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On controllability and observability of fractional continuous-time linear systems with regular pencils

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Abstract. In this paper necessary and sufficient conditions of controllability and observability for solutions of the state equations of fractional continuous time linear systems with regular pencils are proposed. The derivations of the conditions are based on the construction of Gramian matrices.

Key words: controllability, observability, Gramian matrix, regular pencil, Drazin inverse.

1. Introduction

The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19th century [1, 2] and another one was proposed in 20th century by Caputo [3]. This idea has been used by engineers for modeling different processes [4–9, 11]. Mathematical fundamentals of fractional calculus are given in the monographs [1–3, 10].

Descriptor (singular) linear systems have been considered in many papers and books [12, 13]. The popularity of descriptor systems is continuously increasing as these are general enough to provide a solid understanding of inner dynamics for underlying physical problems [24]. Application of the Drazin inverse method to analysis of descriptor fractional continuous-time linear system was described in [25, 26].

The problem of controllability and observability began to attract the attention of mathematicians and engineers since it began to play a significant role in the control theory and engineering, having important applications in these fields. Many contributions on controllability problem have been made in recent years, see for example [27, 28]. However, it should be stressed that the control theory of singular fractional linear systems is not yet adequately explained, compared to that of fractional linear systems. In this regard, it is required and important to study the controllability and observability problems for fractional singular dynamical systems. To the best of our knowledge, there are no applicable reports on controllability and observability of fractional singular dynamical systems as treated in the current literature. Motivated by these considerations, in this paper, we study the controllability and observability of fractional singular continuous time invariant systems.

The paper is prepared as follows. Section 2 recalls some preliminary definitions and formulas. In Section 3, we obtain necessary and sufficient conditions of controllability and the last section concerns observability for descriptor fractional continuous-time linear system.

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2. Preliminaries

In this section we recall some well known fractional operators and special functions, along with a set of properties that will be of use as we proceed in our discussion, for details see [1–3].

Consider the following fractional continuous-time linear system described by the state equation

$$\begin{cases} ED_t^{\alpha}x(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{1}$$

where $0 < \alpha \leq 1$, $x(\cdot) \in \mathbb{R}^n$ is state vector $u(\cdot) \in \mathbb{R}^m$, is input vector and $y(\cdot) \in \mathbb{R}^p$ is output vector, $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$ and D_t^{α} is the Caputo differential operator, defined by

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{d}{ds} f(s) ds, \ 0 < \alpha < 1.$$

It is assumed that the pencil (E, A) of (1) is regular, i. e.,

$$\det(Es - A) \neq 0 \, \text{for some } s \in \mathbb{C}.$$

If $\det E \neq 0$, then by using Laplace transform the general solution of the system (1) with initial condition $x(0) = x_0$ can be written as [29, Theorem 1]

$$\begin{aligned} x(t) = &\Phi_{\alpha,1}\left(E^{-1}A, t\right) x_0 + \\ &+ \int_0^t \Phi_{\alpha,\alpha}\left(E^{-1}A, t-\tau\right) E^{-1}Bu(\tau) d\tau. \end{aligned} \tag{2}$$

where $\Phi_{\alpha,\beta}(E^{-1}A, t) = \sum_{k=0}^{\infty} \frac{(E^{-1}A)^k t^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)}$ is the state transfer matrix and $\Gamma(\cdot)$ is a Gamma-function.

Definition 1. [3] The Mittag-Leffler two-parameter function for an arbitrary square matrix A is

$$E_{\alpha,\beta}(A)=\sum_{k=0}^{\infty}\frac{A^k}{\Gamma(\alpha k+\beta)},\ \alpha,\beta>0, \tag{3}$$

in particular, $E_{\alpha,1}(A)=E_\alpha(A)$, with $\beta=1$.

Mittag-Leffler and state transfer matrices are related as follows

$$\Phi_{\alpha,\beta}(A,t)=t^{\beta-1}E_{\alpha,\beta}(At^\alpha)$$

and it is easy to check that [29, Lemma 2]

$$D_t^\alpha\Phi_{\alpha,\beta}(A,t)=A\Phi_{\alpha,\beta}(A,t). \tag{4}$$

Definition 2. [13] The smallest nonnegative integer q is called index of the matrix $A\in\mathbb{R}^{n\times n}$ if

$$rankA^q=rankA^{q+1}.$$

Definition 3. [24] A matrix $E^D\in\mathbb{R}^{n\times n}$ is called the Drazin inverse of $E\in\mathbb{R}^{n\times n}$ if it satisfies the following conditions:

$$EE^D=E^DE,\ E^DEE^D=E^D\text{ and }E^DE^{q+1}=E^q, \tag{5}$$

where q is the index of a matrix.

