THE MODEL OF THE DIGITAL TERRAIN MAP (DTM)

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Abstract

The true digital terrain map (DTM) is needed to calculate so-called terrain gravity potential. It is one of the fundamental factors in earth geoid shape determination. There are many methods of calculation of the gravity potential. The gravity integral should cover all Earth area. There is no possibility to establish digital terrain map for whole planet. So it must be numerically proved, if the gravity integral converges to its extreme value for some limited part of the earth area. The model of DTM was established mathematically for testing a different way of gravity potential calculation. It contains all features of real DTM and has been build up in two steps processing.

Key words: numerical and functional modeling, gravity terrain potential, digital terrain map

1 Introduction

A modern information technology is widely implemented in present geodesy and cartography. The new measurements technologies like a GPS, laser and radar scanning, gravimetry and gravity gradient measurements allows the precise earth surface mapping with high accuracy that was earlier unavailable. The investigation of an earth figure is impossible by direct measurements and usually has been done by some theoretical models. The fundamental model of earth figure determination is the Stokes method and the modified Stokes – Helmert method [2,9,11,12]. There is no possibility to integrate gravity anomaly reduced to see level over an all geoid's surface, or, like in modified by Molodensky the Stokes–Helmert method, over all physical earth surface. The integration needs some approximate downward continuation of gravity anomaly to see level and surface condensation of the mass over geoid. The results are non accurate [15,16,18]. In modified method Stokes integral is made over physical earth surface gravity anomaly and the result are quasigeoid and height anomaly. The direct and indirect effect of topography must be derived to evaluate quasigeoid height over geoid. The rigorous determination of terrain mass potential is in work [8] and approximate one in work [9]. The correct determination of a topographic potential at any point on Earth needs a knowledge of the mass decomposition over all geoid surface. This is actually unrealistic, so all calculation has been done by assumption of local constant mass density and knowledge of the topographical height mass over geoid. The approximate value of a topographic potential we get by integration of point mass over all volume topographic mass by some assumption about its density. The main problem in this estimation is, how great is an approximate error. Moreover, using different Earth models (sphere, ellipsoid, plane), we may question, if simpler or more complex model of geoid is needed [13,15,16,17], to obtain acceptable accuracy. The answer rarely is possible by theoretical consideration, because of problem complexity. The solution gives a numerical calculations. The potential integration over mass needs a great base of data from area involving territories of many countries. It's accessibility is not always possible, because of the administrative law of different countries. It is difficult to estimate, how great area potential integrals must cover, theoretically over all Earth surface, to obtain sensible accuracy. The similar problems appear by attempts of improving the model of geoid, as a geodetic reference frame for calculation of the topographic potential. The calculations based on assumption of a constant terrain height are far from real situation [8]. To avoid this problems in real testing of different theoretical models, in this work there is proposed some model of digital terrain map (DTM). It may help to test some theoretical solutions of geoid determination. The similar option we find in work of Kryński J. [9].

2 The requirements a Model of the DTM

The digital terrain model must have all characteristic features of real land. It must contain a great topographical features like lowlands, highlands, plateaus, mountains, without lakes or rivers, small depth, sizes and smaller density. The theoretical requirements about size of such model are unknown, but it seems that area size 2000kmx2000km is sufficient. This mean geographical size 15o-20o from south to north and the same from west to east. Besides that great size features it is necessary to build from the ground base local small topographical structures like hills, abrupt, mountain peaks. The numerical calculation of gravity potential on Earth surface with appropriate accuracy non averaging local height must be done on grid range of 20-30m, it means 1" in geographical latitude and longitude. Only such grid will be adequate on terrain with slope inclination great than 20o (100m of distant means 20m or more of height difference). It means construction of at least $4 \cdot 10^9$ points with defined height and density. It seems reasonable to build at first complete DTM, because of time-consuming numerical arithmetic. The basic DTM is to big to put it in one memory set, so we need to divide it into smaller parts and build convenient access tools to required fragments. General coordinates of each point are two integer numbers and number of adequate part of base DTM. It's easy to transform this coordinates to geographical or geodetical longitude and latitude, and consecutively Cartesian or geocentric coordinates at any place on the Earth model (sphere, ellipsoid, plane).

3 The Representations of the Big Topographical Features

The building of topographical feature is like some kind of an artistic work and means creation of fictitious terrain images with all mentioned earlier big elements. Because of its great size first basic map is build on grid sized about 0.5kmx0.5km (16"x16"). The height on denser grid are going to be calculated algorithmically by assumption of constant inclination the planes build on rare ground grid. The plane is defined unambiguously by three points of a grid, so the base of denser grid must be a triangular net. Its going to be defined further.

Now we define ground requirements of DTM.

