solved, including stability or pole distributions that may be achieved while preserving a decoupled structure. The status of non-interacting control was reviewed by Morse and Wonham (1971).

A comeback of the transfer function methods in the study of block decoupling is witnessed through the works of Koussiouris (1979), Hautus and Heymann (1983), and Kučera (1983). A dynamic state feedback was shown to be equivalent to combined dynamic output feedback and feedforward reference compensation, often referred to as a two-degree-of-freedom controller. To address stability issues, the Youla-Kučera parameterization of all stabilizing controllers was invoked. The basic results are reported by Kučera (1983), Hautus and Heymann (1983), and Gómez and Goodwin (2000). The class of all decoupled transfer matrices that can be achieved by a stabilizing controller was parameterized by Desoer and Gündeş (1986) and Lee and Bongiorno (1993). This result has made it possible to derive the H_2-optimal decoupling controller, which minimizes the performance deterioration due to decoupling.

The two-degree-of-freedom controller structure is ideally suited to decoupling since only one of the degrees of freedom is affected by the decoupling requirement. This is not true for a pure feedback, or a one-degree-of-freedom controller. This case is considerably more difficult to solve, as shown by Hammer and Khargonekar (1984), Lin (1997), Youla and Bongiorno (2000), Bongiorno and Youla (2001), and Park (2008a).

Finally, the decoupling in the generalized plant model, which covers a broad range of control problems in a unified setting, was considered by Park (2008b). Such a plant model can accommodate non-square plant and non-unity feedback cases with one-degree-of-freedom or two-degree-of-freedom controller configuration. The benefits of such a general problem formulation consist in a unified treatment rather than in simplicity of the solution. Indeed, matrix operations need to be converted to vector operations with vectors of a much larger dimension, which result from the Kronecker and Khatri-Rao products of matrices.

This paper adopts the most general setting that is meaningful for decoupling: a system in which the measurement output may be different from the output to be decoupled and a dynamic controller that features both feedback and feedforward parts. The class of all such controllers that decouple and stabilize the system is determined in parametric form and the parameter is used to obtain the H_2-optimal controller. The solution is simple and direct. The controller configuration implies that decoupling and stability are two independent issues.

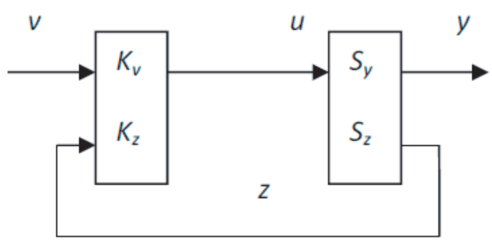

Figure 1. Control system

2. Problem formulation

Consider a linear, time-invariant, differential system governed by the input-output relation

$$y = S_y u, \tag{1}$$

where u is the q-vector input, y is the p-vector output and S_y is the transfer matrix of the system. We assume that S_y is a proper rational matrix over $\mathrm{R}(s)$, the field of rational functions.

Let p_1, ..., p_k be a given set of positive integers that satisfy

$$\sum_{i=1}^{k} p_i = p.$$

System (1) is said to be *decoupled*, or more specifically $(p_1, ..., p_k)$-decoupled, if there exist positive integers q_1, ..., q_k satisfying

$$\sum_{i=1}^{k} q_i = q$$

such that S_y has the block diagonal form

$$S_y := \begin{bmatrix} S_1 & & \\ & \ddots & \\ & & S_k \end{bmatrix},$$

where S_i is $p_i \times q_i$.

This is not a generic property of the system, but it can be achieved by a suitable compensation. To this effect, let z denote the m-vector output of the system that is available for measurement and let it be related with the input by the equation

$$z = S_z u, \tag{2}$$

where S_z is a proper rational matrix over R(s).

The most suitable linear, time-invariant, differential controller can then be described by the equation

$$u = K_v v + K_z z, \tag{3}$$

where v is an external reference input of appropriate dimension, say r. As it is seen in Fig. 1, the transfer matrices K_v and K_z represent the feedforward and the feedback parts of the controller, respectively. We assume that both K_v and K_z are proper rational matrices over R(s).