To compute the Drazin inverse E^D of the matrix $E\in\mathbb{R}^{n\times n}$ the following steps are required [24]:

1. Find the pair of matrices $V\in\mathbb{R}^{n\times r}$, $W\in\mathbb{R}^{r\times n}\}$, such that $rankV=rankW=rankE=r$ and
$$E=VW;$$
2. Compute the nonsingular matrix
$$WEV\in\mathbb{R}^{r\times r};$$
3. The desired Drazin inverse matrix is given by
$$E^D:=V(WEV)^{-1}W.$$

Remark 1. It is easy to see that, if $\det E\neq0$, then $E^D=E^{-1}$.

Example 1. Consider a matrix

$$E=\begin{bmatrix}0&0\\0&4\end{bmatrix}$$

Clearly $\det E=0$ and $rank(E)=1$, moreover

$$E^2=\begin{bmatrix}0&0\\0&16\end{bmatrix}.$$

So, $rank(E^2)=rank(E)$. Hence index of matrix E is 1. By using the Drazin inverse procedure, it follows that

$$E^D=\begin{bmatrix}0&0\\0&1\end{bmatrix}.$$

If the index q of A equals 1, the Drazin inverse A^D is the group inverse and is denoted by $A^\sharp$ (see, e.g., [30, p. 118]). In general the Drazin inverse can be expressed explicitly in terms of Jordan canonical form of A

$$A=S\begin{pmatrix}J&0\\0&N\end{pmatrix}S^{-1},\ A^D=S\begin{pmatrix}J^{-1}&0\\0&0\end{pmatrix}S^{-1},$$

where J contains Jordan blocks corresponding to nonzero eigenvalues, and N is nilpotent with $N^k=0$ and $N^{k-1}\neq0$. With this representation of A^D we can immediately see that [26]

$$\begin{gathered}\mathscr{R}(A^D)=\mathscr{R}(A^q),\ \mathscr{N}(A^D)=\mathscr{N}(A^q)\text{ and}\\ \mathbb{R}^n=\mathscr{R}(A^D)\oplus\mathscr{N}(A^D).\end{gathered} \tag{6}$$

If $\det E=0$ and the pencil of the matrices (E,A) is regular, that is, there exists $c\in\mathbb{C}$ such that, $\det(Ec-A)\neq0$.

Premultiplying equation (1) by $(Ec-A)^{-1}$, we obtain

$$\begin{cases}\bar{E}D_t^\alpha x(t)=\bar{A}x(t)+\bar{B}u(t),\\ y(t)=C(t)+Du(t)\end{cases} \tag{7}$$

where

$$\begin{aligned}&\bar{E}=[Ec-A]^{-1}E,\ \bar{A}=[Ec-A]^{-1}A\text{ and}\\ &\bar{B}=[Ec-A]^{-1}B.\end{aligned} \tag{8}$$

Lemma 1. The matrices $\bar{E}$ and $\bar{A}$ defined in system (8) satisfy the following equalities,

1. $\overline{AE}=\overline{EA}$, $\bar{A}^D\bar{E}=(\overline{EA})^D$, $\bar{E}^D\bar{A}=(\overline{AE})^D$ and $\bar{A}^D\bar{E}^D=\bar{E}^D\bar{A}^D$;
2. $\mathscr{N}(\bar{A})\cap\mathscr{N}(\bar{E})=\{0\}$;
3. $\bar{E}=T\begin{bmatrix}J&0\\0&N\end{bmatrix}T^{-1},\ \bar{E}^D=T\begin{pmatrix}J^{-1}&0\\0&0\end{pmatrix}T^{-1},\ \det T\neq0,$
$J\in\mathbb{R}^{n_1\times m_1}$, is nonsingular, $N\in\mathbb{R}^{n_2\times m_2}$ is nilpotent, $n_1+n_2=n$;
4. $(I-\overline{EE}^D)\overline{AA}^D=I-\overline{EE}^D$ and $(I-\overline{EE}^D)(\overline{EA}^D)^q=0$.

Remark 2. By using equation (5) and Lemma 1, it follows that

$$\begin{aligned}&\mathscr{N}\left(\bar{A}^q\right)\cap\mathscr{N}\left(\bar{E}^q\right)=\{0\}\text{ and}\\ &\mathscr{N}\left(\bar{A}^D\right)\cap\mathscr{N}\left(\bar{E}^D\right)=\{0\}.\end{aligned} \tag{9}$$

By using Drazin inverse method [24], solution of system (7) (and (1)) can be written as.

Theorem 2. The solution of (7) (and (1)) is given by

$$x(t)=\Phi_{\alpha,1}\left(\bar{E}^DA,t\right)v+\bar{E}^D\int_0^t\Phi_{\alpha,\alpha}\left(\bar{E}^DA,t-\tau\right)$$
$$\bar{B}u(\tau)d\tau+(\bar{E}\bar{E}^D-I)\sum_{k=0}^{q-1}(\bar{E}\bar{A}^D)^k\bar{A}^D\bar{B}u^{(k\alpha)}(t), \tag{10}$$

where $u^{(k\alpha)}=\frac{d^{k\alpha}u(t)}{dt^{k\alpha}}$ and the vector $v\in\mathbb{R}^n$ is arbitrary.