A look at any map of scale 1:1000000 shows presence lowlands, highlands, low (800÷1500m) and high mountains (tops over 2000m). On the large scale maps contours are relatively simple, without sharp turns. From lowlands grow up highlands, from highlands grow up mountains. The layers arrangement has multi-pyramid structure. This allows build up similar structure by mathematical methods. The Gauss function of two variables seems to be especially useful:

$$g(x, y) = h \cdot \exp(-[(x - x_0)^2 + (y - y_0)^2] / d^2) \qquad (1)$$

It doesn't has to be normalized for purpose of this work and its values shape on xy plane forms regular circular hill with center in (x_0, y_0) point and height equal h. Of course this regular shape needs some modifications, like irregular changes of slope inclination in different directions and distances from the top. The another modifications must take account, that geological structures have irregular directions against meridians and parallels. H parameter allows to control a maximal height of model's geological structure. Parameter d controls size of a structure. By assuming that 1/16 of maximal height is a border of modeled geological structure we find for this value distance 1.67d. So d parameter is good rating of structure size. We use function g defined by (1) to establish height at any DTM point.

For mathematical reason a rectangular grid was chosen as an integer coordinate base size (N+1)x(M+1). For convenience (real distance) every point coordinate were calculated in km as (x,y)=(D*i,D*j); i∈(1,N+1), j∈(1,M+1), D is size parameter in km (D=0.5km for fundamental map).

Next step is to randomize the regular gaussian hill. It may be achieved by some functional transformation of map area and height's evaluation $h(x,y)$ as g(x',y') at every point of DTM:

$$\begin{aligned} &[x',y'] = f([x,y]); \\ &[x,y] = \vec{r}; \quad [x',y'] = \vec{r}' \\ &h(x,y) = g(x',y'); \end{aligned} \tag{2}$$

4 The Randomized Transformation of the Map Coordinates

A first step of coordinate transformation is rotation around a center point (x_0, y_0) and elongation along one of axes. We rotate reference system by angle φ and multiple one coordinate, for example y', by $\sqrt{k}$. The result is:

$$\begin{aligned} &\begin{cases} x' = (x - x_0)\cos\varphi + (y - y_0)\sin\varphi; \\ y' = -(x - x_0)\sin\varphi + (y - y_0)\cos\varphi; \end{cases} \\ &g(x',y') = h \cdot \exp(-[x'^2 + k \cdot y'^2]/d^2); \end{aligned} \tag{3}$$

We obtain the last effect introducing a changeable rotation angle φ. It may be function of azimuth A, where A is angle in polar coordinates with pole (x_0, y_0). The A angle is periodic, so $\varphi(A)$ must be a periodic function: $\varphi(0) = \varphi(2\pi)$. In this work the following functions were chosen:

$$\begin{aligned} &A = \operatorname{arctg}\left(\frac{y - y_0}{x - x_0}\right); A \in <0, 2\pi> \\ &\varphi(A) = \varphi_0 \cos A; \qquad \varphi(A) = \varphi_0 \sin A; \\ &\varphi(A) = \varphi_0 \frac{(A - \pi)^2}{\pi^2} \cos(A) \end{aligned} \tag{4}$$

Below on Figure 1-4 are shown some effects of transformation (3) and (4) for rectangular area. We see contour graph of function (2) after transformation its rectangular domain.

