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FRACTIONAL OPERATORS AND THEIR COMMUTATORS ON GENERALIZED ORLICZ SPACES

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Abstract. In this paper we examine boundedness of fractional maximal operator. The main focus is on commutators and maximal commutators on generalized Orlicz spaces (also known as Musielak–Orlicz spaces) for fractional maximal functions and Riesz potentials. We prove their boundedness between generalized Orlicz spaces and give a characterization for functions of bounded mean oscillation.

Keywords: maximal operator, commutator, fractional operator, generalized Orlicz, Musielak–Orlicz.

Mathematics Subject Classification: 46E30, 42B25, 42B35.

1. INTRODUCTION

The Hardy–Littlewood maximal operator is one of the most central operators in modern harmonic analysis and theory of partial differential equations. The aim of this paper is to show boundedness result for fractional maximal operators and their commutator variants in generalized Orlicz spaces (also known as Musielak–Orlicz spaces), such as

$$\|[M, b]f\|_{L^\psi(\mathbb{R}^n)} \leqslant C\|f\|_{L^\varphi(\mathbb{R}^n)}$$

where M is a (fractional) maximal function and b belongs to space of bounded mean oscillation $\mathrm{BMO}(\mathbb{R}^n)$ (see Section 2). These type of boundedness properties for commutators have been applied to partial differential equations with discontinuous coefficients to obtain existence and regularity results of solutions, see for instance [6, 20]. On the other hand, a systematic study of various variational integrals or partial differential equations under non-standard growth conditions has often started by proving boundedness of the maximal operator in a related function space, see for instance [10, 14, 23].

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Generalized Orlicz spaces (also known as Musielak–Orlicz spaces) started to gain attraction in study of variational integrals and partial differential equations over the previous decade, see for example [7, 8, 13, 23, 29]. These L^{φ}-spaces include many of the widely studied frameworks under (non-)uniformly elliptic problems such as standard L^p-spaces for $\varphi(x,t) = t^p$, Orlicz-spaces for $\varphi(x,t) = \varphi(t)$, variable exponent spaces for $\varphi(x,t) = t^{p(x)}$ and double phase spaces for $\varphi(x,t) = t^p + a(x)t^q$. They all fall under the framework of so-called (p,q)-growth conditions dating back to Marcellini [28], see also [17]. Developing theory in this general setting allows us to capture many delicate phenomena not present in L^p-scale and describe them with improved accuracy using assumptions intrinsic to the problem at hand.

The fractional variant of the maximal function introduced in [30] and its boundedness properties have also been intensively studied. This version is also related to fractional integrals such as the Riesz potential. Cianchi [9] first proved necessary and sufficient conditions for boundedness of fractional maximal operator between different Orlicz spaces. See also recent work of Musil [31]. In these articles, the condition between Orlicz functions was described by a concept of domination. In [18] this condition was proven to be equivalent to presentation used also in this article, i.e., a relation between the inverses of the (generalized) Orlicz functions. The boundedness results have also been investigated in the variable exponent case, see [5, 27]. The relation between inverses and structural assumptions of the generalized Orlicz function φ required for the boundedness of the maximal function are collected in the following.

Assumptions 1.1. Let $1 < p \leqslant q < \infty$, $\frac{\alpha}{n} = \frac{1}{p} - \frac{1}{q}$ and $\varphi, \psi \in \Phi_w(\mathbb{R}^n)$ be such that

$$\varphi^{-1}(x,t) \approx t^{\frac{\alpha}{n}} \psi^{-1}(x,t). \tag{1.1}$$

Let $r \in (\frac{\alpha}{n}, \frac{1}{p}]$. Assume that φ satisfies (A0), (A1), (A2), $(\mathrm{aInc})_p$ and $(\mathrm{aDec})_{1/r}$.

Assumptions on φ are explained in Section 2. They are widely used in the functional analysis of the generalised Orlicz spaces and the calculus of variations under general growth conditions. See [22] for discussion on their consequences. Note that the assumptions on φ guarantee the existence of described ψ, see [22, Lemma 5.2.3]. We could also state the assumptions starting from ψ satisfying (A0), (A1), (A2), $(\mathrm{aInc})_q$ and $(\mathrm{aDec})_{1/(r-\alpha/n)}$.

In this paper we study boundedness of fractional operators and their commutator variants in the generalized Orlicz spaces. We prove that similar structural assumptions required for boundedness of Hardy–Littlewood maximal operator in generalized Orlicz spaces also imply boundedness of the fractional variant. For definitions of operators M_α, $M_{b,\alpha}$, $[M_\alpha, b]$, I_α, $I_{b,\alpha}$ and $[I_\alpha, b]$ see Section 2. Our first result reads as follows.

Theorem 1.2. *Under Assumptions 1.1 the fractional maximal operator* $M_\alpha : L^{\varphi}(\mathbb{R}^n) \to L^{\psi}(\mathbb{R}^n)$ *is bounded.*

The proof of this result is direct and does not involve extrapolation arguments. We present it in Section 3. Theorem 1.2 is closely related to the boundedness of the Riesz potential, which has already been studied in [21]. Here we work in a slightly different framework for defining the target space and record that these results do not require the generating Φ-functions to be N-functions.

We are in the position to pass to a widely studied topic in the harmonic analysis, that is a boundedness of commutators $[T,b]f = T(bf) - bTf$ of various operators T and functions b. One of the most influential paper regarding commutators in harmonic analysis is due to Coifman, Rochberg and Weiss [11]. They showed that $\|[T,b]f\|_{L^p(\mathbb{R}^n)} \leqslant C\|b\|_*\|f\|_{L^p(\mathbb{R}^n)}$, where $\|\cdot\|_*$ is the BMO-seminorm (see Section 2). Additionally, they gave a characterization of the space $\mathrm{BMO}(\mathbb{R}^n)$ in terms of boundedness of commutator $[T,b]$ in $L^p(\mathbb{R}^n)$ when T is a Calderón–Zygmund singular integral operator. Similar characterization of Lipschitz spaces was given by Janson [26]: $[T,b]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1 < p < q < \infty$ if and only if $b \in \Lambda_\beta$ and $\beta = n\left(\frac{1}{p} - \frac{1}{q}\right)$. These types of commutator boundedness results have also been used successfully for research regarding products of functions from real Hardy space $H^1(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$ [4] and giving new characterization of other well known function spaces such as Campanato spaces [33].

When the operator is a maximal function instead of the Calderón–Zygmund operator, the boundedness of commutators was studied in [3] and for fractional maximal function in [35]. These results have had their generalizations from standard L^p-spaces to their non-standard counterparts such as the variable exponent Lebesgue and Orlicz spaces [18, 36, 37] and more refined Morrey type spaces [2, 19]. To achieve these results in the generalized Orlicz spaces, we first study commutators related to the Riesz potential in Section 4 with the help of the sharp maximal function.

Theorem 1.3. *Under Assumptions 1.1, we have:*

1. $\|[I_\alpha, b]f\|_{L^\psi(\mathbb{R}^n)} \leqslant C\|b\|_*\|f\|_{L^\varphi(\mathbb{R}^n)}$,
2. $\|I_{b,\alpha}f\|_{L^\psi(\mathbb{R}^n)} \leqslant C\|b\|_*\|f\|_{L^\varphi(\mathbb{R}^n)}$

for every $f \in L^\varphi(\mathbb{R}^n)$ *and* $b \in \mathrm{BMO}(\mathbb{R}^n)$.

