COMPUTER MODELING OF SUPERCAPACITOR WITH COLE-COLE RELAXATION MODEL

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Abstract

Electric energy stored insupercapacitors is associated with ion movement between the porous electrodes . This phenomenon can be described by dielectric relaxation model. Cole-Davidson relaxation model application reported in publications is difficult to use for control purposes. In the paper for impedance of the supercapacitors description Cole-Cole relaxation model is applied. For impedance parameters identification Nedler-Mead simplex method is used. Supercapacitor impedance model simplification based on physical properties is presented. Such model can be easy used for calculations in Matlab environment with FOTF toolbox designed to fractional calculus. The example of modeling of dynamic system with supercapacitor impedance model is described. The effects of the simulation show that fractional model of superapacitors is important tool for exact description of its dynamics.

Key words: Supercapacitor modeling, Cole-Cole relaxation model, fractional calculus, control systems

1 Introduction

Supercapacitors are electronic elements having the properties between electrolytic capacitors and accumulators. Capacitance of the supercapacitors reaches several thousands of farads. They can reach energy and power densities of more than 10 Wh/kg and 10 kW/kg respectively. The possibility of large electric charge storage is obtained due to porous electrodes made of active carbon, graphene, carbon nanotubes or aerogel. Supercapacitors are used in many applications: for protection of computers from input power interruptions, as power supply of robots, toys, electric toothbrushes etc. Recently they are increasingly used in electric vehicles for braking energy storage and its delivery during acceleration.

Electric energy stored insupercapacitors is associated with ion movement between the porous electrodes of large surface and relatively large resistance. This phenomenon causes that the typical equivalent models of capacitors that contain one or two lumped parameter RC circuits are not sufficient for accurate representation of dynamic properties of the supercapacitors. In the result, for this purpose, the complex equivalent schemes with many connected RC elements [1] or fractional differential equations [2, 3] are used.

In the paper, for impedance of the supercapacitors description fractional order calculus and model of dielectric relaxation are applied. Dielectric relaxation can be described by few models [4]. It was reported that Cole-Davidson model application is well for exact modeling of the supercapacitors [4, 5, 6] but its application in automation is difficult. The paper presents Cole-Cole model application for such purposes.

2 Cole-Cole and Cole-Davidson models of supercapacitor impedance

Classic Debye model of ideal dielectric relaxation is in practice replaced by its empiric modifications [4]. Such modification is presented by Havriliak-Negami model of complex dielectric constant, expressed as equation

$$\varepsilon_{HN}(j\omega) = \varepsilon_\infty + \frac{\varepsilon_S - \varepsilon_\infty}{\left[1+(j\omega T)^\delta\right]^\gamma}, \quad 0 < \gamma \leq 1 \quad 0 < \delta \leq 1, \tag{1}$$

where

ε_∞ – infinite frequency dielectric constant,

ε_s – static frequency dielectric constant,

T–characteristicrelaxation time of the medium.

For $\gamma=1$ equation (1) becomes Cole-Cole equation

$$\varepsilon_{CC}(j\omega) = \varepsilon_\infty + \frac{\varepsilon_S - \varepsilon_\infty}{1+(j\omega T)^\delta}, \quad \text{where} \quad 0 < \delta \leq 1 \tag{2}$$

and for $\delta=1$ it becomes Cole-Davidson equation

$$\varepsilon_{CD}(j\omega) = \varepsilon_\infty + \frac{\varepsilon_S - \varepsilon_\infty}{(1+j\omega T)^\gamma}, \quad \text{where} \quad 0 < \gamma \leq 1 \tag{3}$$

Parameters δ and γ are determined experimentally.

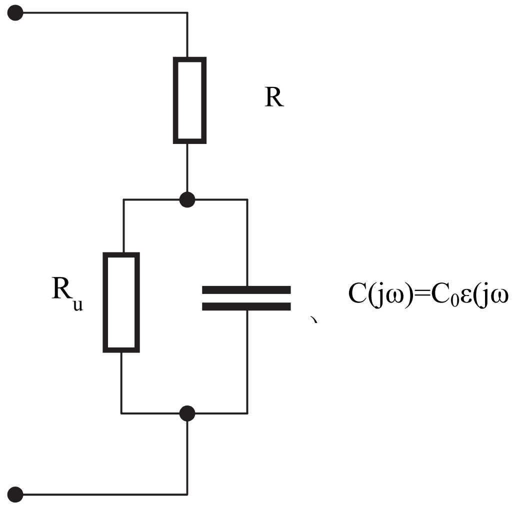

Figure1. Equivalent circuit of supercapacitor

The expression of the real supercapacitor impedance can be based on one of above equations of complex dielectric constants but it should also contain parallel leakage resistance R_u and serial equivalent resistance R_c (Figure 1) [5, 6]. As a result supercapacitor impedance is given by equation