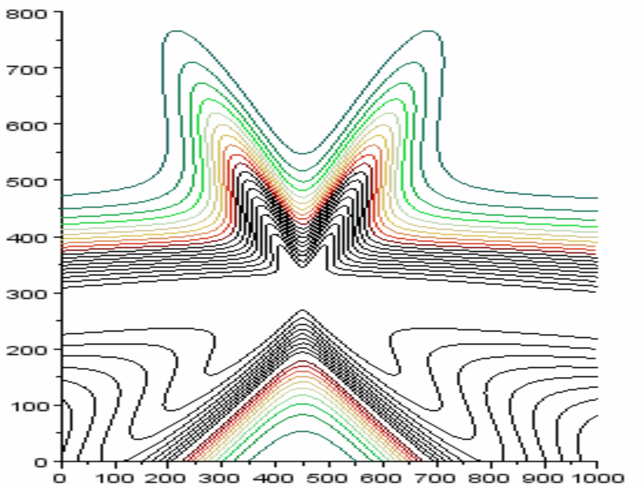

Figure 1. elongation k=4, φ0=1.5, d=200, φ=φ0 cosA

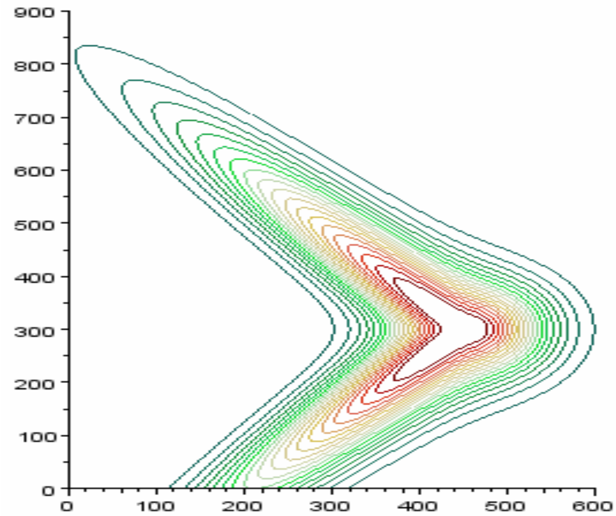

Figure 2. k=0.15, φ0=0.95, d=60, φ=φ0 sinA

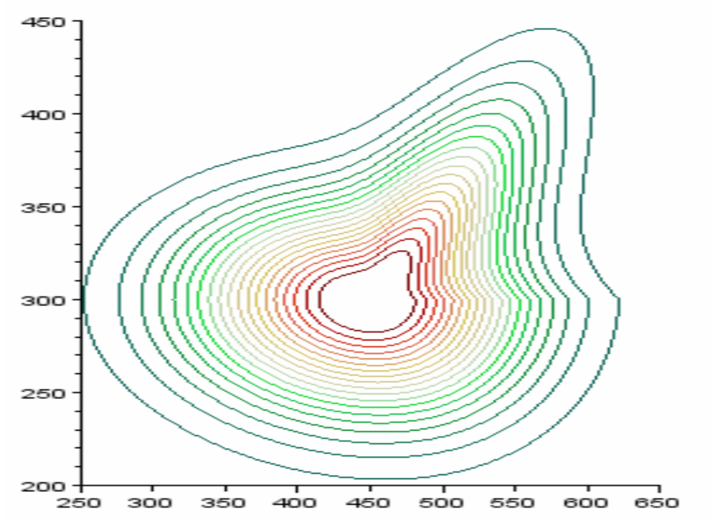

Figure 3. k=5, d=80, φ0=1, φ=φ0 ((A-π)/π)2

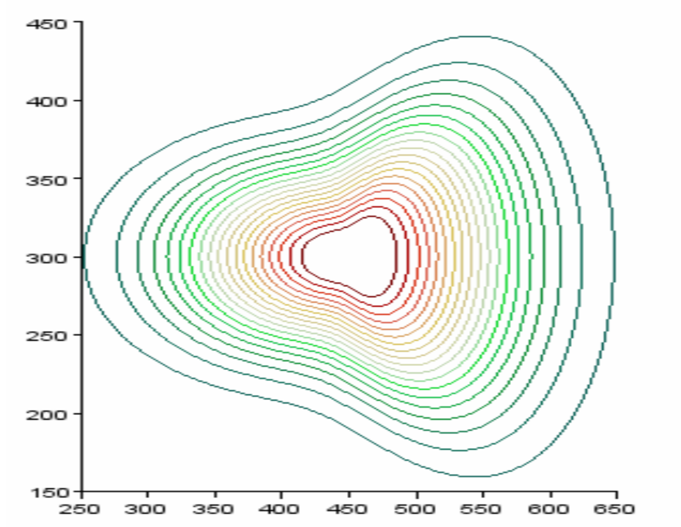

Figure 4. k=5, d=80, φ0=1.1, φ=φ0 ((A-π)/π)2 sinA

The above presented shapes are still too regular in comparison to accidental tectonic forms. The height of the real tectonic forms sometimes grows up, sometimes decreases, moreover they have many tops. In polar coordinates with pole (x_0, y_0) g function is decreasing monotonously and only by appropriate arguments transformation we may achieve multi-tops effect. Let

$v = \sqrt{x^2 + k \cdot y^2}$ means a new g function argument. This is distance from a center of the "hill". We use a following function F(v,k) as a distance transforming function:

$$F(v,k) = v \cdot \{1 - \exp[-(v - v_1)(v - v_2) - (v_1 - v_2)^2 / 4 - -k \cdot \ln((v_1 + v_2) / 2)]\} \quad (5)$$

The parameter k means the same elongation constant k as in (3). We see the plot of function F(v,k) on Figure 5 below.

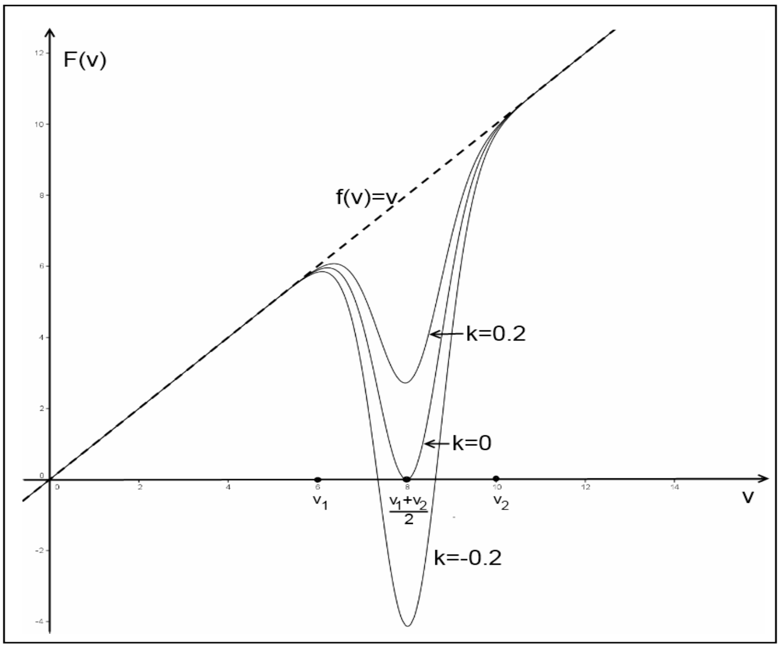

Figure 5. Plot of function F(v): layered three function for k=0 and k=±0.2. For comparison doted line is a plot of linear function f(v)=v.