Theorem 1.4. *Under Assumptions 1.1, the operator* $M_{b,\alpha} : L^\varphi(\mathbb{R}^n) \to L^\psi(\mathbb{R}^n)$ *is bounded and*

$$\|M_{b,\alpha}f\|_\psi \leqslant C\|b\|_*\|f\|_\varphi$$

if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$.

We use the boundedness results from the two theorems above to give a characterization for functions of bounded mean oscillation $\mathrm{BMO}(\mathbb{R}^n)$.

Theorem 1.5. *Suppose Assumptions 1.1 hold and* $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. *Then the following are equivalent:*

1. $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $b^- \in L^\infty(\mathbb{R}^n)$,
2. *the commutator* $[M_\alpha, b] : L^\varphi(\mathbb{R}^n) \to L^\psi(\mathbb{R}^n)$ *is bounded,*
3. *we have*

$$\sup_Q \frac{\|(b - |Q|^{-\alpha/n} M_{\alpha,Q} b)\chi_Q\|_{L^\eta(\mathbb{R}^n)}}{\|\chi_Q\|_{L^\eta(\mathbb{R}^n)}} < \infty \tag{1.2}$$

   for some $\eta \in \Phi_w(\mathbb{R}^n)$ *for which* $M : L^\eta(\mathbb{R}^n) \to L^\eta(\mathbb{R}^n)$ *is bounded,*
4. *condition* (1.2) *holds for all* $\eta \in \Phi_w(\mathbb{R}^n)$ *satisfying* (A0), (A1), (A2), (aInc) *and* (aDec).

To our best knowledge Theorems 1.3-1.5 are new already in many of the special cases such as double phase and its variants.

Due to the structure of the generalized Orlicz spaces, the connection between boundedness of the maximal operator in a generalized Orlicz space and its dual space is not yet fully understood and therefore not possible to utilize here. For this reason, we are forced to state our assumptions in a technical way given in Definition 2.3 instead of just the boundedness of the maximal function. However, some partial results to alleviate this have been obtained in [15]. Additionally, the assumption linking the domain and target space is known to be sharp in Orlicz spaces, but the non-autonomous nature of generalized Orlicz spaces leaves this as an open question. To the best of our knowledge, this is the case already in variable exponent spaces.

2. PRELIMINARIES

By L-almost increasing we mean that a function satisfies the inequality $f(s) \leqslant Lf(t)$ for all $s < t$ and some constant $L \geqslant 1$ and L-almost decreasing is defined analogously. If there exists a constant C such that $f(x) \leqslant Cg(x)$ for almost every x, then we write $f \lesssim g$. Additionally, we write $f \approx g$ if $f \lesssim g \lesssim f$ holds and the constant C may vary from line to line. In the case of a measurable set A, we denote its characteristic function by χ_A and its Lebesgue measure by $|A|$. By $\fint_A f(x)\,dx$ we denote the integral average $\frac{1}{|A|}\int_A f(x)\,dx$. We reserve Q to mean any cube in $\mathbb{R}^n$ with sides parallel to coordinate axes and specify it to have a center x and side-length $2r$, denoted as $Q(x,r)$, when needed. By p' we mean the Hölder conjugate $\frac{p}{p-1}$ of p.

2.1. GENERALIZED Φ-FUNCTIONS

Definition 2.1. We say that $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty]$ is a *weak* Φ*-function*, and write $\varphi \in \Phi_w(\mathbb{R}^n)$, if the following conditions hold:

- for any function $f : \mathbb{R}^n \to \mathbb{R}$ the function $x \mapsto \varphi(x,|f(x)|)$ is measurable and for every $x \in \mathbb{R}^n$ the function $t \mapsto \varphi(x,t)$ is non-decreasing,
- $\varphi(x,0) = \lim\limits_{t\to 0^+} \varphi(x,t) = 0$ and $\lim\limits_{t\to\infty} \varphi(x,t) = \infty$ for every $x \in \mathbb{R}^n$,
- $t \mapsto \frac{\varphi(x,t)}{t}$ is L-almost increasing with L independent of x.

If $\varphi \in \Phi_w(\mathbb{R}^n)$ and additionally $t \mapsto \varphi(x,t)$ is convex and left-continuous for almost every x, then φ is a *convex* Φ*-function*, and we write $\varphi \in \Phi_{\mathrm{c}}(\mathbb{R}^n)$. If $\varphi \in \Phi_w(\mathbb{R}^n)$ and additionally $t \mapsto \varphi(x,t)$ is convex and continuous in the topology of $[0,\infty]$ for almost every x, then φ is a *strong* Φ*-function*, and we write $\varphi \in \Phi_{\mathrm{s}}(\mathbb{R}^n)$. If φ does not depend on the x variable, i.e. is an Orlicz function, we denote $\varphi \in \Phi_w$.

A function $\varphi \in \Phi_c(\mathbb{R}^n)$ is called N-function if for almost every x we have $\varphi(x,t) \in (0,\infty)$ for all $t > 0$, $\lim_{t\to 0^+} \frac{\varphi(x,t)}{t} = 0$ and $\lim_{t\to\infty} \frac{\varphi(x,t)}{t} = \infty$. If φ is an N-function, then $t \mapsto \varphi(x,t)$ is continuous for almost every $x \in \mathbb{R}^n$, since it is finite and convex, and thus a strong Φ-function.

Two functions φ and ψ are *equivalent*, $\varphi \simeq \psi$, if there exists $L \geqslant 1$ such that $\psi\left(x, \frac{t}{L}\right) \leqslant \varphi(x,t) \leqslant \psi(x, Lt)$ for every $x \in \mathbb{R}^n$ and every $t > 0$. We say two functions φ and ψ are *weakly equivalent*, $\varphi \sim \psi$, if there exist $L \geqslant 1$ and $h \in L^1(\mathbb{R}^n)$ such that $\psi(x,t) \leqslant \varphi(x, Lt) + h(x)$ and $\varphi(x,t) \leqslant \psi(x,Lt) + h(x)$ for all $t \geqslant 0$ and almost all $x \in \mathbb{R}^n$. Weakly equivalent Φ-functions give rise to the same space with comparable norms.

By $\varphi^{-1}(x,t)$ we mean the generalized inverse defined by

$$\varphi^{-1}(x,t) := \inf\{\tau \in \mathbb{R} : \varphi(x,\tau) \geqslant t\}.$$

Functions φ and ψ are equivalent if and only if $\varphi^{-1}(x,t) \approx \psi^{-1}(x,t)$.

For $\varphi \in \Phi_w(\mathbb{R}^n)$ we define the conjugate Φ-function $\varphi^* \in \Phi_w(\mathbb{R}^n)$ by

$$\varphi^*(x,t) := \sup_{s>0}\{st - \varphi(x,s)\}.$$

We collect two results on how the generalized inverse behaves. For the proofs see Lemma 2.3.3 and Theorem 2.4.8 in [22].

Lemma 2.2. *Let $\varphi \in \Phi_w(\mathbb{R}^n)$. Then*

a) $\varphi(x, \varphi^{-1}(x,t)) = t$ *when* $\varphi \in \Phi_s(\mathbb{R}^n)$,
b) $\varphi^{-1}(x,t)\left(\varphi^*\right)^{-1}(x,t) \approx t$.

We define the following conditions. They guarantee boundedness of maximal operators and density of smooth functions.