The *decoupling problem* is then to find matrices K_v and K_z such that the transfer matrix

$$T = S_y(I - K_z S_z)^{-1} K_v \tag{4}$$

from v to y be suitably block diagonal.

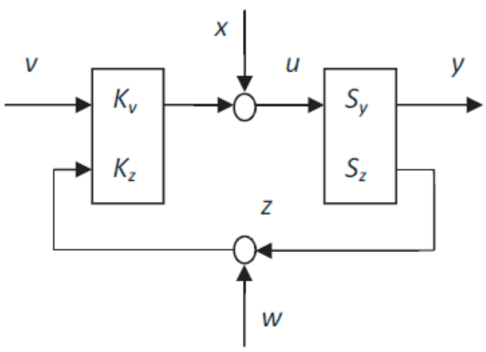

Figure 2. Control system with the complete set of independent inputs and outputs

Obviously, unless additional provisions are made, the decoupling problem is trivial as it could be solved by $K_v = 0$. Thus, it is necessary to impose certain admissibility condition on the decoupling controller to make the problem meaningful, for example

$$\text{rank } T = \text{ rank } S_y \tag{5}$$

over R(s). This condition is equivalent to the preservation of the class of controlled output trajectories. We thus require that no essential loss of control occurs through the decoupling process.

Another requirement, frequently imposed on the decoupled system in practice, is that of stability. This requirement means that the states of the system go to zero from any initial value.

3. Preliminaries

A stable system gives rise to a proper and stable transfer function. In order to study stability of the decoupled system it is convenient to express the transfer matrices of the given system and those of the controller in the following factorized form

$$\begin{bmatrix} S_z \\ S_y \end{bmatrix} := \begin{bmatrix} B \\ C \end{bmatrix} A^{-1}$$

$$\begin{bmatrix} K_z & K_v \end{bmatrix} := P^{-1} \begin{bmatrix} -Q & R \end{bmatrix},$$

where

$$A, \begin{bmatrix} B \\ C \end{bmatrix}$$

are proper and stable rational matrices that are right coprime and

$$P, \begin{bmatrix} -Q & R \end{bmatrix}$$

are proper and stable rational matrices that are left coprime.

These proper and stable fractional representations exist and are unique up to right and left multiplication, respectively, by a unimodular matrix. Recall that a proper and stable rational matrix is said to be unimodular if its inverse exists and is proper and stable.

The system equations (1) and (2) and the controller equation (3) then take the form

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} B \\ C \end{bmatrix} A^{-1} u, \tag{6}$$

$$u = P^{-1} \begin{bmatrix} -Q & R \end{bmatrix} \begin{bmatrix} z \\ v \end{bmatrix}. \tag{7}$$

The overall system transfer function reads

$$T = C(PA + QB)^{-1}R. \tag{8}$$

The fundamental assumption we make here is that the part of the given system that is not controllable from u is stable and the part of the given system that is not jointly observable from y, z is stable. Similarly, we assume that the controller is realized in such a manner that its part that is not jointly controllable from v, z is stable and its part that is not observable from u is stable.

The issue of stability of the overall system is then solved as follows.

Lemma 1. The overall system described by (6) and (7) is stable if and only if the matrix $PA + QB$ is unimodular.

Proof. In the overall system, inject inputs x and w as shown in Fig. 2. Then the overall system is stable if and only if the nine transfer matrices between the inputs v, w, x and the outputs u, y, z, given by

$$\begin{bmatrix} u \\ z \\ y \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} (PA + QB)^{-1} \begin{bmatrix} P & -Q & R \end{bmatrix} \begin{bmatrix} x \\ w \\ v \end{bmatrix}$$

are all well defined and proper and stable rational. This statement follows from the assumption of stability of the uncontrollable and unobservable parts of the system.