This function allows to create multi-top plateaus and mountains by control k values as function of azimuth A. The average arithmetic value of few F(v,k) function allows a precise modeling of a hilly or mountainous area of the DTM (see Figure 6).

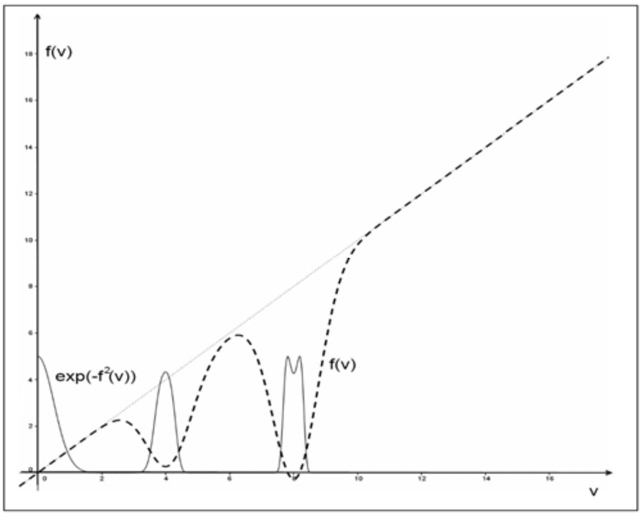

Figure 6. The way of modeling the multiple hills area.

The next step of creating the DTM is controlling a ridge direction and inclination. For our purpose two ridges of any hill are sufficient . We choose two random directions (two azimuths A1 and A2) from interval <0,2π) and two DA1, DA2 from interval < 0, π/2 >. The firsts are direction and the second broadness of a hill ridges. Because the random number generator is determined, action begins with generating a few random numbers. The parameter k is multiplied by function of azimuth fg(A), it is simply sum of two Gauss function:

$$\mathrm{fg(A)} = \mathrm{b} \cdot \exp(-(\mathrm{A}-\mathrm{A}_1)^2 / \mathrm{DA}_1^2) + \exp(-(\mathrm{A}-\mathrm{A}_2)^2 / \mathrm{DA}_2^2) \qquad (6)$$

This function must be periodic and continuous inside <0,2π>, to avoid fault for azimuth zero: fg(0) = fg(2π). The strict formula of function fg(A) is:

$$\begin{aligned}
&\mathrm{fg(A)} = \mathrm{fg(p(A))};\\
&\mathrm{p(A;A_1<\pi)} = (\mathrm{A}-\mathrm{A}_1)[\mathrm{sign(A)}-\mathrm{sign}(\mathrm{A}-\mathrm{A}_1-\pi)]/2-\\
&\qquad -(\mathrm{A}-\mathrm{A}_1-2\pi)[\mathrm{sign}(\mathrm{A}-2\pi)-\mathrm{sign}(\mathrm{A}-\mathrm{A}_1-\pi)]/2+\\
&\qquad +(2\pi-\mathrm{A}_1)[1-\mathrm{sign(A)}];\\
&\mathrm{p(A;A_1>\pi)} = (\mathrm{A}-\mathrm{A}_1)[\mathrm{sign}(\mathrm{A}-\mathrm{A}_1+\pi)-\mathrm{sign}(\mathrm{A}-2\pi)]/2+\\
&\qquad +(\mathrm{A}-\mathrm{A}_1+2\pi)[\mathrm{sign(A)}-\mathrm{sign}(\mathrm{A}-\mathrm{A}_1+\pi)]/2\ ;
\end{aligned} \qquad (7)$$

The plot of periodic Gauss function inside interval <0,2π> shows Figure 7

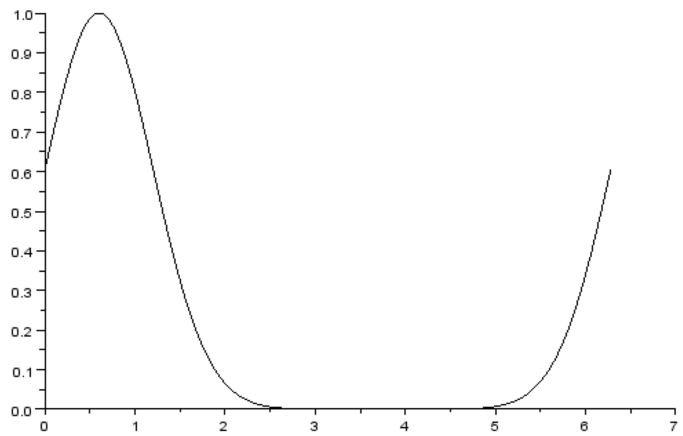

Figure7. The periodic Gauss function inside interval < 0, 2π >, A1=0.6, DA=0.6.

The composition of two fg functions controls k parameter in transformation (5) (see Figure 8 below).