Definition 2.3. We say that $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ satisfies:

(aInc)$_p$ if $t \mapsto \frac{\varphi(x,t)}{t^p}$ is L_p-almost increasing in $(0,\infty)$ for some $L_p \geqslant 1$ and a.e. $x \in \mathbb{R}^n$,
(aDec)$_q$ if $t \mapsto \frac{\varphi(x,t)}{t^q}$ is L_q-almost decreasing in $(0,\infty)$ for some $L_q \geqslant 1$ and a.e. $x \in \mathbb{R}^n$,
(A0) if there exists $\beta \in (0,1]$ such that $\beta \leqslant \varphi^{-1}(x,1) \leqslant \frac{1}{\beta}$ for a.e. $x \in \mathbb{R}^n$,
(A1) if there exists $\beta \in (0,1)$ such that

$$\beta\varphi^{-1}(x,t) \leqslant \varphi^{-1}(y,t)$$

for every $t \in [1, \frac{1}{|Q|}]$, a.e. $x, y \in Q$ and every cube Q with $|Q| \leqslant 1$,
(A2) if there exists $\varphi_\infty \in \Phi_w$, $h \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $\beta \in (0,1]$ and $s > 0$ such that

$$\varphi(x,\beta t) \leqslant \varphi_\infty(t) + h(x) \quad \text{and} \quad \varphi_\infty(\beta t) \leqslant \varphi(x,t) + h(x)$$

for a.e. $x \in \mathbb{R}^n$ when $\varphi_\infty(t) \in [0,s]$ and $\varphi(x,t) \in [0,s]$, respectively.

We say φ satisfies (aInc) if it satisfies (aInc)$_p$ for some $p > 1$ and similarly (aDec) if it satisfies (aDec)$_q$ for some $q < \infty$. Conditions (aInc) and (aDec) correspond to the ∇_2 and Δ_2 conditions respectively from the classical Orlicz space theory and rule out the often problematic L^1 and L^∞ spaces. (A0) rules out degenerate or singular Φ-functions and is required for density of smooth functions in the generalized Orlicz space, for example. (A1) on the other hand is a regularity assumption generalizing for instance the log-Hölder continuity of the variable exponent. It is a key assumption for boundedness of the Hardy–Littlewood maximal function. Lastly, (A2) says that there is an Orlicz function φ_∞ which is asymptotically weakly equivalent to φ. This is likewise required for the boundedness of the maximal function on unbounded sets.

Remark 2.4. We use the following properties without mentioning them explicitly. For the proofs see [22].

a) The conditions (A0), (A1), (A2), $(\mathrm{aInc})_p$ and $(\mathrm{aDec})_q$ are invariant under equivalence ($\simeq$).
b) (A0), (A1) and (A2) hold for φ if and only if they hold also for φ^*. Additionally, φ satisfies $(\mathrm{aInc})_p$ and (aDec) if and only if φ^* satisfies $(\mathrm{aDec})_{p'}$ and $(\mathrm{aInc})_{q'}$, respectively.
c) For any $p, q > 0$ any $\varphi \in \Phi_w(\mathbb{R}^n)$ satisfies $(\mathrm{aInc})_p$ and $(\mathrm{aDec})_q$ if and only if φ^{-1} satisfies $(\mathrm{aDec})_{1/p}$ and $(\mathrm{aInc})_{1/q}$, respectively.
d) If φ satisfies $(\mathrm{aDec})_{q_1}$, then it satisfies $(\mathrm{aDec})_{q_2}$ for all $q_2 > q_1$.

The generalized Orlicz space $L^\varphi(\mathbb{R}^n)$ comprises of measurable functions f that satisfy

$$\int_{\mathbb{R}^n} \varphi(x, \lambda|f(x)|)\, dx < \infty$$

for some $\lambda > 0$. The space $L^\varphi(\mathbb{R}^n)$ is a (quasi)Banach space when equipped with a (quasi)norm

$$\|f\|_{L^\varphi(\mathbb{R}^n)} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{\lambda}\right) dx \leqslant 1\right\}.$$

We often abbreviate $\|f\|_{L^\varphi(\mathbb{R}^n)}$ as $\|f\|_\varphi$. To lighten the notation, we also often omit absolute values in $\varphi(x, |f(x)|)$ since we can without loss of generality consider only non-negative functions. A comprehensive presentation of generalized Orlicz spaces can be found in [22].

We extend the classical Hölder's inequality to generalized Orlicz spaces.

Lemma 2.5 ([22, Lemma 3.2.11])**.** *Let $\varphi \in \Phi_w(\mathbb{R}^n)$, $f \in L^\varphi(\mathbb{R}^n)$ and $g \in L^{\varphi^*}(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} |f(x)||g(x)|\, dx \leqslant 2\|f\|_{L^\varphi(\mathbb{R}^n)}\|g\|_{L^{\varphi^*}(\mathbb{R}^n)}.$$

Here the constant cannot be lower than 2 in general.

We can also have a φ-norm on the left-hand side.

Lemma 2.6 (Generalized Hölder's inequality)**.** *Let $\varphi_i \in \Phi_w(\mathbb{R}^n)$ for $i = 1, 2, 3$. If for all $t \geqslant 0$ and almost every $x \in \mathbb{R}^n$ we have*

$$\varphi_1^{-1}(x,t)\varphi_2^{-1}(x,t) \leqslant \varphi_3^{-1}(x,t),$$

then

$$\|fg\|_{L^{\varphi_3}(\mathbb{R}^n)} \leqslant 2C\|f\|_{L^{\varphi_1}(\mathbb{R}^n)}\|g\|_{L^{\varphi_2}(\mathbb{R}^n)}$$

for every $f \in L^{\varphi_1}(\mathbb{R}^n)$ *and* $g \in L^{\varphi_2}(\mathbb{R}^n)$. *If* $\varphi_i \in \Phi_c(\mathbb{R}^n)$, *then* $C = 1$.

Proof. Without loss of generality, we may assume $\|f\|_{L^{\varphi_1}(\mathbb{R}^n)} = \|g\|_{L^{\varphi_2}(\mathbb{R}^n)} = 1$. If $\varphi_i \in \Phi_c(\mathbb{R}^n)$, then the proofs in [32] apply verbatim. Suppose then we are not in the convex regime, that is $\varphi_i \in \Phi_w(\mathbb{R}^n)$. There exist $\psi_i \in \Phi_c(\mathbb{R}^n)$ such that $\varphi_i \simeq \psi_i$ [22, Lemma 2.2.1]. Now using the Young's inequality for convex functions ψ_i and the equivalence, we end up with

$$\varphi_3\left(x, \frac{st}{L^3}\right) \leqslant \psi_3\left(x, \frac{st}{L^2}\right) \leqslant \psi_1\left(x, \frac{s}{L}\right) + \psi_2\left(x, \frac{t}{L}\right) \leqslant \varphi_1(x,s) + \varphi_2(x,t)$$

where L is the largest equivalence constant of the three Φ-functions. Denoting the $(\mathrm{aInc})_1$-constant of φ_3 as L_1, we have

$$\int_{\mathbb{R}^n} \varphi_3\left(x, \tfrac{f(x)g(x)}{2L_1L^3(1+\varepsilon)^2}\right) dx \leqslant \frac{1}{2}\int_{\mathbb{R}^n} \varphi_1\left(x, \tfrac{f(x)}{1+\varepsilon}\right) + \varphi_2\left(x, \tfrac{g(x)}{1+\varepsilon}\right) dx \leqslant 1.$$

Thus $\|fg\|_{\varphi_3} \leqslant 2L_1L^3(1+\varepsilon)^2 \to 2L_1L^3$ as $\varepsilon \to 0$ and the inequality is proven. □

The explicit norms of functions are often difficult to calculate, but under suitable assumptions this can be done for characteristic functions of simple sets such as cubes.