Now, in view of the coprimeness assumptions on A, B, C and P, Q, R these transfer matrices are well defined and stable if and only if $PA + QB$ is a unimo–dular matrix. □

4. Problem solvability

A necessary and sufficient condition will now be established for a system to be decoupled and stable.

Based on the partition (p_1, ..., p_k), write

$$C := \begin{bmatrix} C_1 \\ \vdots \\ C_k \end{bmatrix}, \tag{9}$$

where C_i is a $p_i \times q$ submatrix.

Theorem 1. Given system (1), (2) in fractional form (6) and partition (9), there exists an admissible controller (3) such that the overall system is

(i) *stable if and only if* A and B are right coprime,

(ii) *decoupled if and only if*

$$\sum\nolimits_{i=1}^{k} \text{rank } C_i = \text{rank } C. \tag{10}$$

Proof. (i) Let the overall system be stable. By Lemma 1, the matrix $PA + QB$ is unimodular whence A and B must be right coprime.

Conversely, let the matrices A and B of (6) be right coprime. Then there exist proper and stable rational matrices P and Q such that

$$PA + QB = I \tag{11}$$

with P invertible and the inverse of P proper.

Then, controller (3) in fractional form (7) that is defined by the matrices P and Q from (11) and by an arbitrary proper and stable rational matrix R satisfying rank CR = rank C is admissible since, by (8),

$$\text{rank } T = \text{rank } CR = \text{rank } C = \text{rank } S_y.$$

The resulting system (1), (2) and (3) is stable in view of Lemma 1 and identity (11).

(ii) Let (7) be an admissible decoupling controller for system (6). Denote

$$K := (PA + QB)^{-1}R.$$

The block diagonal property of the matrix T then implies

$$\text{rank } CK = \sum\nolimits_{i=1}^{k} \text{rank } C_iK$$

and the admissibility of the controller gives

$$\text{rank } C_iK = \text{rank } C_i\,, \quad i = 1,\ ...,\ k.$$

Therefore (10) holds.

The sufficiency will be proved by constructing a suitable R. Denote

$$r_i := \text{rank } C_i, \quad i = 1,\ ...,\ k.$$

Then there exists a $p_i \times p_i$ unimodular proper and stable rational matrix U_i such that

$$C_i = U_i \begin{bmatrix} C_i' \\ 0 \end{bmatrix}, \tag{12}$$

where the rows of C_i' are linearly independent over $\mathrm{R}(s)$ and where the zero matrix has $p_i - r_i$ rows and may be empty. If (10) holds, then

$$C' := \begin{bmatrix} C_1' \\ \vdots \\ C_k' \end{bmatrix}$$

has linearly independent rows over $R(s)$. Hence, there exists a $q \times q$ unimodular proper and stable rational matrix U'such that

$$C'U' := \begin{bmatrix} D_1 & & & 0 \\ & \ddots & & \vdots \\ & & D_k & 0 \end{bmatrix}, \tag{13}$$

where D_i is an $r_i \times r_i$ diagonal proper and stable rational matrix and where the zero matrices have $q - r$ columns with r defined by

$$r := \sum\nolimits_{i=1}^{k} r_i.$$

Define an admissible controller (7) by the matrices P and Q from (11) and by the matrix R formed by the first r columns of U'. The transfer matrix (8)

$$T = CR = \begin{bmatrix} U_1 & & \\ & \ddots & \\ & & U_k \end{bmatrix} \begin{bmatrix} \begin{bmatrix} D_1 \\ 0 \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} D_k \\ 0 \end{bmatrix} \end{bmatrix} \tag{14}$$

is block diagonal. The resulting system is therefore decoupled and the external reference input v has dimension r. □

The interpretation of these solvability conditions is as follows. Condition (11) corresponds to the stability of the subsystem of the given system that is not observable at the measured output z. Condition (11) calls for the linear independence of any two outputs of the given system that belong to different blocks. The solvability of the decoupling problem thus strongly depends on the partition $(p_1, ..., p_k)$, that is to say, upon the allocation of the outputs into the blocks.