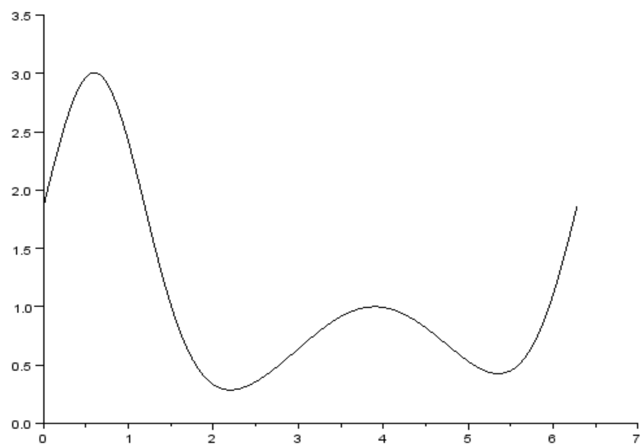

Figure 8. The sum of two periodic Gauss function inside < 0, 2π >, A1=0.6, DA1=0.6, A2=3.9, DA2=0.95, b=3.

We calculate height at point (x,y), so we need unambiguously evaluate the azimuth A at every point relative polar reference system with the center in a top of the hill. Easily to check, that function:

$$A = f(x,y) = \frac{\pi}{2}\cdot \mathrm{sign}(y)\cdot[1-\mathrm{sign}(|x|)] + \mathrm{arctg}\left(\frac{y}{x}\right)\cdot \mathrm{sign}(|x|) + \\ +\frac{\pi}{2}\cdot\{\mathrm{sign}(y)\cdot[1-\mathrm{sign}(x)] - [1-\mathrm{sign}(|y|)]\cdot[1+\mathrm{sign}(x)] - \\ -[1-\mathrm{sign}(|x|)]\cdot[1+\mathrm{sign}(y)]\}; \quad (8)$$

satisfied this conditions.

The constants v1 and v2 are multiplied by function fg(A) and next f(F(v)) is evaluated. Finally to increase impression of naturalness f(v) it is multiplied by polynomial of degree 4 of A variable:

$$z(A) = a - c\cdot A(A-A_1)(A-A_2)(A-2\pi)/(\max(z)-\min(z)) \quad (9)$$

where a and c – free constants.

5 Building a Model of the DTM.

There is shown action of the above functions in building a model of the DTM. In first step it is set a number of highlands and mountains at area of DTM. For our purpose we set four highlands with maximal height range 100-250m, two middle mountains height range 1200-1600m and one mountains with maximal height 2500-3000m. We set central coordinates of each formation, its size, elongation, initial rotation angle and v1 and v2 constants. Then the grid and scale size D is established, in this work N=4000, M=4000 and D =0,5km. So basic DTM covers area 2000x2000 km and contains 16008001 points with evaluated height. Numerical program sum up heights of all elements evaluated over all DTM area and saves it in matrix h(x,y). Figure 9 shows summarized colored contour plot of matrix h(x,y) evaluated for smaller grid size 800x600km.

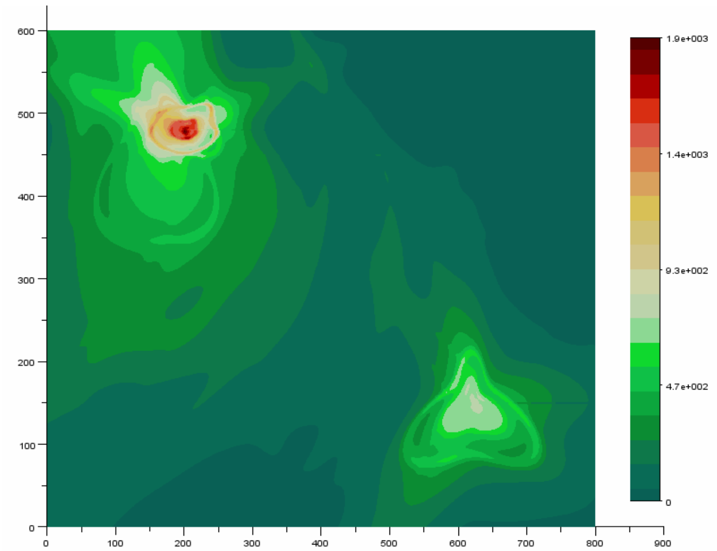

Figure 9. Summarized colored contour plot of matrix h(x,y).

Because of great scale of this map, some of fragments seems to be poor, without details. But enlarged left lower and upper right quadrant looks much better and shows appreciable diversity (Figure 10 and 11).