$$Z(j\omega) = R_c + \frac{R_u \frac{1}{j\omega C(j\omega)}}{R_u + \frac{1}{j\omega C(j\omega)}} \tag{4}$$

where capacitance C(jω) is proportional to complex dielectric constant (1). Additionally, for the supercapacitors, can by assumed that

$$\varepsilon_\infty \ll \varepsilon_s \tag{5}$$

Let us replace Fourier transform with Laplace transform. Impedance of supercapacitor Z(s) can be treated as fractional transfer function G(s) with current input signal transform I(s) and voltage output signal transform V(s). On the basis of Cole-Davidson model (3), equations (4) and (5) one obtains the expression of supercapacitor impedance [5, 6]

$$G_{CD}(\mathrm{s}) = \frac{\mathrm{V(s)}}{\mathrm{I(s)}} = \mathrm{R_c} + \frac{R_u \frac{(1+\mathrm{sT})^{\gamma}}{sC}}{R_u + \frac{(1+\mathrm{sT})^{\gamma}}{sC}} = \frac{\left(1+\frac{R_c}{R_u}\right)(1+Ts)^{\gamma} + sR_cC}{\frac{1}{R_u}(1+sT)^{\gamma} + sC} \tag{6}$$

Transfer function is commonly in automation presented as [2, 3]

$$G(s) = \frac{b_0 s^{\beta_0} + b_1 s^{\beta_1} + \cdots + b_{m-1} s^{\beta_{m-1}} + b_m s^{\beta_m}}{a_0 s^{\propto_0} + a_1 s^{\propto_1} + \cdots + a_{n-1} s^{\beta_{n-1}} + a_n s^{\propto_n}} \tag{7}$$

Such a form of fractional transfer function can be directly used for calculation e.g. applying numerical computing environment Matlab with FOTF toolbox [7, 8] designed for fractional calculus.

Unfortunately equation (6) can't be directly expressed in form (7) because of presence binomial to a fractional power γ [6]. The same complications are connected with Havriliak-Negami model.

To avoid that issue one can apply Cole-Cole model of dielectric relaxation given by expression (2). Using the same transformation as for Cole-Davidson model, one can obtain equation

$$G_{CC}(\mathrm{s}) = \frac{\left(1+\frac{R_c}{R_u}\right) + s^{\delta}\left(1+\frac{R_c}{R_u}\right)T^{\delta} + sR_cC}{\frac{1}{R_u} + s^{\delta}\frac{T^{\delta}}{R_u} + sC} \tag{8}$$

Taking into consideration parameters of the supercapacitor equation (8) can be simplified. At the beginning it is worth to notice that serial resistance Rc is several order of magnitude lower than parallel leakage resistance Ru

$$\frac{R_c}{R_u} \ll 1 \tag{9}$$

This inequality leads to expression

$$G_{CC}(\mathrm{s}) = \frac{\left(1+\frac{R_c}{R_u}\right) + s^{\delta}\left(1+\frac{R_c}{R_u}\right)T^{\delta} + sR_cC}{\frac{1}{R_u} + s^{\delta}\frac{T^{\delta}}{R_u} + sC} \cong \frac{1 + s^{\delta}T^{\delta} + sR_cC}{\frac{1}{R_u} + s^{\delta}\frac{T^{\delta}}{R_u} + sC} \tag{10}$$

Generally transfer function (10) can be written in form

$$G_{CC}(\mathrm{s}) = \frac{1 + s^{\delta}T^{\delta} + sR_cC}{\frac{1}{R_u} + s^{\delta}\frac{T^{\delta}}{R_u} + sC} = \frac{1 + b_1 s^{\delta} + b_2 s}{a_0 + a_1 s^{\delta} + a_2 s} \tag{11}$$

which corresponds to (7).

$R_u$value can be determined from supercapacitor self-discharge curve. As a result the value of a0 coefficient is known

$$a_0 = \frac{1}{R_u} \tag{12}$$

Taking into account the value of a_0 and the following equality

$$b_1 = T^{\delta} \tag{13}$$

it can be written that

$$a_1 = \frac{T^{\delta}}{R_u} = a_0 b_1 \tag{14}$$

Summarizing, one can find that omitting R_c for determination of model (11) only 4 parameters should be identified:a_2, b_1, b_2 and δ. This identification can be based on the measurements of complex impedance values for the appropriate frequency range.