If the index q of matrix E is 1, then the solution (10) of system (7) have the following form

$$x(t)=\Phi_{\alpha,1}\left(\bar{E}^DA,t\right)v+\bar{E}^D\int_0^t\Phi_{\alpha,\alpha}\left(\bar{E}^DA,t-\tau\right)$$
$$\bar{B}u(\tau)d\tau+(\bar{E}\bar{E}^D-I)\bar{A}^D\bar{B}u(t). \tag{11}$$

3. Controllability

Definition 4. System (1) (and (7)) is called state controllable on $[0,\,t_f]$ with $t_f>0$, if given any state $x_0\in\mathbb{R}^n$, there exists an input signal $u(\cdot):[0,\,t_f]\to\mathbb{R}^m$ such that the corresponding solution of (1) satisfies $x(t_f)=0$.

In this section, we give necessary and sufficient conditions of controllability for (1) (and (7)).

Case 1: $\det E\neq 0$

We consider the following matrix

$$W_c[0,t_f]:=\int_0^{t_f}\Phi_{\alpha,\alpha}\left(E^{-1}A,t_f-\tau\right)\left(E^{-1}B\right)$$
$$\times\left(E^{-1}B\right)^*\Phi^*_{\alpha,\alpha}\left(E^{-1}A,t_f-\tau\right)d\tau, \tag{12}$$

where $*$ denotes the matrix transpose. Now we are formulating the results for controllability.

Theorem 3. System (1) is controllable on $[0,\,t_f]$ if and only if the controllability Gramian matrix (12) is non-singular.

Proof. Suppose that $W_c[0,\,t_f]$ is non-singular, then $W_c^{-1}[0,\,t_f]$ is well defined. For a given $x_0\in\mathbb{R}^n$, choose

$$u(t)=-(E^{-1}B)^*\Phi^*_{\alpha,\alpha}\left(E^{-1}A,t_f-t\right)$$
$$\times W_c^{-1}[0,t_f]\Phi_{\alpha,1}\left(E^{-1}A,t_f\right)x_0. \tag{13}$$

Obviously, the control input $u(\cdot)$ is continuous on $[0,\,t_f]$. Substituting $t=t_f$ in equation (2), we have

$$x(t_f)=\Phi_{\alpha,1}\left(E^{-1}A,t_f\right)x_0$$
$$+\int_0^{t_f}\Phi_{\alpha,\alpha}\left(E^{-1}A,t_f-\tau\right)\left(E^{-1}B\right)u(\tau)d\tau. \tag{14}$$

By using (13) in (14), we have

$$x(t_f)=\Phi_{\alpha,1}\left(E^{-1}A,t_f\right)x_0-\int_0^{t_f}\Phi_{\alpha,\alpha}\left(E^{-1}A,t_f-\tau\right)\left(E^{-1}B\right)$$
$$\times(E^{-1}B)^*\Phi^*_{\alpha,\alpha}\left(E^{-1}A,t_f-\tau\right)W_c^{-1}[0,t_f]\Phi_{\alpha,1}\left(E^{-1}A,t_f\right)x_0d\tau$$
$$=\Phi_{\alpha,1}\left(E^{-1}A,t_f\right)x_0-W_c[0,t_f]W_c^{-1}[0,t_f]\Phi_{\alpha,1}\left(E^{-1}A,t_f\right)x_0$$
$$=0.$$

Thus the system (1) is controllable on $[0,\,t_f]$.

On the other hand, if $W_c[0,\,t_f]$ is singular, without loss of generality, there exist a nonzero vector $z\in\mathbb{R}^n$ such that $z^*W_c[0,\,t_f]z=0$, that is

$$z^*\int_0^{t_f}\Phi_{\alpha,\alpha}\left(E^{-1}A,t_f-\tau\right)E^{-1}B$$
$$\times\left(\Phi_{\alpha,\alpha}\left(E^{-1}A,t_f-\tau\right)E^{-1}B\right)^*zd\tau=0.$$

It yields

$$z^*\Phi_{\alpha,\alpha}\left(E^{-1}A,t_f-\tau\right)E^{-1}B=0.$$

We consider $x_0=\Phi^{-1}_{\alpha,1}(E^{-1}A,\,t_f)z$. By the assumption, there exist an input u such that it steers x_0 to the origin in the interval $[0,\,t_f]$. It follows that

$$x(t_f)=\Phi_{\alpha,1}\left(E^{-1}A,t_f\right)\Phi^{-1}_{\alpha,1}\left(E^{-1}A,t_f\right)z$$
$$+\int_0^{t_f}\Phi_{\alpha,\alpha}\left(E^{-1}A,t_f-\tau\right)\left(E^{-1}B\right)u(\tau)d\tau$$
$$=z+\int_0^{t_f}\Phi_{\alpha,\alpha}\left(E^{-1}A,t_f-\tau\right)\left(E^{-1}B\right)u(\tau)d\tau=0$$