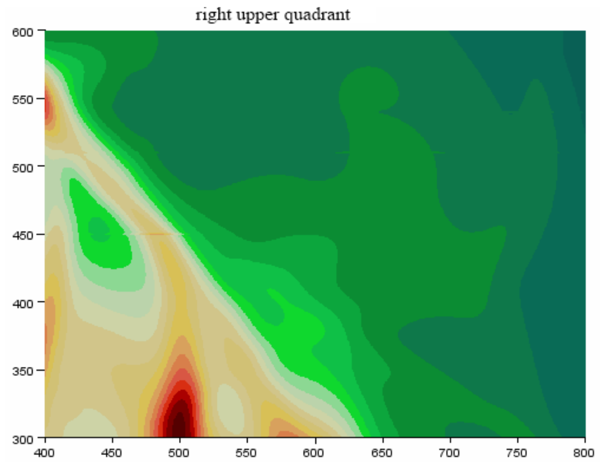

Figure 10 Enlarged upper right quadrant of the map from Figure 9

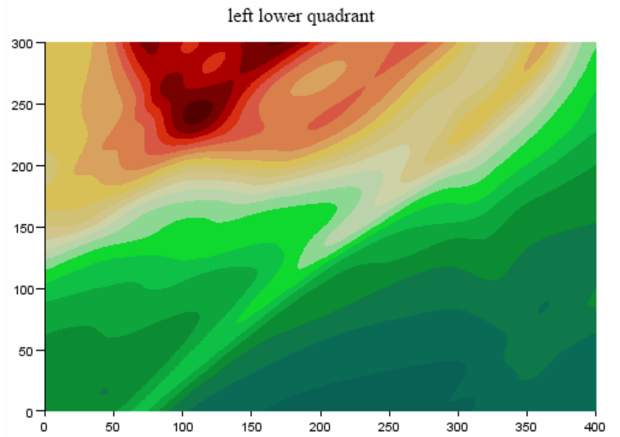

Figure 11. Enlarged left lower quadrant of the map from Figure 9.

The final sparse DTM (grid 0,5x0,5km or 16”x16”) is a sum of four above operations with the center coordinates and rotation angle randomly changed from 0 to ±50% of their initial values. The function φ(A) has been randomly chosen from three options (4) for every step of calculation.

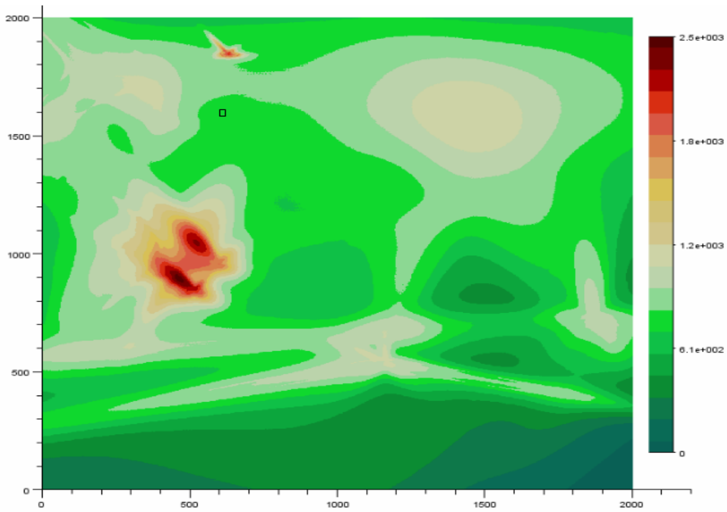

Figure 12. The colored contour plot of basic sparse height’s matrix DTM. Maximal height h=2345.95m.

The layered effect will be seen on Figure 12 above.

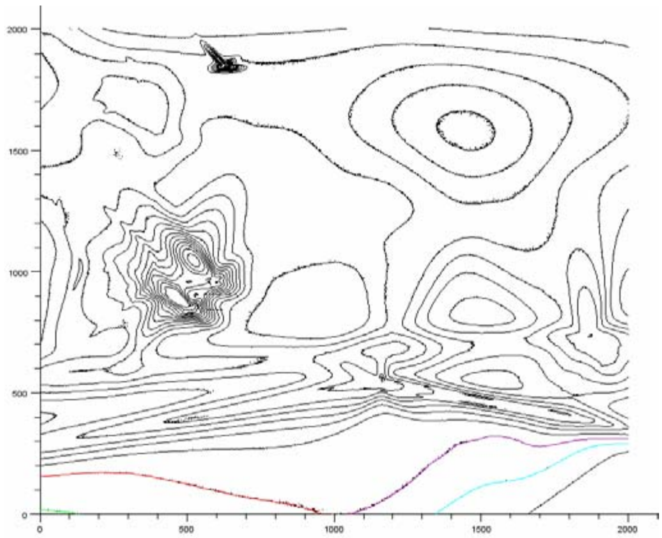

Figure 13. The contour plot of the basic sparse matrix DTM from Figure12

Below are shown the meridian profiles of terrain height.