Identification of model (11) parameters can be performed on basis of minimization of performance index

$$J_f = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\left|G_{CC}(j\omega_i) - G_p(j\omega_i)\right|}{\left|G_p(j\omega_i)\right|}\right)^2 \tag{15}$$

where

G_{CC} – transfer function (11),

G_p – measured frequency response of the supercapacitor,

ω_i – frequency of measured point.

Chosen performance index corresponds to the variance of moduli of relative errors of the frequency response points, related to appropriate points of approximation function (11). For minimization purpose Nelder-Mead simplex method was used. This optimization problem is multi-modal so proper start point should be chosen. Fortunately the coefficients in expression (11) can be roughly estimated on the basis of estimation of supercapacitor physical parameters.

Measured frequency responses of supercapacitors presented in the paper, are based on data published in [5, 9, 10]. The example of transfer function calculated for 2700 F supercapacitor using data [10] is

$$G_{CC}(s) = \frac{1 + 0.869s^{0.846} + 0.632s}{0.00200 + 0.00174s^{0.846} + 2020s} \tag{16}$$

The result of the approximation of the frequency response (16) is presented in Figure 2. Another example is the impedance of the supercapacitorof 0.047 F capacitance [5]. Its transfer function is

$$G_{CC}(s) = 1000\frac{1 + 2.44s^{0.735} + 1.65s}{0.010 + 0.024s^{0.735} + 58.7s} \tag{17}$$

The frequency diagram of (17) is shown in Figure 3.

The basis for comparison of the accuracy of approximation for different supercapacitors can be performance index J_f (15).The square root of J_f corresponds to standard deviation of the error. For supercapacitors taken into consideration standard deviation of error is equal a few percent.

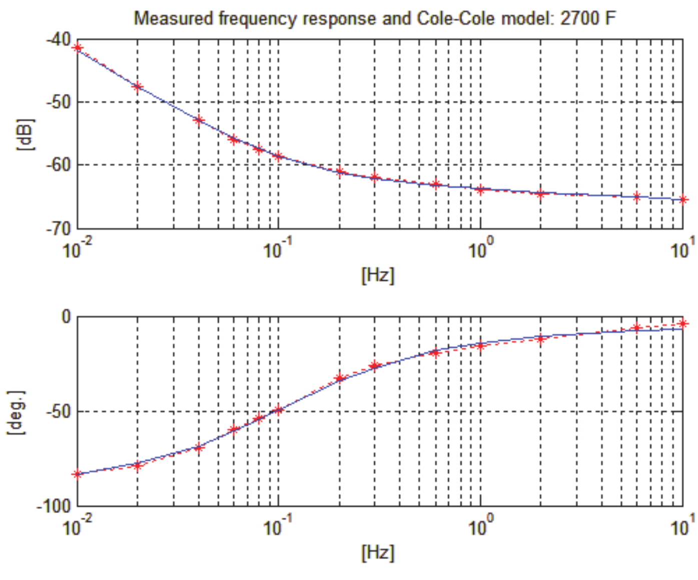

Figure 2. Measured frequency response points (asterisks) and approximating function (16) for 2700 F supercpacitor

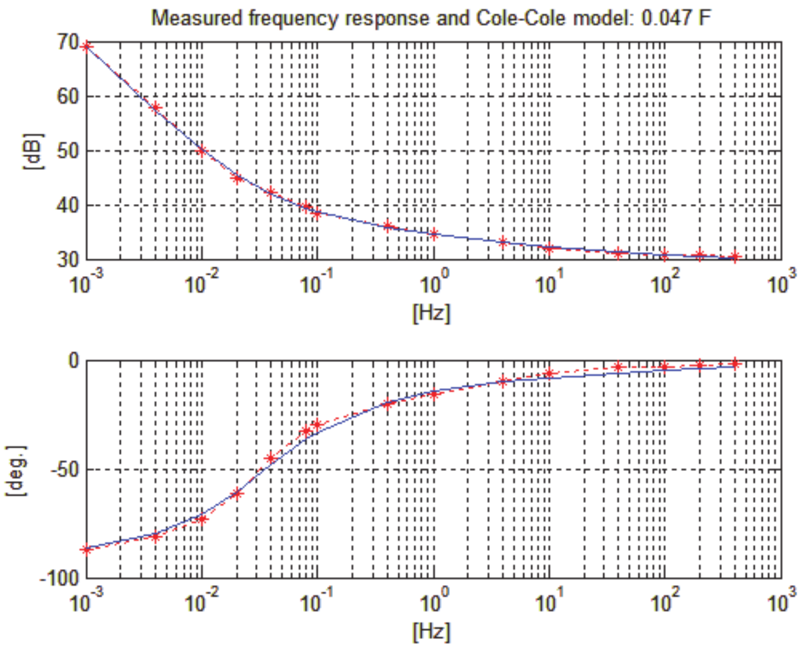

Figure 3. Measured frequency response points (asterisks) and approximating function (17) for 47 mF supercapacitor