Lemma 2.7 ([22, Proposition 4.4.8])**.** *Let* $\varphi \in \Phi_w(\mathbb{R}^n)$ *satisfy* (A0), (A1) *and* (A2). *Then for every cube* $Q \subset \mathbb{R}^n$ *we have*

$$\|\chi_Q\|_{L^{\varphi}(\mathbb{R}^n)}\|\chi_Q\|_{L^{\varphi^*}(\mathbb{R}^n)} \approx |Q|.$$

Here the implicit constant is independent of the cube Q.

The previous lemma combined with boundedness of averaging operator yields the following result (see [22, Proposition 4.4.11]).

Lemma 2.8. *Let* $\varphi \in \Phi_w(\mathbb{R}^n)$ *satisfy* (A0), (A1) *and* (A2). *Then for every cube* $Q \subset \mathbb{R}^n$ *we have*

$$\|\chi_Q\|_{L^{\varphi}(\mathbb{R}^n)} \approx \frac{1}{⨍_Q \varphi^{-1}\left(x, \frac{1}{|Q|}\right) dx}.$$

2.2. OTHER FUNCTION SPACES AND OPERATORS

Next we introduce other function spaces appearing in this paper. Functions of bounded mean oscillation, that is functions $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying

$$\sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|}\int_Q |b(x) - b_Q|\, dx < \infty, \quad \text{where} \quad b_Q = ⨍_Q b(x)\, dx$$

and Q is a cube, play an essential role in the theory of commutators. They form a function space $\mathrm{BMO}(\mathbb{R}^n)$ when equipped with a seminorm

$$\|b\|_* = \sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|}\int_Q |b(x)-b_Q|\,dx.$$

We define the negative part of a function $b^-(x) := -\min\{b(x),0\}$ and similarly the positive part $b^+(x) = \max\{b(x),0\}$. It immediately follows that for every x we have $|b(x)| - b(x) = 2b^-(x)$.

The space of compactly supported and smooth functions is denoted as $C_0^\infty(\mathbb{R}^n)$. This space is dense in $L^\varphi(\mathbb{R}^n)$ if φ satisfies (A0) and (aDec) [22, Theorem 3.7.15].

Lastly, we define various operators investigated in this paper. All of the maximal operators are derived from non-centred Hardy–Littlewood maximal function

$$Mf(x) := \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f(y)|\,dy.$$

This is comparable with the centred variant

$$M^c f(x) := \sup_{Q(x,r)} \frac{1}{|Q(x,r)|}\int_{Q(x,r)} |f(y)|\,dy$$

as they enjoy pointwise inequalities $M^c f(x) \leqslant Mf(x) \leqslant 2^n M^c f(x)$. We can also have a restricted maximal function in each cube Q_0 as

$$M_{Q_0} f(x) := \sup_{\substack{Q\ni x\\ Q\subset Q_0}} \frac{1}{|Q|}\int_Q |f(y)|\,dy.$$

We also define the sharp maximal function as

$$M^\sharp f(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx.$$

If a measurable function b is given, we define the maximal commutator as

$$M_b f(x) := \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |b(x)-b(y)||f(y)|\,dy$$

and the commutator of a maximal function is defined as

$$[M,b]f(x) := M(bf)(x) - b(x)Mf(x).$$

For $\alpha \in (0,n)$, the focus of this article is the fractional maximal function

$$M_\alpha f(x) := \sup_{Q\ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}}\int_Q |f(y)|\,dy$$

and its variations similar to previously given to the Hardy–Littlewood maximal function, denoted as $M_{\alpha,Q_0}f(x), M_{b,\alpha}f(x)$ and $[M_\alpha,b]f(x)$.

Closely related to fractional maximal function is the Riesz potential of a function f

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

For the Riesz potential we also have the variants

$$[I_\alpha, b]f(x) = I_\alpha(bf)(x) - b(x)I_\alpha f(x) \text{ and } I_{b,\alpha}f(x) = \int_{\mathbb{R}^n} \tfrac{|b(x)-b(y)|}{|x-y|^{n-\alpha}} f(y)\, dy.$$

Note that $I_{b,\alpha}$ is often denoted in the literature as $|I_\alpha, b|$ or $|b, I_\alpha|$, but to unify notation with maximal functions we use a different notation.

Boundedness of Hardy–Littlewood maximal operator in generalized Orlicz spaces with the current framework was proven by Hästö [23]. In particular, if $\varphi \in \Phi_w(\mathbb{R}^n)$ satisfies (A0), (A1), (A2) and (aInc), we have

$$\|Mf\|_\varphi \leqslant C_M \|f\|_\varphi \tag{2.1}$$

for all $f \in L^\varphi(\mathbb{R}^n)$.

3. BOUNDEDNESS OF THE FRACTIONAL MAXIMAL OPERATOR

In this section we prove boundedness properties of the fractional maximal operator. The result follows also from pointwise inequality $M_\alpha f(x) \lesssim I_\alpha f(x)$ and [21] or from extrapolation arguments [12] but here we give a direct proof of the result and simplify the presentation of function spaces. To obtain this, we exploit the fact that φ can be regularized to a function $\tilde{\varphi}$, while preserving the function spaces and having equivalent norms. This construction was developed in [21] but we demonstrate the steps involved and show that our approach, that is assuming Assumptions 1.1 instead of using compositions of the type $\varphi \circ \left(\lambda^{-1}\right)$, is the same. The main reason to consider regularized Φ-function is to extend the range of t in (A1) to $\left[0, \frac{1}{|Q|}\right]$ (see [21, Proposition 4.5]). Note that we do not need to assume that φ is an N-function as was done in [21], since the critical result, $\varphi^{-1}(x,t)\left(\varphi^*\right)^{-1}(x,t) \approx t$, is now known to hold for weak Φ-functions, too.

1. We start with a generalized Orlicz function φ satisfying (A0), (A1), (A2), $(\text{aInc})_p$ and $(\text{aDec})_{1/r}$, where $r \in (\frac{\alpha}{n}, \frac{1}{p}]$ and $p > 1$.
2. Choose $\varphi_1 \in \Phi_{\mathrm{s}}(\mathbb{R}^n)$ such that $\varphi_1(x,1) = \varphi_1^{-1}(x,1) = 1$ and $\varphi_1 \simeq \varphi$ [22, Lemma 3.7.3].
3. Define $\varphi_2(x,t) = \max\{\varphi_1(x,t), 2t-1\}$. We have $\varphi_2 \simeq \varphi_1 \simeq \varphi$ and therefore φ_2 satisfies (A0), (A1), (A2), $(\text{aInc})_p$ and $(\text{aDec})_{1/r}$.
4. Define
$$\tilde{\varphi}(x,t) = \begin{cases} (\varphi_2)_\infty(t) & \text{when } t < 1, \\ 2\varphi_2(x,t) - 1 & \text{when } t \geqslant 1, \end{cases}$$
where $(\varphi_2)_\infty(t) = \limsup_{|x|\to\infty} \varphi_2(x,t)$.

In [21, Proposition 4.2] it was shown that $L^{\varphi}(\mathbb{R}^n) = L^{\tilde\varphi}(\mathbb{R}^n)$ with comparable norms and thus $\varphi \sim \tilde\varphi$.

