Jan Długosz University in Częstochowa
Scientific Issues, Mathematics XVI, Częstochowa 2011

ON SOME STABILITY PROPERTIES OF POLYNOMIAL FUNCTIONS

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Abstract. In this paper we present conditions under which a function F with a control function f, in the following sense

$$\|\Delta_y^{n+1}F(x)\| \leq \Delta_y^{n+1}f(x), \quad x \in \mathbb{R},$$

can by uniformly approximated by a polynomial function of degree at most n.

1. Introduction

We start with the notation and definitions used in this paper.

Definition 1. Let $(G, +)$ stand for an Abelian group. Let $f : \mathbb{R} \to G$ be a given function and let $y \in \mathbb{R}$ be fixed. Then a difference operator Δ_y is defined by the formula

$$\Delta_y f(x) = f(x + y) - f(x), \quad x \in \mathbb{R},$$

and, for a positive integer n, by

$$\Delta_y^{n+1} f(x) = \Delta_y \Delta_y^n f(x), \quad x \in \mathbb{R}.$$

Definition 2. A map $f : \mathbb{R} \to G$ is called a polynomial function of degree at most n if and only if

$$\Delta_y^{n+1} f(x) = 0$$

for all $x, y \in \mathbb{R}$.

Definition 3. A map $f : \mathbb{R} \to G$ is called a monomial function of order n if and only if

$$\Delta_y^n f(x) = n! f(y)$$

for all $x, y \in \mathbb{R}$.

It is easy to see that a monomial function of order n is a polynomial function of degree at most n.

Definition 4. Let $I \subset \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be a function. A function f is called convex of order n, or shortly n–convex ($n \in \mathbb{N}$), if and only if
$$\Delta_y^{n+1} f(x) \geq 0$$
for every $x \in I$ and every $y \in (0, +\infty)$ such that $x + (n+1)y \in I$.

A function $f : I \to \mathbb{R}$ is concave of order n, or shortly n–concave, if and only if $-f$ is n–convex. The above notions are due to [3–5].

In [1] we have proved the following

Theorem 1. Let $(S, +)$ be an Abelian semingroup and let $(Y, \| \cdot \|)$ be a k–dimensional real normed linear space. Let further $f : S \to \mathbb{R}$ be a function such that $\Delta_y^n f(x) \geq 0$ for all $x, y \in S$, and $F : S \to Y$ be a mapping such that the inequality
$$\|n! F(y) - \Delta_y^n F(x)\| \leq n! f(y) - \Delta_y^n f(x)$$
holds for all $x, y \in S$.

Then there exists a monomial mapping $M : S \to Y$ of order n such that
$$\|F(x) - M(x)\| \leq k \cdot f(x)$$
for all $x \in S$.

In this paper we consider the functional inequality
$$\|\Delta_y^{n+1} F(x)\| \leq \Delta_y^{n+1} f(x),$$
and we look for the conditions implying the existence of a polynomial function P such that
$$\|F(x) - P(x)\| \leq k \cdot f(x).$$

We shall use the following theorem which was proved in [5]:

Theorem 2. Let $n \in \mathbb{N}$ and let $I \subset \mathbb{R}$ be an interval. If $f : I \to \mathbb{R}$ is n–concave, $g : I \to \mathbb{R}$ is n–convex and $f(x) \leq g(x), x \in I$, then there exists a polynomial w of degree at most n such that $f(x) \leq w(x) \leq g(x), x \in I$.

2. Results

Theorem 3. Let $(Y, \| \cdot \|)$ be a k-dimensional real normed linear space. Let further $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) \geq 0, x \in \mathbb{R}$, and $F : \mathbb{R} \to Y$ be a mapping such that the following inequality
$$\|\Delta_y^{n+1} F(x)\| \leq \Delta_y^{n+1} f(x) \tag{1}$$
holds for all $x, y \in \mathbb{R}$.