5. Controller parameterization

When a decoupling and stabilizing controller exists, we shall parameterize the class of all such controllers.

The control system (6), (7) is stable if and only if $PA + QB$ is a unimodular matrix by Lemma 1. Thus, stabilization involves only the feedback part K_z of the controller, which surrounds the measurement subsystem S_z. As a result, the parameterization of K_z amounts to the well-known Youla-Kučera parameterization of feedback stabilizing controllers. For details, see Kučera (1975), Youla, Jabr and Bongiorno (1976), Kučera (1979), Desoer et al. (1980), and Vidyasagar (1985).

Let $\bar{P}$, $\bar{Q}$ be any solution pair of equation (11). Then the solution class of (11) is given by

$$P = \bar{P} + W\bar{B}, \quad Q = \bar{Q} - W\bar{A}, \tag{15}$$

where $\bar{A}$ and $\bar{B}$ are left coprime, proper and stable rational matrices such that

$$\bar{A}^{-1}\bar{B} = BA^{-1} \tag{16}$$

and W is an arbitrary proper and stable rational matrix parameter.

The class of all stabilizing proper rational K_z is then obtained in the form

$$K_z = -P^{-1}Q = -(\bar{P} + W\bar{B})^{-1}(\bar{Q} - W\bar{A}), \tag{17}$$

where the parameter W is constrained so that the inverse of $\bar{P}+W\bar{B}$ exists and is proper rational.

Once the control system (6) and (7) is stabilized, it is decoupled if and only if $T = CR$ by (8). Thus, decoupling involves only the feedforward part K_v of the controller.

Partition the $q \times q$ unimodular matrix U'defined in (13) as

$$U' = \begin{bmatrix} U'_r & U'_{q-r} \end{bmatrix},$$

where U'_r has r columns and U'_{q-r} has $q - r$ columns and may be empty. The class of all decoupling proper rational K_v is then given by $K_v = P^{-1}R$ with P determined in (14) and

$$R = U'_r \begin{bmatrix} V_1 & & \\ & \ddots & \\ & & V_k \end{bmatrix}, \tag{18}$$

where V_i is an arbitrary $r_i \times r_i$ proper and stable rational matrix parameter. The matrices V_1, ..., V_k, in turn, parameterize the class of achievable block-diagonal transfer matrices (8) as follows

$$T = \begin{bmatrix} U_1 & & \\ & \ddots & \\ & & U_k \end{bmatrix} \begin{bmatrix} \begin{bmatrix} D_1 \\ 0 \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} D_k \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} V_1 & & \\ & \ddots & \\ & & V_k \end{bmatrix}. \tag{19}$$

The parameterization of decoupling stabilizing controllers reveals that decoupling and stabilization are two independent issues. That is why the controller described by (3) is called the two-degree-of-freedom controller. However, this is no longer true for one-degree-of-freedom controllers, e.g., for the error-actuated controllers described by $u = -P^{-1}Q(v - w)$ in place of (7).

6. Optimal controllers

The decoupling constraint can deteriorate system's performance. The bonus of having a parameterized solution set is that the lost performance can easily be optimized. Optimal decoupling controllers can be obtained by an appropriate choice of the parameters V_1, ..., V_k and W.

Suppose that the control objective is for each block of outputs y_i to track the corresponding block of reference inputs v_i. Thus, we suppose that $p_i = r_i$ for $i = 1$, ..., k, i.e., there are as many reference inputs as controlled outputs in each block. The tracking error for each block is given by

$$e_i := v_i - y_i = H_i v_i.$$

In view of (19), H_i has the generic form

$$H_i = I - F_i V_i, \tag{20}$$

where $F_i := U_i D_i$ and V_i are proper and stable rational matrices with F_i fixed and V_i an arbitrary parameter to be specified.

The benefits of controller parameterization will now be demonstrated in the case of H_2 control design, see Kučera (2011). It turns out that only the parameters V_1, ..., V_k are subject to selection, whereas W is free and can be independently selected to accommodate additional design specifications.