Then,

$$z^*z+\int_0^{t_f}z^*\Phi_{\alpha,\alpha}\left(E^{-1}A,t_f-\tau\right)\left(E^{-1}B\right)u(\tau)d\tau=0.$$

But the second term is zero, leading to the conclusion that $z^*z=0$. This is a contradiction to $z\neq 0$. Thus $W_c[0,\,t_f]$ is non-singular. This completes the proof. □

Theorem 4. System (1) is controllable on $[0,\,t_f]$ if and only if

$$rankQ_c=n,$$

where

$$Q_c=\left[(E^{-1}B)|(E^{-1}A)(E^{-1}B)|\ldots|(E^{-1}A)^{n-1}(E^{-1}B)\right].$$

Proof. By Cayley-Hamilton's Theorem, we can write

$$\Phi_{\alpha,\alpha}\left(E^{-1}A,t\right)=\sum_{i=0}^{n-1}c_i(t)(E^{-1}A)^i,$$

where $c_i(t)$ is the polynomial in t. Thus, it follows

$$\Phi_{\alpha,\alpha}\left(E^{-1}A,t_f-\tau\right)=\sum_{i=0}^{n-1}c_i(t_f-\tau)(E^{-1}A)^i. \tag{15}$$

By using the relation (15), solution of the system (1) has the form

$$x(t_f)=\sum_{i=0}^{n-1}\int_0^{t_f}c_i(t_f-\tau)(E^{-1}A)^i\left(E^{-1}B\right)u(\tau)d\tau$$
$$+\Phi_{\alpha,1}\left(E^{-1}A,t_f\right)x_0.$$

It follows that

$$x(t_f) - \Phi_{\alpha,1}\left(E^{-1}A, t_f\right)x_0 = \sum_{i=0}^{n-1}(E^{-1}A)^i E^{-1}B \times \int_0^t c_i(t-\tau)u(\tau)d\tau.$$
$$= \left[(E^{-1}B)|(E^{-1}A)(E^{-1}B)|\cdots|(E^{-1}A)^{n-1}(E^{-1}B)\right] \times \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{n-1} \end{pmatrix}, \tag{16}$$

where

$$d_i = \int_0^{t_f} c_i(t_f - \tau)u(\tau)d\tau$$

for $i = 0, 1, 2, \cdots, n-1$. To have a unique solution of $u(t)$, the necessary and sufficient condition is clearly that

$$rank Q_c = n.$$

This completes the proof. □

Case 2: $\det E = 0$

If the matrix E is not nonsingular, then consider the solution in Drazin inverse setting as follows

$$x(t) = \Phi_{\alpha,1}\left(\bar{E}^D A, t\right)\upsilon + \int_0^t \Phi_{\alpha,\alpha}\left(\bar{E}^D A, t-\tau\right) \bar{E}^D\bar{B}u(\tau)d\tau + (\bar{E}\bar{E}^D - I)\sum_{k=0}^{q-1}(\bar{E}\bar{A}^D)^k \bar{A}^D\bar{B}u^{(k\alpha)}(t).$$

Now let us define controllability Gramian matrix for the following case

$$W_c[0,t_f] = \int_0^{t_f} \bar{E}^D\Phi_{\alpha,\alpha}\left(\bar{E}^D A, t_f-\tau\right)\bar{E}^D\bar{B} \times \left(\bar{E}^D\Phi_{\alpha,\alpha}\left(\bar{E}^D A, t_f-\tau\right)\bar{E}^D\bar{B}\right)^* d\tau. \tag{17}$$

Theorem 5. System (7) is controllable on $[0, t_f]$ if and only if the controllability Gramian matrix (17) is non-singular.

Proof. If the index q of matrix E is 1, then the solution (10) of system (7) has the following form

$$x(t) = \Phi_{\alpha,1}\left(\bar{E}^D A, t\right)\upsilon + \int_0^t \Phi_{\alpha,\alpha}\left(\bar{E}^D A, t-\tau\right) \bar{E}^D\bar{B}u(\tau)d\tau + (\bar{E}\bar{E}^D - I)\bar{A}^D\bar{B}u(t). \tag{18}$$

Premultiplying both sides of equation (18) by $\bar{E}^D$, and using the properties of Drazin inverse, we have

$$\bar{E}^D x(t) = \bar{E}^D\Phi_{\alpha,1}\left(\bar{E}^D A, t\right)\upsilon + \int_0^t \bar{E}^D\Phi_{\alpha,\alpha}\left(\bar{E}^D A, t-\tau\right)\bar{E}^D\bar{B}u(\tau)d\tau. \tag{19}$$