3 Cole-Cole model simplification and time response

On the basis of the results of the impedance approximation of supercapacitors of capacitance between 0.047 F and 2700 F it can be stated that for all those examples model (11) can be simplified. The denominator of expression (11) can be written as

$$G_{CCd} = G_{CCd1} + G_{CCd2} \tag{18}$$

where

$$G_{CCd1} = a_0 + a_2 s \tag{19a}$$

$$G_{CCd2} = a_1 s^{\delta} \tag{19b}$$

It was proved that the ratio of

$$S(\omega) = \frac{|G_{CCd2}(\omega)|}{|G_{CC1}(\omega)|} \ll 1 \tag{20}$$

which means that the term G_{CCd2} practically has no influence on frequency response of the supercapacitor. In Figure 4 are shown graphs of S(ω) for various supercapacitors which frequency responses are presented in [5, 10].

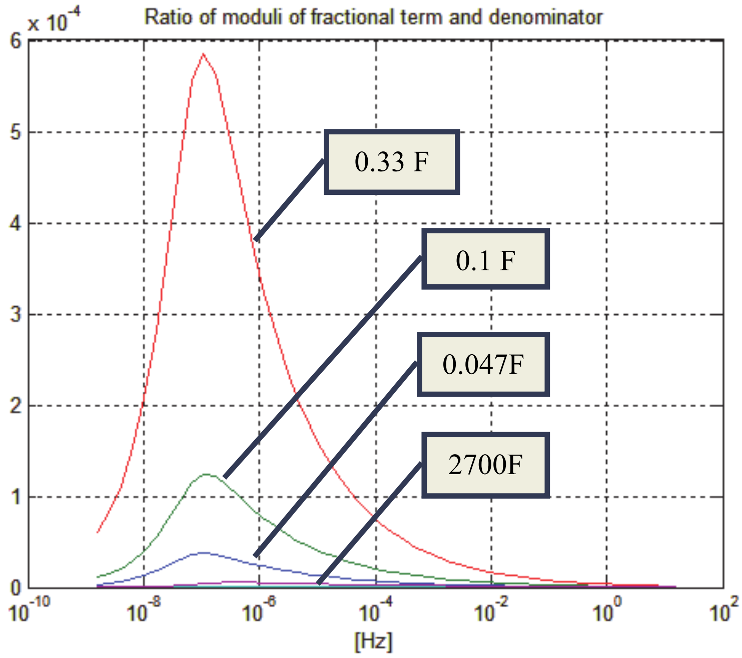

Figure 4. Frequency dependence of ratio S (18) for various supercapacitors

It can be mentioned that S(ω) strongly depends on exponent δ value. Typical value of δ for the capacitors is between 0.5 and 0.9. Graph of S(ω) for 0.6 F supercapacitor [10] is presented in Figure 5. Identified value of δ for this supercapacitor is 0.82. Other plots were calculated for hypothetical cases with lower values of δ.

Basing on current analysis one can determine the simpler model of the impedance of the supercapacitor. Omitting term G_{CCd2} the simplified expression is given as