Suppose that for each block, the reference-to-error transfer function H_i is to have the least H_2 norm defined by

$$\|H_i\|_2 := \left(\text{trace } \frac{1}{2\pi} \int_{-\infty}^{\infty} H_i^*(j\omega) H_i(j\omega) d\omega\right)^{1/2},$$

where the asterisk denotes the conjugate transpose. Thus, $H_i^*(s) := H_i^T(-s)$ for any complex argument s.

To achieve this task, determine the inner-outer factorization of F_i,

$$F_i = F_{iI} F_{iO},$$

where F_{iI} is inner and F_{iO} is outer. Since F_i is square and nonsingular, F_{iI} satisfies $F_{iI}^* F_{iI} = I$ and F_{iO} is free of zeros in Res >0.

Since F_{iI} is inner, left multiplication by F_{iI}^* preserves the H $_2$ norm,

$$\|H_i\|_2 = \|F_{iI}^* H_i\|_2 = \|F_{iI}^* - F_O V_i\|_2 .$$

Observe that $F_{iI}^*(\infty) = I$. Separate the strictly proper part, F_{iIsp}^*, of F_{iI}^* as follows

$$F_{iI}^* = I + F_{iIsp}^*$$

and note that, by definition, F_{iIsp}^* has poles only in Res >0. Then

$$\begin{aligned} \|H_i\|_2^2 &= \left\|F_{iIsp}^* + (I - F_{iO} V_i)\right\|_2^2 \\ &= \left\|F_{iIsp}^*\right\|_2^2 + \| I - F_{iO} V_i\|_2^2 \end{aligned}$$

because the cross terms contribute nothing to the norm. This is a complete square in which only the second term depends on V_i. Therefore, a unique V_i that attains the minimum of the norm for subsystem i is

$$V_i = F_{iO}^{-1}. \tag{21}$$

However, only a proper and stable V_i is admissible. It follows that the H_2 control problem for subsystem i has a solution if and only if F_{iO} is unimodular. The minimum norm is then given by

$$\min_{V_i} \|H_i\|_2 = \|F_{iIsp}\|_2 .$$

Other performance requirements can be addressed by the choice of W; for example, Kučera (2012) studies asymptotic tracking of a reference.

7. An example

Consider a system defined by (1), (2) with the transfer matrices

$$S_y = \begin{bmatrix} 1 & \frac{s+2}{s-1} \\ \frac{s-1}{s+2} & 2 \end{bmatrix}, \quad S_z = \begin{bmatrix} \frac{2s+1}{s+2} & \frac{3s}{s-1} \\ \frac{s-1}{s+2} & 2 \end{bmatrix}.$$

Thus, the measurement output z is different from the output y to be decoupled in that it involves a non-unity feedback sensor.

The task is to determine a two-degree-of-freedom controller (3) that (1, 1)-decouples and stabilizes the system.

The first step is to obtain a proper and stable fractional representation (6) for the system. Standard calculations yield

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{s-1}{s+2} \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{2s+1}{s+2} & \frac{3s}{s+2} \\ \frac{s-1}{s+2} & 2\frac{s-1}{s+2} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ \frac{s-1}{s+2} & 2\frac{s-1}{s+2} \end{bmatrix}.$$

Now apply Theorem 1. Since A is right coprime to B, a stabilizing controller exists. Since the rank of C equals the sum of the ranks of the rows of C, an admissible decoupling controller exists as well.