If $W_c[0, t_f]$ is non-singular, then $W_c^{-1}[0, t_f]$ is well defined. For a given υ, we define the control function as

$$u(t) = -\left(\bar{E}^D\Phi_{\alpha,\alpha}\left(\bar{E}^D A, t_f - t\right)\bar{E}^D\bar{B}\right)^* W_c^{-1}[0,t_f] \times \bar{E}^D\Phi_{\alpha,1}\left(\bar{E}^D A, t_f\right)\upsilon.$$

then at $t = t_f$ (19) becomes

$$\bar{E}^D x(t_f) = \bar{E}^D\Phi_{\alpha,1}\left(\bar{E}^D A, t_f\right)\upsilon - W_c[0,t_f]W_c^{-1}[0,t_f] \times (\bar{E}^D\Phi_{\alpha,1}\left(\bar{E}^D A, t_f\right)\upsilon).$$

It follows that $\bar{E}^D x(t_f) = 0$. Premultiplying by $\bar{A}^D$, we have

$$\bar{A}^D\bar{E}^D x(t_f) = 0. \tag{20}$$

Since $\bar{A}^D\bar{E}^D = \bar{E}^D\bar{A}^D$, it implies that $\bar{E}^D x(t_f) \in \ker(\bar{A}^D)$ and $\bar{A}^D x(t_f) \in \ker(\bar{E}^D)$. By using equation (9), we obtain $x(t_f) = 0$. Thus the system is controllable on $[0, t_f]$.

On the other hand, if $W_c[0, t_f]$ is singular, without loss of generality, there exist a nonzero vector z such that $z^* W_c[0, t_f]z = 0$, that is,

$$\int_0^{t_f} z^*(\bar{E}^D\Phi_{\alpha,\alpha}\left(\bar{E}^D A, t_f-\tau\right)\bar{E}^D\bar{B}) \times(\bar{E}^D\Phi_{\alpha,\alpha}\left(\bar{E}^D A, t_f-\tau\right)\bar{E}^D\bar{B})^* z d\tau = 0,$$

which implies that

$$z^*(\bar{E}^D\Phi_{\alpha,\alpha}\left(\bar{E}^D A, t_f-\tau\right)\bar{E}^D\bar{B}) = 0. \tag{21}$$

Since system (7) is controllable. For $\upsilon = \Phi_{\alpha,1}^{-1}(\bar{E}^D A, t_f)z$ (19) yields that

$$\bar{E}^D x(t_f) = 0 = \bar{E}^D\Phi_{\alpha,1}\left(\bar{E}^D A, t_f\right)\Phi_{\alpha,1}^{-1}\left(\bar{E}^D A, t_f\right)z + \bar{E}^D\int_0^{t_f}\Phi_{\alpha,\alpha}\left(\bar{E}^D A, t_f-\tau\right)\bar{E}^D\bar{B}u(\tau)d\tau$$
$$= \bar{E}^D z + \bar{E}^D\int_0^{t_f}\Phi_{\alpha,\alpha}\left(\bar{E}^D A, t_f-\tau\right)\bar{E}^D\bar{B}u(\tau)d\tau$$

Premultiplying and using (21), we obtain that $\bar{E}^D\|z\|^2 = 0$. Since $\bar{A}^D\bar{E}^D = \bar{E}^D\bar{A}^D$, it implies that $\bar{E}^D\|z\|^2 \in \ker(\bar{A}^D)$ and $\bar{A}^D\|z\|^2 \in \ker(\bar{E}^D)$. By using (9), we obtain $\|z\|^2 = 0$, which leads to a contradiction that, that is $z = 0$.

For the matrix index $q = 2$, equation (10) becomes

$$x(t) = \Phi_{\alpha,1}\left(\bar{E}^D A, t\right)\upsilon + \int_0^t \Phi_{\alpha,\alpha}\left(\bar{E}^D A, t-\tau\right)\bar{E}^D\bar{B}u(\tau)d\tau + (\bar{E}\bar{E}^D - I)\sum_{k=0}^{1}(\bar{E}\bar{A}^D)^k\bar{A}^D\bar{B}u^{(k\alpha)}(t),$$

It follows that

$$x(t) = \Phi_{\alpha,1}\left(\bar{E}^D A, t\right)\upsilon + \int_0^t \Phi_{\alpha,\alpha}\left(\bar{E}^D A, t-\tau\right)\bar{E}^D\bar{B}u(\tau)d\tau + (\bar{E}\bar{E}^D - I)\left[\bar{A}^D\bar{B}u(t) + (\bar{E}\bar{A}^D)\bar{A}^D\bar{B}u^{(\alpha)}(t)\right]. \tag{22}$$

Premultiplying $\bar{E}^D$ both sides of (22), we obtain equation (19). Therefore, the proof goes similar as $q = 1$. This completes the proof. □