$$G_{CC}(\mathrm{s}) = \frac{1 + b_1 s^{\delta} + b_2 s}{a_0 + a_2 s} \tag{21}$$

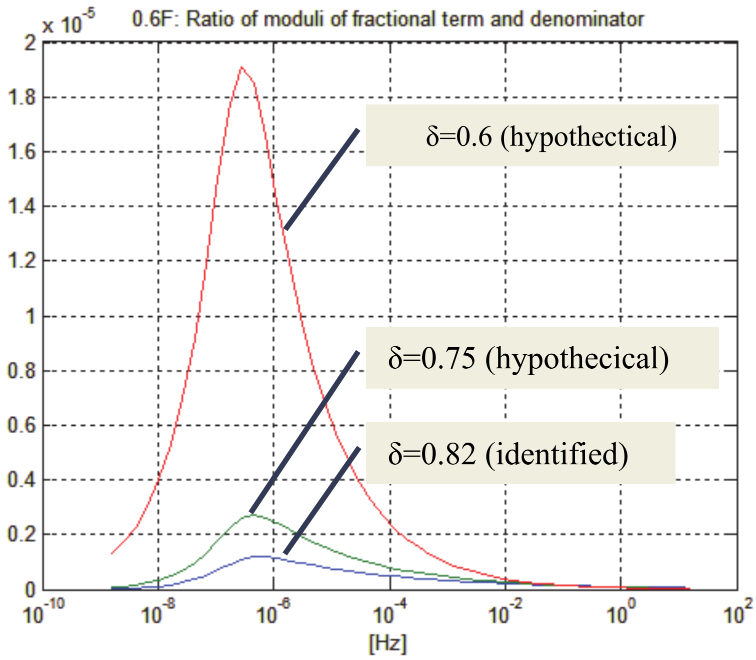

Figure 5. Ratio S(ω) for supercapacitor 0.6F

Consequently the impedance of e.g. 0.33 F supercapacitor [5] can be written as

$$\mathrm{G_{CC}(s)} = \frac{1 + 13.5\mathrm{s}^{0.670} + 0.632\mathrm{s}}{1.65\mathrm{e} - 07 + 0.340\mathrm{s}} \tag{22}$$

For the further analysis expression (21) can be decomposed into three simple fractions

$$G_{cc}(s) = \frac{1 + b_1 s^{\delta} + b_2 s}{a_0 + a_2 s} = G_{cc1}(s) + G_{cc2}(s) + G_{cc3}(s) \tag{23}$$

where

$$G_{cc1}(s) = R_c \tag{24a}$$

$$G_{cc2}(s) = \frac{R_u}{1 + sR_u C} \tag{24b}$$

$$G_{cc3}(s) = \frac{s^{\partial} T^{\partial} R_u}{1 + sR_u C} \tag{24c}$$

In Figure 6 are shown moduli of frequency responses of each term of (23) and modulus of $G_{cc.}$ The terms are asymptotes of $G_{cc}(s)$. The slope of logarithmic plots for G_{cc2} is -20 dB per decade of frequency and the slope of G_{cc3} is -20*(1-δ) dB per decade of frequency.

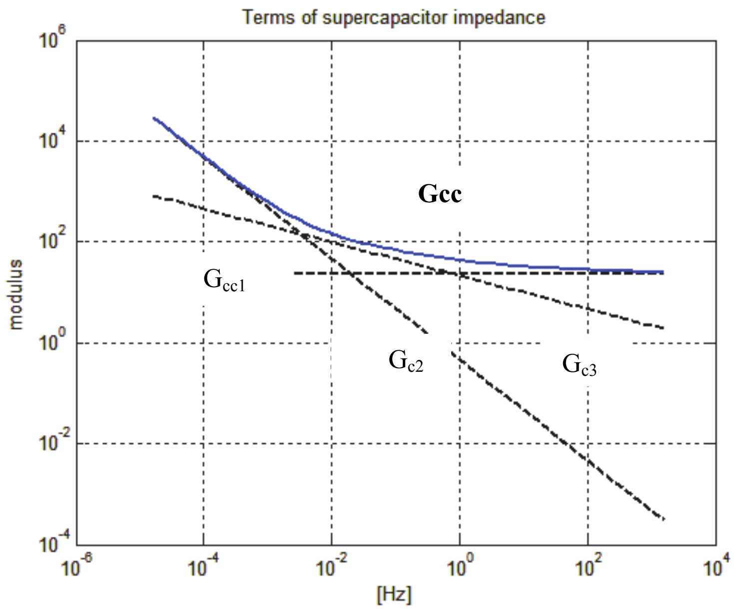

Figure6. Moduli of terms of equation (22) and modulus Gcc for supercapacitor 0.33 F

Voltage response of impedance (23) to current step is a sum of responses of mentioned 3 terms: proportional step, exponential response of large time constant $R_u C$, and response dependent on fractional order term. The voltage response for I_0 magnitude of current step can be written as

$$v_{cc}(t) = \mathcal{L}^{-1}\left\{\frac{I_0}{s}\left[R_c + \frac{R_u}{1 + sR_u C} + \frac{s^{\partial} T^{\partial} R_u}{1 + sR_u C}\right]\right\} = v_{cc1}(t) + v_{cc2}(t) + v_{cc3}(t) \tag{25}$$

where

$$v_{cc1}(t) = \mathcal{L}^{-1}\left\{\frac{I_0}{s}R_c\right\} = I_0 R_c \qquad (26a)$$

$$v_{cc2}(t) = \mathcal{L}^{-1}\left\{\frac{I_0}{s}\left[\frac{R_u}{1+sR_uC}\right]\right\} = I_0 R_u\left[1 - exp\left(\frac{-t}{R_uC}\right)\right] \qquad (26b)$$

$$v_{cc3}(t) = I\mathcal{L}^{-1}\left\{\frac{I_0}{s}\left[\frac{s^{\partial}T^{\partial}R_u}{1+sR_uC}\right]\right\}_0 = f_T(t, R_u, C, T, \partial) \qquad (26c)$$

For time $t \ll R_uC$ the two first terms causes step summed with quasi-linear increase. The third term is responsible for initial non-linearity – Figure 7.