[21, Proposition 4.5] shows that the if φ satisfies (A0), (A1) and (A2), then so does $\tilde\varphi$ (condition (A0) follows automatically, since $\tilde\varphi(x,1) = 1$ by construction). We add to this by noting that regularization preserves also (aInc) and (aDec): for $t < s < 1$ we have

$$\frac{\tilde\varphi(x,t)}{t^p} = \limsup_{|x|\to\infty} \frac{\varphi_2(x,t)}{t^p} \lesssim \limsup_{|x|\to\infty} \frac{\varphi_2(x,s)}{s^p} = \frac{\tilde\varphi(x,s)}{s^p},$$

for $1 < t < s$ we have

$$\frac{\tilde\varphi(x,t)}{t^p} = \frac{2\varphi_2(x,t)-1}{t^p} \lesssim \frac{2\varphi_2(x,s)}{s^p} \leqslant 2\frac{2\varphi_2(x,s)-1}{s^p} = 2\frac{2\tilde\varphi(x,s)}{s^p}$$

and finally for $t < 1 < s$ (aInc) follows since $\tilde\varphi$ is increasing. (aDec) is proven similarly.

There is still the question whether there exists a regularized function $\tilde\psi$ which is weakly equivalent to ψ and satisfies the relation (1.1) with $\tilde\varphi$. Since $\tilde\varphi$ satisfies (A0), (A1), (A2), $(\text{aInc})_p$ and $(\text{aDec})_{1/r}$, then by [22, Lemmas 3.7.3 and 5.2.3] there exists $\psi_0 \in \Phi_s(\mathbb{R}^n)$ satisfying $(\text{aInc})_q$ and $(\text{aDec})_{1/(r-\alpha/n)}$ such that $\tilde\varphi^{-1}(x,t) \approx t^{\frac{\alpha}{n}}\psi_0^{-1}(x,t)$. Let us show that $\psi_0 \sim \psi$ when $\varphi^{-1}(x,t) \approx t^{\frac{\alpha}{n}}\psi^{-1}(x,t)$. Equivalently this means that $\psi_0^{-1}(x,t) \approx \psi^{-1}(x,t+h(x))$, where $h \in L^1(\mathbb{R}^n)$. Since $(\text{aDec})_{n/\alpha}$ of φ implies $(\text{aInc})_{\alpha/n}$ of φ^{-1} and we know that $\varphi \sim \tilde\varphi$, we estimate

$$\begin{aligned}\psi_0^{-1}(x,t) &\approx \frac{\tilde\varphi^{-1}(x,t)}{t^{\alpha/n}} \lesssim \frac{\tilde\varphi^{-1}(x,t+h(x))}{[t+h(x)]^{\alpha/n}} \approx \frac{\varphi^{-1}(x,t+2h(x))}{[t+h(x)]^{\alpha/n}} \\ &\lesssim \frac{\varphi^{-1}(x,t+2h(x))}{[t+2h(x)]^{\alpha/n}} \approx \psi^{-1}(x,t+2h(x)),\end{aligned}$$

i.e. $\psi_0 \sim \psi$. From now on, we denote $\tilde\psi := \psi_0$ since it possesses all the relevant properties of a regularized function: weak equivalence, (A0) and (A1) in range $t \in \left[0, \frac{1}{|Q|}\right]$.

Remark 3.1. In the paper of Harjulehto and Hästö [21], the target space of I_α was constructed in a slightly different way than in our presentation. Let us show that they are in fact equivalent: We show that under suitable structural conditions on φ, the Riesz potential maps functions from $L^{\varphi}(\mathbb{R}^n) \to L^{\psi}(\mathbb{R}^n)$, where φ and ψ satisfy the relation $\varphi^{-1}(x,t) \approx t^{\frac{\alpha}{n}}\psi^{-1}(x,t)$. In [21] the target function space is $L^{\varphi^\sharp_\alpha}(\mathbb{R}^n)$, where $\varphi^\sharp_\alpha$ is a convex Φ-function equivalent to $\varphi(x,\lambda^{-1}(x,t))$ with $\lambda(x,t) := t\varphi(x,t)^{-\frac{\alpha}{n}}$. Recall that equivalence of Φ-functions is an equivalent condition to comparability of their inverses. The claim is invariant under equivalence of Φ-functions so we can assume that $\varphi, \psi \in \Phi_s(\mathbb{R}^n)$ and therefore by Lemma 2.2 (a)

$$\begin{aligned}\left(\varphi^\sharp_\alpha\right)^{-1}(x,t) \approx \lambda(x,\varphi^{-1}(x,t)) &= \varphi^{-1}(x,t)\ \varphi(x,\varphi^{-1}(x,t))^{-\frac{\alpha}{n}} \\ &= t^{-\frac{\alpha}{n}}\varphi^{-1}(x,t) \approx \psi^{-1}(x,t).\end{aligned}$$

Thus $\varphi^\sharp_\alpha \simeq \psi$ and they generate the same function spaces with comparable norms and we resume to denote ψ for the function generating the target space.

We collect these observations to the following lemma. Note that (A1) is now valid for cubes of all sizes, not just those with $|Q| \leqslant 1$.

Lemma 3.2. *Let φ and ψ satisfy Assumptions 1.1. Then there exist $\tilde\varphi, \tilde\psi \in \Phi_{\mathrm{s}}(\mathbb{R}^n)$ such that $\varphi \sim \tilde\varphi$, $\psi \sim \tilde\psi$, $\tilde\varphi^{-1}(x,t) \approx t^{\frac{\alpha}{n}} \tilde\psi^{-1}(x,t)$ and $\tilde\psi^{-1}(x,t) \lesssim \tilde\psi^{-1}(y,t)$ for every cube Q, almost every $x, y \in Q$ and every $t \in \left[0, \frac{1}{|Q|}\right]$.*

Proof of Theorem 1.2. Recall the centred fractional maximal function

$$M^c_\alpha f(x) := \sup_{r>0} \tfrac{1}{|Q(x,r)|^{1-\alpha/n}} \int_{Q(x,r)} |f(y)|\,dy$$

and recall that $M^c_\alpha f(x) \approx M_\alpha f(x)$. Thus, we can prove the boundedness for the centred fractional maximal operator.

Let Φ-functions $\tilde\varphi$ and $\tilde\psi$ be as in Lemma 3.2. Let $Q(x,r)$ be any cube and split a function f as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{Q(x,r)}(y)$, and $f_2(y) = f(y)\chi_{Q(x,r)^c}(y)$. By sublinearity of the maximal operator we have $M^c_\alpha f(x) \leqslant M^c_\alpha f_1(x) + M^c_\alpha f_2(x)$. We start by estimating the last term

$$\begin{aligned} M^c_\alpha f_2(x) &\lesssim \sup_{t>0} \tfrac{1}{|Q(x,t)|^{1-\alpha/n}} \int_{Q(x,t)\cap Q(x,r)^c} |f(y)|\,dy \\ &\lesssim \sup_{t>r} \tfrac{1}{|Q(x,t)|^{1-\alpha/n}} \int_{Q(x,t)} |f(y)|\,dy. \end{aligned}$$

We proceed with Hölder's inequality, Lemma 2.8 and Lemma 2.2 b)

$$\begin{aligned} M^c_\alpha f_2(x) &\lesssim \sup_{t>r} \|f\|_{\tilde\varphi} |Q(x,t)|^{\alpha/n-1} \|\chi_{Q(x,t)}\|_{\tilde\varphi^*} \\ &\lesssim \sup_{t>r} \|f\|_{\tilde\varphi} |Q(x,t)|^{\alpha/n-1} \left(\fint_{Q(x,t)} (\tilde\varphi^*)^{-1}\left(y, |Q(x,t)|^{-1}\right) dy \right)^{-1} \\ &\lesssim \sup_{t>r} \|f\|_{\tilde\varphi} |Q(x,t)|^{\alpha/n-1} \left(\fint_{Q(x,t)} \tfrac{1}{|Q(x,t)|} \tfrac{1}{\tilde\varphi^{-1}(y,|Q(x,t)|^{-1})}\, dy \right)^{-1}. \end{aligned}$$