Theorem 6. System (7) is controllable on $[0, t_f]$ if and only if

$$rank Q_c = n,$$

where

$$Q_c = \left\{ \bar{E}^D \left[\left(\bar{E}^D \bar{B} | (\bar{E}\bar{A}^D)\bar{E}^D\bar{B} | ... | (\bar{E}\bar{A}^D)^{n-1}\bar{E}^D\bar{B} \right) \right] \right\}$$

Proof. Suppose that the system (7) is controllable on $[0, t_f]$. If the rank condition does not hold, then there exists $z \in \mathbb{R}^n$ with $z \neq 0$ such that

$$z^\star \bar{E}^D \left(\bar{E}\bar{A}^D\right)^j \bar{E}^D\bar{B} = 0, \; j = 0, 1, \cdots n-1. \tag{23}$$

By using relation (15) in controllable Gramian matrix (17), it follows that

$$\begin{aligned} z^\star W_c\left[0, t_f\right] &= \int_0^{t_f} \bar{E}^D \Phi_{\alpha,\alpha}\left(\bar{E}^D A, t_f - \tau\right)\bar{E}^D\bar{B} \\ &\quad \times \left(\bar{E}^D \Phi_{\alpha,\alpha}\left(\bar{E}^D A, t_f - \tau\right)\bar{E}^D\bar{B}\right)^* d\tau \\ &\quad \int_0^{t_f} \bar{E}^D \sum_{j=0}^{n-1} \left(\bar{E}\bar{A}^D\right)^j \bar{E}^D\bar{B} d_j\left(t_f - \tau\right) \\ &\quad \times \left(\bar{E}^D \Phi_{\alpha,\alpha}\left(\bar{E}^D A, t_f - \tau\right)\bar{E}^D\bar{B}\right)^* d\tau \\ &= 0. \end{aligned}$$

It follows that $rank W_c[0, t_f] < n$. This contradicts conclusion of Theorem 5 and therefore, we can conclude that $rank Q_c = n$.

Conversely, suppose that $rank Q_c = n$. If the system (7) is not controllable on $[0, t_f]$, then the controllable Gramian matrix (17) is not invertible. Thus there exists $z \in \mathbb{R}^n$ with $z \neq 0$ such that

$$z^\star \bar{E}^D \Phi_{\alpha,\alpha}\left(\bar{E}^D A, t_f - \tau\right)\bar{E}^D\bar{B} = 0. \tag{24}$$

In particular, for $\tau = t_f$, it follows that $z^\star \bar{E}^D(\bar{E}^D\bar{B}) = 0$.

Taking Caputo's fractional derivative for the equation (24), from (4) we have

$$z^\star \bar{E}^D\left(\bar{E}^D A\right)\Phi_{\alpha,\alpha}\left(\bar{E}^D A, t_f - \tau\right)\bar{E}^D\bar{B} = 0.$$

For $\tau = t_f$, we have $z^\star \bar{E}^D(\bar{E}^D A)\bar{E}^D\bar{B} = 0$. Repeting this argument $n-1$ times, we have

$$z^\star \bar{E}^D\left(\bar{E}^D A\right)^j \bar{E}^D\bar{B} = 0 \text{ for } j = 0, 1, \cdots n-1.$$

Therefore

$$z^\star \bar{E}^D \left(\begin{array}{l} \bar{E}^D\bar{B} | \left(\bar{E}^D A\right)\bar{E}^D\bar{B} | \left(\bar{E}^D A\right)^2 \bar{E}^D\bar{B} | \ldots \\ \left(\bar{E}^D A\right)^{n-1}\bar{E}^D\bar{B} \end{array} \right) = 0,$$

which implies that the rank condition (6) fails. This contradiction proves that the system (7) is controllable on $[0, t_f]$. This completes the proof. □

4. Observability

In this section we establish necessary and sufficient conditions of observability for the systems (1) and (7).

Definition 5. Systems (1) (and (7) are called state observable on $[0, t_f]$ if any initial state $x(0) = x_0 \in \mathbb{R}^n$ is uniquely determined by the corresponding system input $u(t)$ and system output $y(t)$, for $t \in [0, t_f]$; $t_f \in [0, T]$.

Case 1: $\det E \neq 0$

Theorem 7. The system (1) is observable on $[0, t_f]$ if and only if the observability Gramian matrix

$$W_o[0, t_f] := \int_0^{t_f} \Phi^*_{\alpha,1}\left(E^{-1}A, t\right) C^* C \Phi_{\alpha,1}\left(E^{-1}A, t\right) dt$$

is nonsingular for some $t_f > 0$.