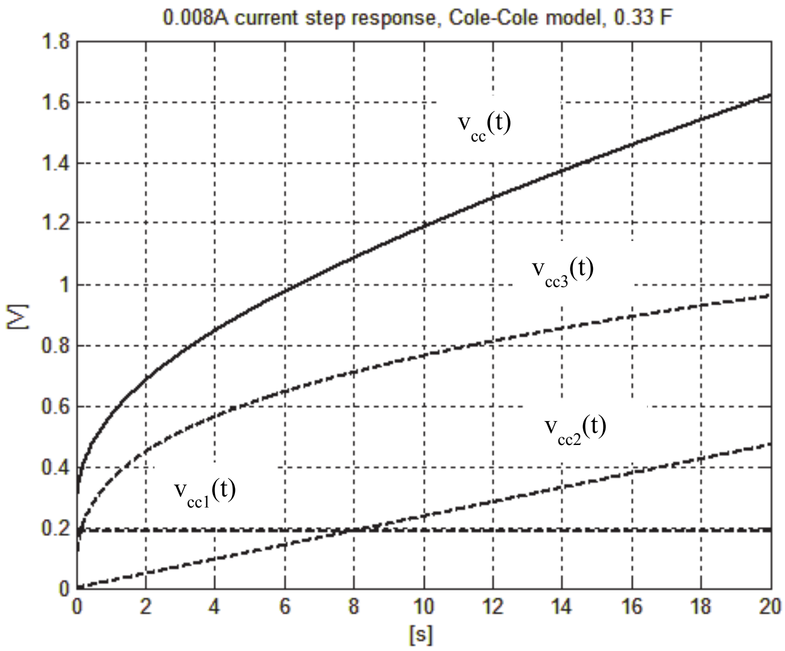

Figure 7. Current step response of supercapacitor of 0.33 F

4 Cole-Cole model application in control systems analysis

It has been mentioned that for fractional calculus the numerical computing environment Matlab with FOTF toolbox [7] can be applied. Matlab environment is well known and widely used tool for modeling and simulation of

physical systems. Using Control Toolbox one can study and design control systems. FOTF toolbox enables fractional calculus providing functions for:
- fractional transfer function object creation,
- presentation of Bode and Nyquist plots of this transfer function,
- calculation of time response on basis of transfer function and time input signal,
- addition, subtraction, multiplication and inversion of created models,
- feedback connection of such models,
- determination whether system is stable.

Presented example of FOTF toolbox application is design of resistor /capacitor voltage divider of inertial properties consisting of resistor R_0=5kΩ and supercapacitor of C=0.1F (Figure 8).

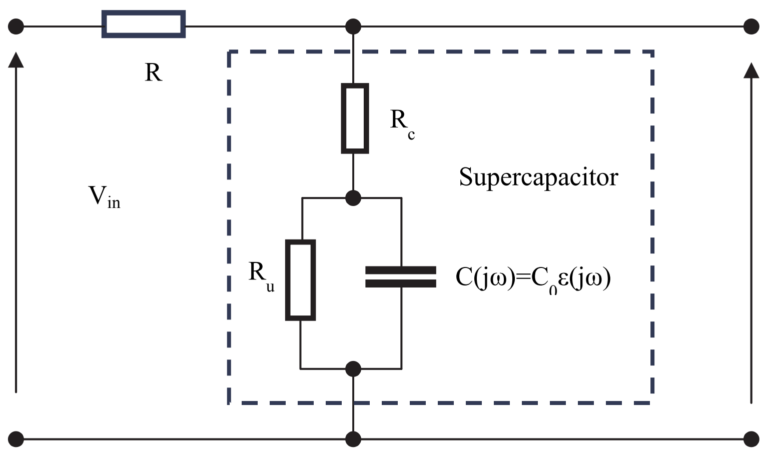

Figure 8. Scheme of the voltage divider

This divider shown in Figure 8 is described by equation

$$G_d(s) = \frac{G_{CC}(s)}{R_0 + G_{CC}(s)} = G_{CC}(s)\big(R_0 + G_{CC}(s)\big)^{-1} \tag{27}$$