Then there exists a polynomial mapping $P : \mathbb{R} \to Y$ such that

$$\|F(x) - P(x)\| \le kf(x), \quad x \in \mathbb{R}.$$

Proof. Assume that $F : \mathbb{R} \to Y$ and $f : \mathbb{R} \to \mathbb{R}$ satisfy (1). Then for every $y^* \in Y^*, \|y^*\| = 1$ and for all $x, y \in \mathbb{R}$ we have

$$-\Delta_y^{n+1} f(x) \le \Delta_y^{n+1} y^* \circ F(x) \le \Delta_y^{n+1} f(x).$$

Hence

$$\Delta_y^{n+1}(y^* \circ F + f)(x) \ge 0 \tag{2}$$

and

$$\Delta_y^{n+1}(y^* \circ F - f)(x) \le 0 \tag{3}$$

for every $y^* \in Y^*, \|y^*\| = 1$ and for all $x, y \in \mathbb{R}$.

Let $\{e_1, \dots , e_k\}$ be a basis of Y such that $\|e_i\| = 1$ for all $i \in \{1, \dots , k\}$. Let further $y_i^* : Y \to \mathbb{R}$ be a projection onto the ith axis, i.e. $y_i^*(y_1 e_1 + \dots y_k e_k) = y_i$ for $(y_1, \dots , y_k) \in \mathbb{R}^k$, $i \in \{1, \dots , k\}$. Clearly, $y_i^* \in Y^*$ and $\|y_i^*\| = 1$ for all $i \in \{1, \dots , k\}$.

For every $i \in \{1, \dots , k\}$, we define functions $p_i : \mathbb{R} \to \mathbb{R}$ and $q_i : \mathbb{R} \to \mathbb{R}$ by the following formulas:

$$p_i(x) := y_i^* \circ F(x) + f(x), \ x \in \mathbb{R} \tag{4}$$

and

$$q_i(x) := y_i^* \circ F(x) - f(x), \ x \in \mathbb{R}. \tag{5}$$

Since $f(x) \ge 0$ for all $x \in \mathbb{R}$, we infer that

$$p_i(x) \ge q_i(x)$$

for every $i \in \{1, \dots , k\}$ and for all $x \in \mathbb{R}$.

From (2) we deduce that for every $i \in \{1, \dots , k\}$ the function p_i is n–convex. From (3) we have that for every $i \in \{1, \dots , k\}$ the function q_i is n–concave.

By virtue of Theorem 2, we infer that for every $i \in \{1, \dots , k\}$ there exists a polynomial function w_i of degree at most n such that

$$q_i(x) \le w_i(x) \le p_i(x), \ x \in \mathbb{R}. \tag{6}$$

Then, by (4), (5) and (6), we obtain

$$|y_i^* \circ F(x) - w_i(x)| \le f(x) \tag{7}$$

for all $i \in \{1, \dots , k\}$ and for all $x \in \mathbb{R}$.

Now, we define a function $P : \mathbb{R} \to Y$ by the formula

$$P(x) = w_1(x) \cdot e_1 + \dots + w_k(x) \cdot e_k, \quad x \in \mathbb{R}.$$

The function P is, of course, a polynomial function of degree at most n. We have also, by (7),

$$\begin{aligned}\|F(x)-P(x)\| &= \|\sum_{i=1}^{k}(y_i^*(F(x))-w_i(x))e_i\| \\ &\leq \sum_{i=1}^{k}|y_i^*(F(x))-w_i(x)|\cdot\|e_i\| \leq k\cdot f(x)\end{aligned}$$

for all $x \in \mathbb{R}$.

Ger [2] considered the operator

$$\delta_y^n f(x) := \Delta_{\frac{y-x}{n+1}}^{n+1} f(x).$$

Then f is n–convex if and only if

$$x \leq y \Rightarrow \delta_y^n f(x) \geq 0.$$

Analogically we can prove Theorem 4.

Theorem 4. Let $I \subset \mathbb{R}$ be an open interval and let $(Y, \|\cdot\|)$ be a k–dimensional real normed linear space. Let further $F : I \to Y$ and $f : I \to \mathbb{R}$ be mappings such that the following inequality

$$\|\delta_y^n F(x)\| \leq \delta_y^n f(x)$$

holds for all $x, y \in I$.

If $f(x) \geq 0, x \in I$, then there exists a polynomial mapping P of degree at most n such that

$$\|F(x)-P(x)\| \leq kf(x),\ x \in I.$$

References

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