Proof. We know that

$$\begin{aligned} x(t) = &\int_0^t \Phi_{\alpha,\alpha}\left(E^{-1}A, t-\tau\right)E^{-1}Bu(\tau)d\tau \\ &+ \Phi_{\alpha,1}\left(E^{-1}A, t\right)x_0. \end{aligned}$$

The output will become

$$\begin{aligned} y(t) = C &\int_0^t \Phi_{\alpha,\alpha}\left(E^{-1}A, t-\tau\right)E^{-1}Bu(\tau)d\tau \\ &+ C\Phi_{\alpha,1}\left(E^{-1}A, t\right)x_0 + Du(t). \end{aligned}$$

We define

$$\bar{y}(t) = y(t) - C\int_0^t \Phi_{\alpha,\alpha}\left(E^{-1}A, t-\tau\right)E^{-1}Bu(\tau)d\tau - Du(t).$$

Then

$$\bar{y}(t) = C\Phi_{\alpha,1}\left(E^{-1}A, t\right)x_0.$$

It is obvious that observability of system (1) is equivalent to the estimation of x_0 from $y(t)$.

Since $\bar{y}(t)$ and x_0 are arbitrary, this returns in the estimation of x_0 from $y(t)$ given by

$$y(t) = C\Phi_{\alpha,1}\left(E^{-1}A, t\right)x_0$$

as $u(t) \equiv 0$.

If $W_o[0, t_f]$ is nonsingular then $W_o^{-1}[0, t_f]$ is well defined. Hence for arbitrary $y(t)$, for $t_f > 0$, we have the following expression

$$\begin{aligned} &W_o^{-1}[0, t_f]\int_0^{t_f} \Phi^*_{\alpha,1}\left(E^{-1}A, t\right)C^* y(t)dt \\ &= W_o^{-1}[0, t_f]\int_0^{t_f} \Phi^*_{\alpha,1}\left(E^{-1}A, t\right)C^* C\Phi_{\alpha,1}\left(E^{-1}A, t\right)x_0 dt \\ &= W_o^{-1}[0, t_f]W_o[0, t_f]x_0. \end{aligned}$$

It follows that

$$W_o^{-1}[0,t_f]\int_0^{t_f}\Phi_{\alpha,1}^*\left(E^{-1}A,t\right)C^*y(t)dt = x_0. \qquad (25)$$

The left side of (25) depends on $y(t) \in [0, t_f]$, and (25) is a linear algebraic equation of x_0. Since $W_o[0, t_f]$ is invertible, then the initial state $x(0) = x_0$ is uniquely determined by the corresponding system output $y(t)$, for $t \in [0, t_f]$.

Conversely, if the Gramian matrix $W_o[0, t_f]$ is singular for some $t_f > 0$, there exists a non zero x_α such that

$$x_\alpha^* W_o[0,t_f]x_\alpha = 0.$$

Choosing $x_\alpha = x_0$, we have

$$\int_0^{t_f} y^*(t)y(t)dt = x_0^*\int_0^{t_f}\Phi_{\alpha,1}^*\left(E^{-1}A,t\right)C^*C\Phi_{\alpha,1}\left(E^{-1}A,t\right)x_0 dt = 0.$$

It follows that

$$\int_0^{t_f}\|y(t)\|^2dt = 0.$$

Therefore,

$$y(t) = C\Phi_{\alpha,1}\left(E^{-1}A,t\right)x_0 = 0.$$

Since the system is observable, it implies that $x_0 = 0$. This is a contradiction, hence $W_o[0, t_f]$ is nonsingular. □

Theorem 8. The system (1) is observable on $[0, t_f]$ if and only if

$$rankQ_o = rank\begin{pmatrix} C \\ C(E^{-1}A) \\ \vdots \\ C(E^{-1}A)^{n-1}\end{pmatrix} = n.$$

Proof. From Theorem 7, we have

$$y(t) = C\Phi_{\alpha,1}\left(E^{-1}A,t\right)x_0$$

x_0 is uniquely determined by $y(t)$ if and only if $C\Phi_{\alpha,1}(E^{-1}A, t)$ is nonsingular. Using Cayley Hamilton theorem, we have

$$C\Phi_{\alpha,1}\left(E^{-1}A,t\right) = C\sum_{i=0}^{n-1}\beta_i(t)(E^{-1}A)^i.$$

It follows that

$$C\Phi_{\alpha,1}\left(E^{-1}A,t\right) = \textstyle\sum_{i=0}^{n-1}\beta_i(t_f)C(E^{-1}A)^i$$
$$= \begin{pmatrix}\beta_0(t_f) & \beta_1(t_f) & \dots & \beta_{n-1}(t_f)\end{pmatrix}\begin{pmatrix} C \\ C(E^{-1}A) \\ \vdots \\ C(E^{-1}A)^{n-1}\end{pmatrix}$$

$C\Phi_{\alpha,1}(E^{-1}A, t)$ is nonsingular if and only if

$$rank\begin{pmatrix} C \\ C(E^{-1}A) \\ \vdots \\ C(E^{-1}A)^{n-1}\end{pmatrix} = n.$$

Thus, the system (1) is observable on $[0, t_f]$ if and only if

$$rankQ_o = n. \qquad □$$

Case 2: $\det E = 0$

Theorem 9. The system (9) is observable on $[0, t_f]$ if and only if the observability Gramian matrix

$$W_o[0,t_f] := \int_0^{t_f}\Phi_{\alpha,1}^*\left(\bar{E}^DA,t\right)C^*C\Phi_{\alpha,1}\left(\bar{E}^DA,t\right)dt$$

is nonsingular for some $t_f > 0$.