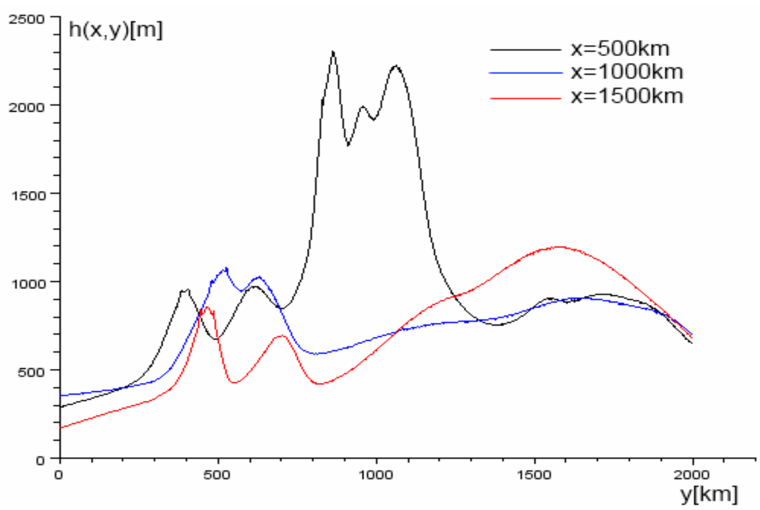

Figure14. The meridian profiles of the terrain height of the map from Figure12

Before operation of building height's matrix on the denser grid (1"x1" or 31.25mx31.25m) a local random change of height was introduced. The following algorithm has been used. A generator of random number is too predictable, so another algorithm of randomization has been used. The first five elements from two consecutive columns of pz matrix were calculated as follows:

```
z(j+1,k)=exp(v(1,j));
z(j+1,k)=z(j+1,k)-int(z(j+1,k));
z(j+1,k+1)=exp(v(2,j));
z(j+1,k+1)=z(j+1,k+1)-int(z(j+1,k+1));
```

where j=1:5 and v(2,5) is random matrix.

Next all elements of the matrix pz(4000,4000) were calculated accordingly to the algorithm:

```
z(j,k)=exp(z(j-5,k+1));
z(j,k)=z(j,k)-int(z(j,k));
z(j,k+1)=exp(z(j-5,k));
z(j,k+1)=z(j,k+1)-int(z(j,k+1));
```

j=6:4000 step 5 and k=1:4000 step 2.

All values of this matrix lies inside (0,1) interval. The new local height was calculated according to formula:

$$h = h * [1 + (0.5 - pz)/100]$$

so the height's changes doesn't exceed 1% of the initial value.

6 The Dense Grid and the High Resolution DTM

The base of the model of DTM is grid of points. As it was pointed out earlier, the DTM doesn't have to be completely real to main purpose. For this reason earth surface may be represented by planes(facets) defined for every three points of basic sparse grid. From a rectangular grid we must move to the triangle grid.

Simple dividing of each rectangular by diagonal is ambiguous (see Figure 15).

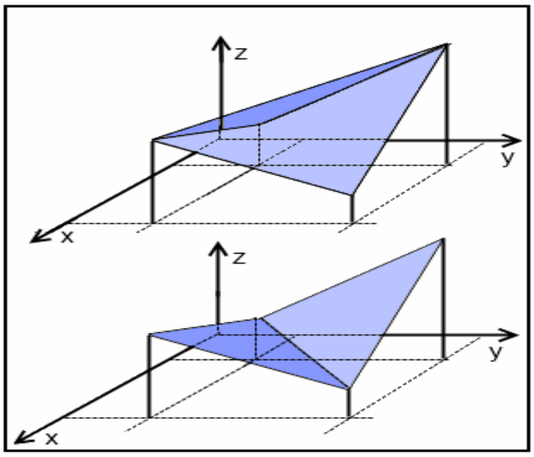

Figure 15. Two different triangle facets from rectangular grid. .

To avoid this problem, we have divided every rectangle into four triangles by cross-point of its two diagonal (Figure 16).

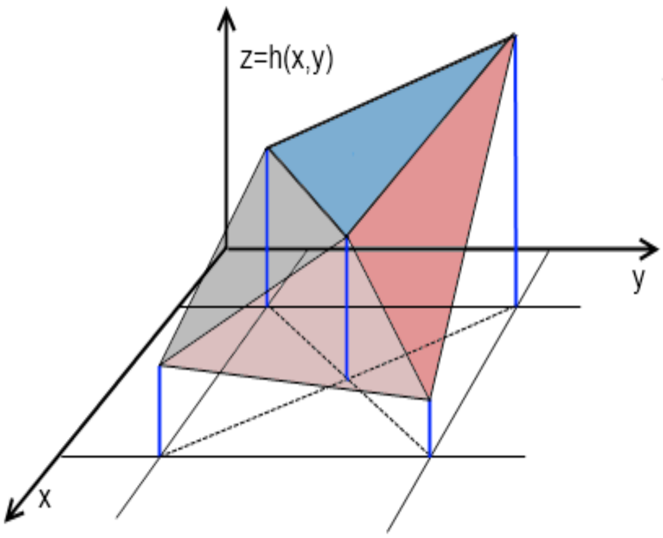

Figure16. A rectangular dividing into the triangle.

We fix the height at the cross-point (center of rectangle) as an arithmetic average of heights at rectangle apexes. The terrain surface is built from isosceles rectangular triangles (see Figure 17)

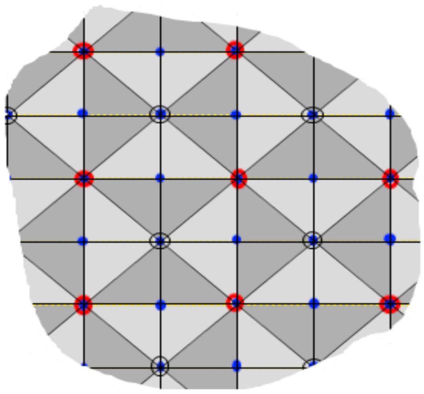

Figure 17. Part of a triangle grid. Red points belongs to sparse basic grid. Dark and bright triangles constitute area surface.