By convexity of $z \mapsto \frac{1}{z}$ with Jensen's inequality and Lemma 3.2 we continue

$$\begin{aligned} M^c_\alpha f_2(x) &\lesssim \sup_{t>r} \|f\|_{\tilde\varphi} |Q(x,t)|^{\alpha/n} \fint_{Q(x,t)} \tilde\varphi^{-1}(y, |Q(x,t)|^{-1})\,dy \\ &\lesssim \|f\|_{\tilde\varphi} \sup_{t>r} t^\alpha \fint_{Q(x,t)} \tilde\varphi^{-1}(y, t^{-n})\,dy \\ &\leqslant \|f\|_{\tilde\varphi} \sup_{t>r} \tilde\psi^{-1}(x, t^{-n}) \leqslant \|f\|_{\tilde\varphi} \tilde\psi^{-1}(x, r^{-n}). \end{aligned}$$

For f_1 we have $M^c_\alpha f_1(x) \lesssim r^\alpha M^c f(x) \approx r^\alpha M f(x)$ (see for example [24, Lemma. (b)]), so returning to non-centred maximal functions we see

$$\begin{aligned} |M_\alpha f(x)| &\lesssim r^\alpha M f(x) + \|f\|_{\tilde\varphi} \tilde\psi^{-1}(x, r^{-n}) \\ &\lesssim M f(x) \tfrac{\tilde\psi^{-1}(x,r^{-n})}{\tilde\varphi^{-1}(x,r^{-n})} + \|f\|_{\tilde\varphi} \tilde\psi^{-1}(x, r^{-n}). \end{aligned}$$

Recall that $\tilde\varphi \in \Phi_s(\mathbb{R}^n)$ so it is increasing and continuous, therefore surjective from $[0,\infty)$ onto $[0,\infty)$. Thus we can choose r in such a way that $\tilde\varphi^{-1}(x, r^{-n}) = \frac{Mf(x)}{C_M\|f\|_{\tilde\varphi}}$, where C_M is the constant in (2.1), and get

$$|M_\alpha f(x)| \lesssim \|f\|_{\tilde\varphi}\tilde\psi^{-1}\left(x, \tilde\varphi\left(x, \tfrac{Mf(x)}{C_M\|f\|_{\tilde\varphi}}\right)\right).$$

Since $\tilde\psi \in \Phi_s(\mathbb{R}^n)$, by Lemma 2.2 a) it follows that

$$\tilde\psi\left(x, \tfrac{|M_\alpha f(x)|}{C_M\|f\|_{\tilde\varphi}}\right) \lesssim \tilde\psi\left(x, \tilde\psi^{-1}\left(x, \tilde\varphi\left(x, \tfrac{Mf(x)}{C_M\|f\|_{\tilde\varphi}}\right)\right)\right) = \tilde\varphi\left(x, \tfrac{Mf(x)}{C_M\|f\|_{\tilde\varphi}}\right).$$

Integrating both sides we have

$$\int_{\mathbb{R}^n} \tilde\psi\left(x, \tfrac{|M_\alpha f(x)|}{C_M\|f\|_{\tilde\varphi}}\right) dx \lesssim \int_{\mathbb{R}^n} \tilde\varphi\left(x, \tfrac{Mf(x)}{C_M\|f\|_{\tilde\varphi}}\right) dx \leqslant 1.$$

In other words $\|M_\alpha f\|_{\tilde\psi} \lesssim \|f\|_{\tilde\varphi}$. But $\tilde\varphi$ and $\tilde\psi$ are weakly equivalent to φ and ψ so they have comparable norms, respectively, and all in all

$$\|M_\alpha f\|_{L^\psi(\mathbb{R}^n)} \lesssim \|f\|_{L^\varphi(\mathbb{R}^n)}. \qquad \square$$

Remark 3.3. Contrary to the regular Orlicz case, we do not know if the assumption $\varphi^{-1}(x,t) \approx t^{\frac{\alpha}{n}}\psi^{-1}(x,t)$ is necessary. To the best of our knowledge, this is an open question already in the variable exponent case. Following the proof of Orlicz case in [18], the problem arises as $\|\chi_Q\|_\varphi$ cannot always be estimated pointwise as $\left(\varphi^{-1}(x, |B|^{-1})\right)^{-1}$ but only in the integral from as in Lemma 2.8.

4. COMMUTATORS OF RIESZ POTENTIAL

It turns out that the boundedness of $[I_\alpha, b]$ can be tackled with the sharp maximal function. Our methods are based on [25]. There is no pointwise inequality between a function f and the sharp maximal function $M^\sharp f$ but a norm estimate can be achieved as is done in Lemma 4.1. Due to density of $C_0^\infty(\mathbb{R}^n)$ in $L^\varphi(\mathbb{R}^n)$ (see [22, Corollary 3.7.10]) and (2.1), the proof follows as in [16, Lemma 6.2.4].

Lemma 4.1. *If* $\varphi \in \Phi_w(\mathbb{R}^n)$ *satisfies* (A0), (A1), (A2), (aInc) *and* (aDec), *then*

$$\|f\|_{L^\varphi(\mathbb{R}^n)} \lesssim \|M^\sharp f\|_{L^\varphi(\mathbb{R}^n)}$$

for all $f \in L^\varphi(\mathbb{R}^n)$.

The next lemma has been proven by Shirai in the case of $[I_\alpha, b]$ but the proof works without changes also for the operator $I_{b,\alpha}$.

Lemma 4.2 ([34, Lemma 4.2])**.** *Suppose* $0 < \alpha < n$ *and* $s \in (1,\infty)$. *Then for every* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $f \in C_0^\infty(\mathbb{R}^n)$ *we have pointwise estimates:*

1. $M^\sharp([I_\alpha, b]f)(x) \lesssim \|b\|_* \left(I_\alpha|f|(x) + I_{\alpha s}(|f|^s)(x)^{1/s}\right)$,
2. $M^\sharp(I_{b,\alpha}f)(x) \lesssim \|b\|_* \left(I_\alpha|f|(x) + I_{\alpha s}(|f|^s)(x)^{1/s}\right)$.

The following lemma is the main result of [21] taking account Remark 3.1.

Lemma 4.3. *If φ and ψ satisfy Assumptions 1.1, then the Riesz potential $I_\alpha : L^\varphi(\mathbb{R}^n) \to L^\psi(\mathbb{R}^n)$ is bounded.*

Combining these lemmas, we get the boundedness of Riesz type.