where

$$G_{CC}(s) = \frac{1 + 4.67s^{0.705} + 5.01s}{5e - 08 + 0.1s} \tag{28}$$

One can specify FOFT object for (28) and enter it into MATLAB. Vectors formulated according to form (7) are input parameters of such an object. They contain coefficients a_i, b_i and exponents of s defined in (21). For (28) these vectors (in the reverse order) are equal

$$wa = [a_2 \quad a_0] = [0.1 \quad 5e-08] \tag{29a}$$

$$pa = [1 \quad 0] \tag{29b}$$

$$wb = [b_2 \quad b_1 \quad 1] = [5.01 \quad 4.67 \quad 1] \tag{29c}$$

$$pb = [1 \quad \delta \quad 0] = [1 \quad 0.705 \quad 0] \tag{29d}$$

In the next step the FOFT objects of (28) and R_o should be created

```
Gcc=fotf(wa,pa,wb,pb);
R0=fotf([[1],[0],[5000],[0]);
```

Then according to (26) these objects should be added

```
G1=plus(R0,Gcc);
```

inverted

```
G1i=inv(G1);
```

and multiplied

```
Gd=mtimes(Gcc,G1i);
```

The calculated transfer function of considered divider is equal to

$$G_d(s) = \frac{5e-08 + 2.34e-07s^{0.705} + 0.1s + 0.470s^{1.705} + 0.504s^2}{5e-08 + 2.34e-07s^{0.705} + 0.1s + 0.470s^{1.705} + 51s^2} \tag{30}$$

This transfer function has been compared with transfer function of divider with capacitor in which the relaxation phenomenon can be neglected. Impedance of such idealized capacitor of capacitance C_i=0.1 F is similar to (10)

$$G_i(s) = R_c + \frac{R_u \frac{1}{sC_i}}{R_{u+} \frac{1}{sC_i}} \cong \frac{1 + sR_cC_i}{\frac{1}{R_u} + sC_i} = \frac{1+5\text{s}}{5\text{e}-08+0.1\text{s}} \tag{31}$$

Transfer function of the divider with this capacitor is

$$G_{id}(s) = \frac{R_0}{R_0 + G_i(s)} = \frac{R_0(1 + SR_uC_i)}{R_0(1 + SR_uC_i) + R_u(1 + SR_cC_i)} = \frac{1+5s}{1+505s} \tag{32}$$

Bode plots of $G_d(s)$ and $G_i(s)$ are compared in Figure 9. The influence of relaxation phenomenon on frequency response is distinct for higher frequencies.

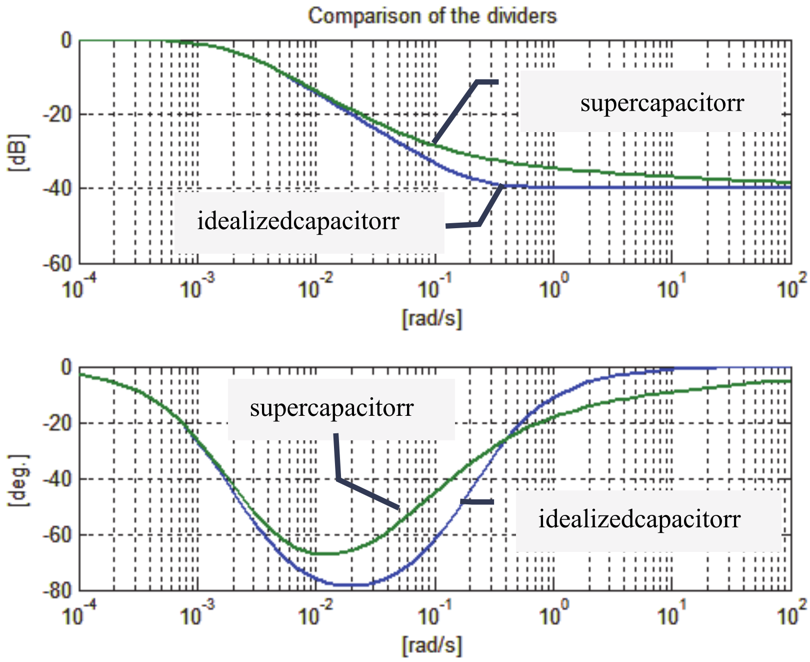

Figure 9. Bode plots of transfer functions of the dividers with supercapacitor and idealized capacitor