From practical point of view, the most convenient rising of map density is multiplication by 2n, then n controls density growth process. The matrix size increase 22n-time. For n=4 it means 256 time greater set to save and proportional calculating-time growth. It is wise to calculate only one time the whole DTM, save it and use, when needed.

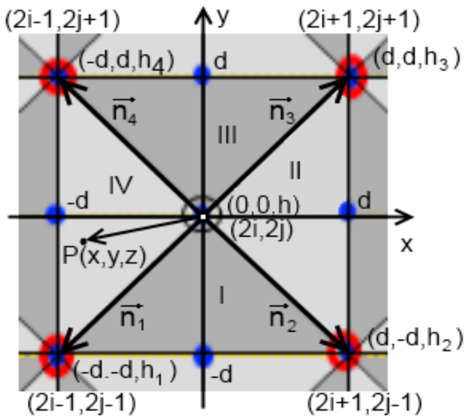

Figure18. Local coordinate system at basic sparse grid; (d,d,h) –Cartesian coordinates, (i,j)- numerical integer coordinates.

Choosing local coordinate system like on Figure18, we set four triangles with common apex and four vectors constituting that triangles: $\vec{n}_1, \vec{n}_2, \vec{n}_3$ and $\vec{n}_4$ with common initial point. A height z=h(x,y) is the third coordinate of each apex point. The coordinates of any point $\vec{r}$ inside closed area of each triangle fulfill plane equation:

$$(\vec{r} - \vec{r}_0) \cdot \vec{N}_{ij} = 0; \quad \vec{N}_{ij} = \vec{n}_i \times \vec{n}_j \tag{10}$$

where $\vec{n}_i, \vec{n}_j$ - vectors constituting a triangle with point P(x,y,z), $\vec{r}_0 = [0, 0, h]$ - common points of all triangles. After short calculations we obtain required result:

$$z = h(P) = h - \frac{xN_{ij}^x + yN_{ij}^y}{2d}; \tag{11}$$

where d=D/2, h is an arithmetic average height at center of rectangle and x,y are the coordinates on local grid:

$$\begin{cases} x = k \cdot D / 2^n; \\ y = l \cdot D / 2^n; \end{cases} \quad k, l \in (-2^{n-1}; 2^{n-1}); \tag{12}$$

The integer coordinates of a points of the dense grid are:

$$\begin{cases} ix = (i-1) \cdot 2^2 + k; \\ iy = (j-1) \cdot 2^2 + k; \end{cases} \quad i \in <1; N>; \; j \in <1, M>; \; k \in <1; 2^n> \tag{13}$$

The Figure18 below shows colored contour plot of a small part of the dense high resolution DTM.

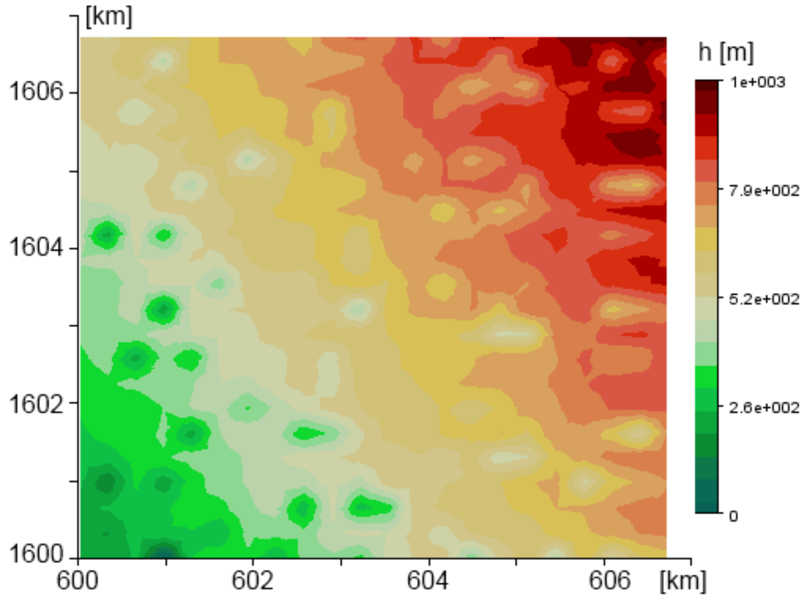

Figure19. High resolution 16-time enlarged signed fragment of the map from Figure12.

7 Summary

The model of digital terrain map was established. The destination of this model is testing some theoretical problems of earth potential and geoid determination. Starting from low resolution grid (0.5x0.5km or 16”x16”) the model containing all features of real terrain was build up mathematically. The high resolution (1”x1”) grid for testing the theoretical models of geoid determination is needed. The earth surface may be represented acceptably by triangular facets 250x500m and heights estimated at that planes. We have achieved high resolution of DTM (31x31m or 1”x1”), sufficient for numerical integration of terrain geophysical potential . The DTM features simulates natural real mass composition over earth geoid. The size of DTM is sufficient for numerical calculations of potential integrals over big areas, noted by some authors [8,9,16,17]. The model of DTM is going to be useful for testing accuracy of the methods applied by some authors to geoid determination, especially to determine the ellipsoidal correction of terrain potential [3,4,5,6,7,10,14]. The functional dependence of an ellipsoidal correction from geodetic coordinates may be established too.

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