Proof of Theorem 1.3. We give the proof for (1) as the proof of (2) is identical. Let us denote $\varphi_s = \varphi\left(x, t^{1/s}\right)$ for any weak Φ-function. Recall that ψ satisfies similar assumptions to φ but with different parameters in (aInc) and (aDec). Therefore, choosing any $s \in (1,p)$ and applying first Lemma 4.1 and then Lemma 4.2, we see that for any $f \in C_0^\infty(\mathbb{R}^n)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$

$$\begin{aligned} \|[I_\alpha, b]f\|_\psi &\lesssim \|M^\sharp([I_\alpha, b]f)\|_\psi \lesssim \|b\|_* \|I_\alpha|f| + I_{\alpha s}(|f|^s)^{1/s}\|_\psi \\ &\leqslant \|b\|_* \left(\|I_\alpha|f|\|_\psi + \|I_{\alpha s}(|f|^s)\|_{\psi_s}^{1/s}\right). \end{aligned} \tag{4.1}$$

By Lemma 4.3, $I_\alpha : L^\varphi(\mathbb{R}^n) \to L^\psi(\mathbb{R}^n)$ is bounded so let us show boundedness of $I_{\alpha s} : L^{\varphi_s}(\mathbb{R}^n) \to L^{\psi_s}(\mathbb{R}^n)$. It is easy to check that $(\psi_s)^{-1}(x,t) = \psi^{-1}(x,t)^s$. Next, it immediately follows that

$$(\psi_s)^{-1}(x,t) = \psi^{-1}(x,t)^s \approx t^{-\frac{\alpha s}{n}} \varphi^{-1}(x,t)^s \approx t^{-\frac{\alpha s}{n}} \varphi_s^{-1}(x,t).$$

In [22, Proposition 5.2.2] it is shown that φ_s satisfies (A0), (A1), (A2), $(\mathrm{aInc})_{p/s}$ and $(\mathrm{aDec})_{1/rs}$. Since $\frac{p}{s} > 1$ and $rs \in (\frac{\alpha s}{n}, \frac{s}{p}]$, Lemma 4.3 shows that $I_{\alpha s}$ is bounded from $L^{\varphi_s}(\mathbb{R}^n)$ to $L^{\psi_s}(\mathbb{R}^n)$ and likewise

$$\|I_{\alpha s}(|f|^s)\|_{\psi_s}^{\frac{1}{s}} \lesssim \||f|^s\|_{\varphi_s}^{\frac{1}{s}}.$$

Coming back to (4.1) and definition of φ-norm, we have

$$\|[I_\alpha, b]f\|_\psi \lesssim \|b\|_* \left(\|f\|_\varphi + \|f^s\|_{\varphi_s}^{1/s}\right) \lesssim \|b\|_* \|f\|_\varphi.$$

Since φ satisfies (A0) and (aDec), we know that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^\varphi(\mathbb{R}^n)$ [22, Corollary 3.7.10]. Now we get the result for all $f \in L^\varphi(\mathbb{R}^n)$ since the commutator is linear and bounded on a dense set. □

5. MAXIMAL COMMUTATORS

In this section we show boundedness of both maximal commutators

$$M_b : L^\varphi(\mathbb{R}^n) \to L^\varphi(\mathbb{R}^n) \quad \text{and} \quad M_{b,\alpha} : L^\varphi(\mathbb{R}^n) \to L^\psi(\mathbb{R}^n).$$

We start with the simpler non-fractional version but first we collect some lemmas. The first lemma was essentially proven in [1, Corollaries 1.11, 1.12]. Authors' main results lead to pointwise estimates

$$M_b f(x) \lesssim \|b\|_* M^2 f(x) \quad \text{and} \quad |[M,b]f(x)| \lesssim (\|b^+\|_* + \|b^-\|_\infty) M^2 f(x)$$

where $b \in \mathrm{BMO}(\mathbb{R}^n)$, so the following lemma holds for any (quasi) Banach space where M is bounded, especially for $L^\varphi(\mathbb{R}^n)$.

Lemma 5.1. *If $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $\varphi \in \Phi_w(\mathbb{R}^n)$ satisfies* (A0), (A1), (A2) *and* (aInc), *then*

$$\|M_b f\|_\varphi \lesssim \|b\|_* \|f\|_\varphi.$$

If additionally $b^- \in L^\infty(\mathbb{R}^n)$, then also the commutator $[M, b] : L^\varphi(\mathbb{R}^n) \to L^\varphi(\mathbb{R}^n)$ is bounded.

It is straightforward to prove that $b \in \mathrm{BMO}(\mathbb{R}^n)$ is also a necessary condition for boundedness of the maximal commutator and we get the following characterization of $\mathrm{BMO}(\mathbb{R}^n)$.

Proposition 5.2. *Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\varphi \in \Phi_w(\mathbb{R}^n)$ satisfy* (A0), (A1), (A2) *and* (aInc). *Then $M_b : L^\varphi(\mathbb{R}^n) \to L^\varphi(\mathbb{R}^n)$ is bounded if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$.*

Proof. If $b \in \mathrm{BMO}(\mathbb{R}^n)$, the claim follows directly from Lemma 5.1.

Let us then assume that $M_b : L^\varphi(\mathbb{R}^n) \to L^\varphi(\mathbb{R}^n)$ is bounded and take any cube $Q \subset \mathbb{R}^n$. Using Hölder's inequality (Lemma 2.5), the assumption and Lemma 2.7 in this order, we see that

$$\begin{aligned}\fint_Q |b(x) - b_Q|\, dx &\leqslant \fint_Q \left(\int_Q |b(x) - b(y)| \chi_Q(y)\, dy \right) dx \leqslant \frac{1}{|Q|} \int_{\mathbb{R}^n} M_b \chi_Q(x)\, dx \\ &\lesssim \frac{1}{|Q|} \|M_b \chi_Q\|_\varphi \|\chi_Q\|_{\varphi^*} \leqslant \frac{1}{|Q|} \|\chi_Q\|_\varphi \|\chi_Q\|_{\varphi^*} \leqslant C.\end{aligned}$$

Taking the supremum over all cubes Q, we see that $b \in \mathrm{BMO}(\mathbb{R}^n)$. □

As the fractional commutators are closely related, we get the boundedness of maximal fractional commutator also in a similar fashion.

Proof of Theorem 1.4. Let us first assume that $b \in \mathrm{BMO}(\mathbb{R}^n)$. Since we have the pointwise estimate $M_{b,\alpha} f \lesssim I_{b,\alpha}(|f|)$, Theorem 1.3 (2) immediately yields

$$\|M_{b,\alpha} f\|_\psi \lesssim \|I_{b,\alpha}(|f|)\|_\psi \lesssim \|b\|_* \|f\|_\varphi.$$

Let us then assume that $M_{b,\alpha}$ is bounded and Q is an arbitrary cube. This and Hölder's inequality give

$$\begin{aligned}\fint_Q |b(x) - b_Q|\, dx &\leqslant \frac{1}{|Q|^{1+\frac{\alpha}{n}}} \int_Q \left(\frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |b(x) - b(y)| \chi_Q(y)\, dy \right) dx \\ &\leqslant \frac{1}{|Q|^{1+\frac{\alpha}{n}}} \|M_{b,\alpha} \chi_Q\|_\psi \|\chi_Q\|_{\psi^*} \lesssim \frac{1}{|Q|^{1+\frac{\alpha}{n}}} \|\chi_Q\|_\varphi \|\chi_Q\|_{\psi^*}.\end{aligned}$$

Now we need to estimate $\|\chi_Q\|_\varphi$ in terms of $\|\chi_Q\|_\psi$. This is done with the help of Lemma 2.8:

$$\|\chi_Q\|_\varphi \approx \frac{1}{\fint_Q \varphi^{-1}\left(x, \frac{1}{|Q|}\right) dx} \approx \frac{|Q|^{\alpha/n}}{\fint_Q \psi^{-1}\left(x, \frac{1}{|Q|}\right) dx} \approx |Q|^{\alpha/n} \|\chi_Q\|_\psi. \tag{5.1}$$

Therefore Lemma 2.7 and the previous two displays yield

$$\frac{1}{|Q|}\int_Q |b(x) - b_Q|\,dx \lesssim \frac{1}{|Q|}\|\chi_Q\|_\psi \|\chi_Q\|_{\psi^*} < C.$$

Taking supremum over all cubes Q we see that $b \in \mathrm{BMO}(\mathbb{R}^n)$. □

6. COMMUTATORS OF FRACTIONAL MAXIMAL OPERATORS

In this section we combine previous estimates of maximal operators to show boundedness of commutators of fractional maximal operators generalizing the ideas of Zhang, Si and Wu in [36, 37]. In [3] the authors showed the following characterization for functions with bounded mean oscillation and negative part. The proof of $b \in \mathrm{BMO}(\mathbb{R}^n)$ is similar to the proof of Lemma 6.3. However, the L^1 nature of assumption with non-fractional maximal function is needed for $b^- \in L^\infty(\mathbb{R}^n)$ and it is not known if assumption like the norm estimate in Lemma 6.3 with fractional maximal function implies this.