Proof. Let us consider the solution of (7)

$$x(t) = \Phi_{\alpha,1}\left(\bar{E}^DA,t\right)\upsilon + \int_0^t\Phi_{\alpha,\alpha}\left(\bar{E}^DA,t-\tau\right)\bar{E}^D\bar{B}u(\tau)d\tau + (\bar{E}\bar{E}^D - I)\textstyle\sum_{k=0}^{q-1}(\bar{E}\bar{A}^D)^k\bar{A}^D\bar{B}u^{(k\alpha)}(t)$$

and the corresponding output is as follows

$$y(t) = C\Big[\Phi_{\alpha,1}\left(\bar{E}^DA,t\right)\upsilon + \int_0^t\Phi_{\alpha,\alpha}\left(\bar{E}^DA,t-\tau\right)\bar{E}^D\bar{B}u(\tau)d\tau + (\bar{E}\bar{E}^D - I)\textstyle\sum_{k=0}^{q-1}(\bar{E}\bar{A}^D)^k\bar{A}^D\bar{B}u^{(k\alpha)}(t)\Big] + Du(t).$$

Let us define

$$\bar{y}(t) = y(t) - C\int_0^t\Phi_{\alpha,\alpha}\left(\bar{E}^DA,t-\tau\right)\bar{E}^D\bar{B}u(\tau)d\tau + C\left[(\bar{E}\bar{E}^D - I)\sum_{k=0}^{q-1}(\bar{E}\bar{A}^D)^k\bar{A}^D\bar{B}u^{(k\alpha)}(t)\right] - Du(t).$$

Then it follows that

$$\bar{y}(t) = C\Phi_{\alpha,1}\left(\bar{E}^DA,t\right)\upsilon.$$

It is obvious that observability of system (7) is equivalent to the estimation of x_0 from $\bar{y}(t)$. Since $\bar{y}(t)$ and x_0 are arbitrary, this returns in the estimation of x_0 from $y(t)$ given

$$y(t) = C\Phi_{\alpha,1}\left(\bar{E}^DA,t\right)\upsilon$$

as $u(t) \equiv 0$.

If $W_o[0, t_f]$ is nonsingular then $W_o^{-1}[0, t_f]$ is well defined. Hence for arbitrary $y(t)$, for $t_f > 0$, we can construct

$$\begin{aligned} & W_o^{-1}[0,t_f]\int_0^{t_f}(\Phi_{\alpha,1}\left(\bar{E}^D A,t\right))^* C^* y(t)dt \\ & = W_o^{-1}[0,t_f]\int_0^{t_f}(\Phi_{\alpha,1}\left(\bar{E}^D A,t\right))^* C^* C\Phi_{\alpha,1}\left(\bar{E}^D A,t\right)dt\upsilon \\ & = W_o^{-1}[0,t_f]W_o[0,t_f]\upsilon = \upsilon \end{aligned}$$

therefore, □

$$W_o^{-1}[0,t_f]\int_0^{t_f}(\Phi_{\alpha,1}\left(\bar{E}^D A,t\right))^* C^* y(t)dt = \upsilon. \tag{26}$$

The left side of (26) depends on $y(\cdot)$, and (26) is a linear algebraic equation of υ. Since $W_o[0, t_f]$ is invertible, then the initial state υ is uniquely determined by the corresponding system output $y(t)$, for $t \in [0, t_f]$.

On the other hand, if the Gramian matrix $W_o[0, t_f]$ is singular for some $t_f > 0$, there exists a non zero x_α such that

$$x_\alpha^* W_o[0,t_f]x_\alpha = 0.$$

Choose $x_\alpha = \upsilon$, then we have

$$\begin{aligned} \int_0^{t_f}(y(t))^* y(t)dt = \upsilon^* \int_0^{t_f}(\Phi_{\alpha,1}\left(\bar{E}^D A,t\right))^* C^* \\ C\Phi_{\alpha,1}\left(\bar{E}^D A,t\right)dt\upsilon \\ = \upsilon^* W_o[0,t_f]\upsilon = 0. \end{aligned}$$

This implies that

$$\int_0^{t_f}\|y(t)\|^2 dt = 0.$$

Since the system is observable, therefore, it follows that $v = 0$. This is a contradiction, hence $W_o[0, t_f]$ is nonsingular.

Theorem 10. The system (7) is observable on $[0, t_f]$ if and only if

$$rank\begin{pmatrix} C \\ C(\bar{E}\bar{A}^D) \\ \vdots \\ C(\bar{E}\bar{A}^D)^{n-1} \end{pmatrix} = n.$$

Proof. Proof steps are the same as in Theorem 6. □

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