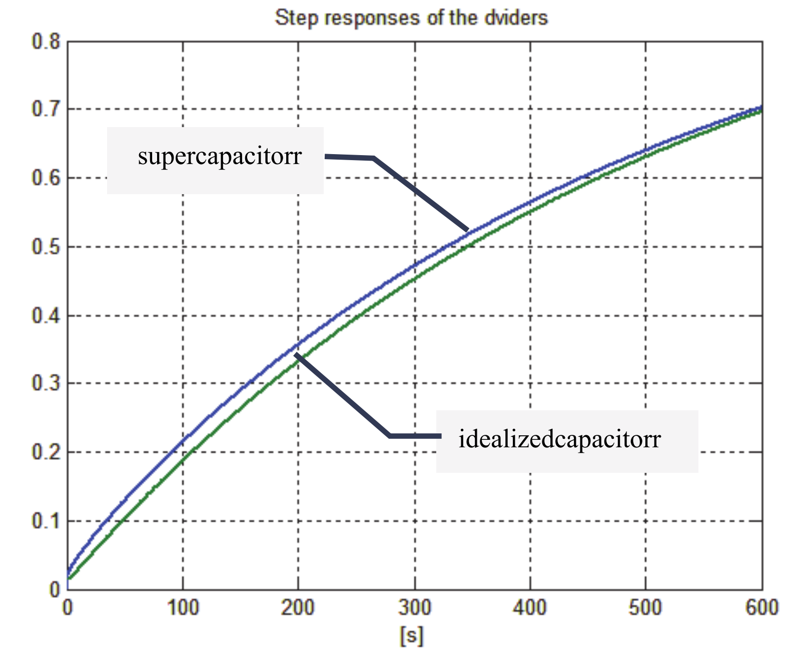

Figure 10. Step responses of the dividers

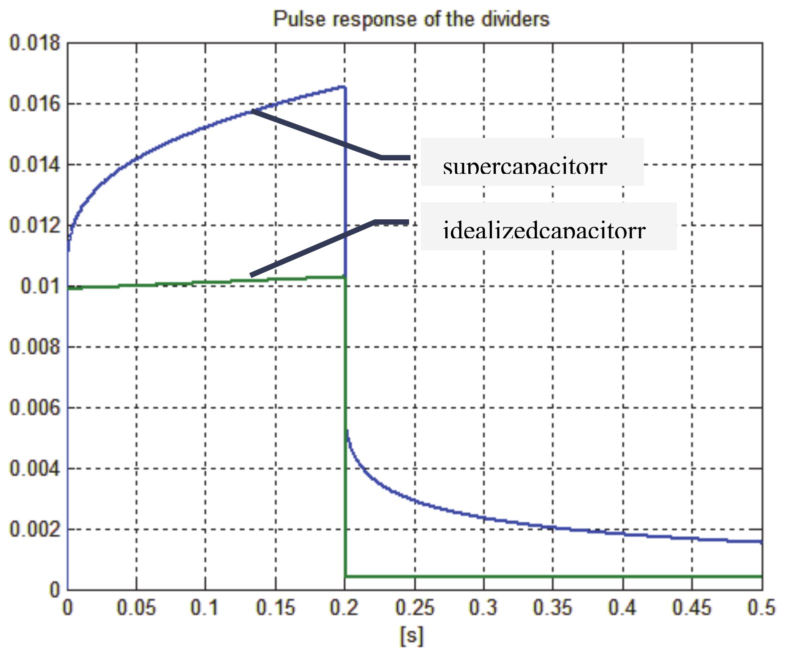

Figure 11. Pulse responses of the dividers

The step responses of both dividers are presented in Figure 10. They are convergent with the time rise. The essential difference at the beginning of the time responses is presented by the plots of time responses to short input pulse of 0.2 s duration (Figure 11).

Taking into account the difference between time and frequency responses of models of fractional and lumped parameters one can state that the fractional model of superapacitors can be important for exact description of its dynamics.

5 Conclusions

The technical literature mostly concerns the supercapacitor models with Cole-Davidson relaxation model application. In the paper the computer model of supercapacitor impedance based on Cole-Cole relaxation modelis presented. Consequently the impedance has polynomial form commonly used in automation. It enables the analysis of various control systems containing supercapacitors. For this purpose Matlab environment with FOTF toolbox designed to fractional calculus can be applied.

In the studied examples Cole-Davidson model in general is a bit more accurate for frequency and time responses of real supercapacitor approximation but advantages connected with easy analysis and simulation of control systems is essential. The comparison of practical effects of both relaxation models application in control systems analysis will be subject of the next publications.

In general it can be stated that the fractional model of superapacitors can be important tool for exact description of its dynamics.

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