All stabilizing and decoupling controllers will be parameterized using the fractional representation (7). To obtain the feedback part of the controller, we consider any particular solution of equation (11), for example

$$\bar{P} = \begin{bmatrix} 1 & 0 \\ -\frac{2s+1}{s+2} & -2 \end{bmatrix}, \quad \bar{Q} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

A left coprime fractional representation that satisfies (15) is given by

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ -\frac{s-1}{s+2} & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \frac{s-1}{s+2} & 2 \\ -\frac{(s-1)(2s+1)}{(s+2)^2} & -\frac{3s}{s+2} \end{bmatrix}.$$

Thus, the solution class (15) of equation (11) is

$$\begin{aligned} P &= \begin{bmatrix} 1 & 0 \\ -\frac{2s+1}{s+2} & -2 \end{bmatrix} + W \begin{bmatrix} \frac{s-1}{s+2} & 2 \\ -\frac{(s-1)(2s+1)}{(s+2)^2} & -\frac{3s}{s+2} \end{bmatrix} \\ Q &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - W \begin{bmatrix} 0 & 1 \\ -\frac{s-1}{s+2} & 0 \end{bmatrix}. \end{aligned} \tag{22}$$

To obtain the feedforward part of the controller, note that $U_1 = U_2 = 1$ and the unimodular matrix defined in (13) equals

$$U' = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Thus, (17) yields

$$R = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}. \tag{23}$$

The matrices P, Q in (22) and R in (23) define the class of all controllers that solve the given problem. The parameters V_1, V_2 are free proper and stable rational functions and W is permitted to range over proper and stable rational 2×2 matrices so that the inverse of P exists and is proper. Obviously, this means that $P(\infty)$ is to be a nonsingular matrix.

The decoupled transfer matrices that can be achieved in this example are given by (19) as

$$T = \begin{bmatrix} 1 & 0 \\ 0 & \frac{s-1}{s+2} \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}.$$

The optimal controller that minimizes the H_2 norm of the reference-to-error transfer matrix is determined from (21), channel by channel. Clearly, $V_1 = 1$. To optimize V_2, the inner-outer factorization of

$$F_2 = \frac{s-1}{s+2}$$

is seen to be

$$F_{2I} = \frac{s-1}{s+1}, \quad F_{2O} = \frac{s+1}{s+2}$$

and the strictly proper part of

$$F_{2I}^* = \frac{s+1}{s-1}$$

equals

$$F^*_{2Isp} = \frac{2}{s-1}\,.$$

Thus, from (21),

$$V_2 = \frac{s+2}{s+1}\,.$$

It follows from (23) that the unique optimal R is

$$R = \begin{bmatrix} 2 & -\frac{s+2}{s+1} \\ -1 & \frac{s+2}{s+1} \end{bmatrix}$$

and the overall system has the transfer function

$$T = \begin{bmatrix} 1 & 0 \\ 0 & \frac{s-1}{s+1} \end{bmatrix}.$$

The parameter W does not affect the transfer function T. It is related to the internal structure of the decoupling controller. For example, $W = 0$ results in a fairly simple feedback controller. Another choice of W may provide a robustly stabilizing controller.

8. Conclusion

An optimal H_2 decoupling control problem has been studied in the most general setting, for systems in which the measurement output may be different from the output to be decoupled and for dynamic controllers that feature both feedback and feedforward parts. The class of all such controllers that decouple and stabilize the system has been determined in parametric form and the parameter has been used to obtain the H_2-optimal controller.

The main contribution of the present paper is in a streamlined and transparent exposition and a simple and direct solution of the optimal decoupling control problem, see Kučera (2011). This is primarily because of the following facts. The adopted controller configuration is ideally suited to decoupling since stability and noninteraction can be treated as two independent constraints. The problem is formulated and solved using an algebraic approach, namely the notion of proper and stable fractional representations for system transfer matrices. The parameterization of the decoupling controllers is achieved via the Youla-Kučera parameterization of all stabilizing controllers. Finally, the H_2 norm involved in the optimization is minimized using the completion of the squares, which is a simple algebraic technique.

A large body of literature exists on decoupling and related topics. In technical details, the present paper draws inspiration from the work of Hautus and Heymann (1983) for the formulation of the problem, from Kučera (1983) for the algebraic treatment of stability, from Desoer and Gündeş (1986) for the parameterization of the decoupled system, and from Lee and Bongiorno (1993) for the optimal control.

The matrix fraction approach presented in this paper can be also applied to a class of linear, time-invariant, singular systems that possess a transfer function. This is a subject of current research.

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