Lemma 6.1. *Suppose $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and*

$$\sup_Q \frac{1}{|Q|}\int_Q |b(x) - M_Q b(x)|\,dx < \infty.$$

Then $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.

The next result is a weaker form of Proposition 6.4, but it will be used later when we have commutator with $|b|$ which is clearly non-negative.

Lemma 6.2. *Let Assumptions 1.1 hold. If $0 \leqslant b \in \mathrm{BMO}(\mathbb{R}^n)$, then the commutator $[M_\alpha, b] : L^\varphi(\mathbb{R}^n) \to L^\psi(\mathbb{R}^n)$ is bounded.*

Proof. As $|b(x)| = b(x)$ for all $x \in \mathbb{R}^n$, we see that

$$|[M_\alpha, b]f(x)| \leqslant \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}}\int_Q |b(x) - b(y)||f(y)|\,dy = M_{b,\alpha}f(x).$$

By Theorem 1.4 we see that the commutator is bounded. □

Let us separate the following result as a lemma.

Lemma 6.3. *Let Assumptions 1.1 hold and suppose $[M_\alpha, b] : L^\varphi(\mathbb{R}^n) \to L^\psi(\mathbb{R}^n)$ is bounded, where $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then $b \in \mathrm{BMO}(\mathbb{R}^n)$ and*

$$\frac{\|(b - |Q|^{-\alpha/n} M_{\alpha,Q} b)\chi_Q\|_\psi}{\|\chi_Q\|_\psi} < C$$

for every cube Q and some finite constant C independent of Q.

Proof. Let us first prove the norm estimate. First of all, we have the following identities for maximal functions of a characteristic function

$$M_\alpha \chi_Q(x) = M_{\alpha,Q}\chi_Q(x) = \sup_{\tilde{Q}\ni x} \frac{1}{|\tilde{Q}|^{1-\alpha/n}} \int_{\tilde{Q}} \chi_Q(x)\,dx = |Q|^{\alpha/n} \tag{6.1}$$

and

$$M_\alpha(b\chi_Q)(x) = M_{\alpha,Q} b(x) = \sup_{\tilde{Q}\ni x} \frac{1}{|\tilde{Q}|^{1-\alpha/n}} \int_{\tilde{Q}} b(x)\chi_Q(x)\,dx \tag{6.2}$$

for $x \in Q$. These combined with the assumption that $[M_\alpha, b] : L^\varphi(\mathbb{R}^n) \to L^\psi(\mathbb{R}^n)$ is bounded, we have the following estimate

$$\begin{aligned}\|(b - |Q|^{-\alpha/n} M_{\alpha,Q}(b))\chi_Q\|_\psi &\leqslant |Q|^{-\alpha/n}\|bM_\alpha\chi_Q - M_\alpha(b\chi_Q)\|_\psi\\ &= |Q|^{-\alpha/n}\|[M_\alpha, b]\chi_Q\|_\psi \lesssim |Q|^{-\alpha/n}\|\chi_Q\|_\varphi.\end{aligned}$$

We proceed as in (5.1) to estimate $\|\chi_Q\|_\varphi \lesssim |Q|^{\alpha/n}\|\chi_Q\|_\psi$. Thus

$$\|(b - |Q|^{-\alpha/n} M_{\alpha,Q} b)\chi_Q\|_\psi \lesssim \|\chi_Q\|_\psi \tag{6.3}$$

for every cube Q, so we have proved the desired norm estimate.

Now we are ready to prove that $b \in \mathrm{BMO}(\mathbb{R}^n)$. Let us denote for any cube Q sets

$$E := \{x \in Q : b(x) \leqslant b_Q\} \quad \text{and} \quad F := \{x \in Q : b(x) > b_Q\}.$$

It is immediate that $E = F^c$ and this combined with

$$\int_Q b(x) - b_Q\,dx = 0$$

yields

$$\int_E |b(x) - b_Q|\,dx = \int_F |b(x) - b_Q|\,dx.$$

Also, for any $x \in E$ it holds that $b(x) \leqslant b_Q \leqslant |b_Q| \leqslant |Q|^{-\alpha/n} M_{\alpha,Q} b(x)$. Therefore

$$|b(x) - b_Q| \leqslant |b(x) - |Q|^{-\alpha/n} M_{\alpha,Q}(b)(x)|$$

in the set E.

Let us estimate the mean oscillation of the function b

$$\begin{aligned}\fint_Q |b(x) - b_Q|\,dx = 2\fint_E |b(x) - b_Q|\,dx &\leqslant 2\fint_E |b(x) - |Q|^{-\alpha/n} M_{\alpha,Q} b(x)|\,dx\\ &\leqslant 2\fint_Q |b(x) - |Q|^{-\alpha/n} M_{\alpha,Q} b(x)|\,dx.\end{aligned}$$

We estimate the last integral with Hölder's inequality, (6.3) and Lemma 2.7 to get

$$\fint_Q |b(x) - b_Q|\,dx \lesssim \frac{1}{|Q|}\|(b - |Q|^{-\alpha/n} M_{\alpha,Q} b)\chi_Q\|_\psi \|\chi_Q\|_{\psi^*} \leqslant C. \tag{6.4}$$

Thus, by definition, $b \in \mathrm{BMO}(\mathbb{R}^n)$. □

In the next proposition we characterize $\mathrm{BMO}(\mathbb{R}^n)$ in terms of boundedness of commutators with fractional maximal operators. In Theorem 1.5 we improve condition (3) to a more intrinsic condition containing the fractional maximal operator.

Proposition 6.4. *Let Assumptions 1.1 hold. If $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then the following are equivalent:*

1. $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $b^- \in L^\infty(\mathbb{R}^n)$.
2. *The commutator* $[M_\alpha, b] : L^\varphi(\mathbb{R}^n) \to L^\psi(\mathbb{R}^n)$ *is bounded.*
3. *We have*
$$\sup_Q \frac{\|(b - M_Q b)\chi_Q\|_{L^\psi(\mathbb{R}^n)}}{\|\chi_Q\|_{L^\psi(\mathbb{R}^n)}} < \infty.$$

Proof. We proceed to prove (1) ⇒ (2) ⇒ (3) ⇒ (1). For the first implication we start with the following pointwise estimates

$$\begin{aligned}
&|[M_\alpha, b]f(x) - [M_\alpha, |b|]f(x)| \\
&\qquad \leqslant |M_\alpha(bf)(x) - M_\alpha(|b|f)(x)| + |2b^-(x)M_\alpha f(x)| \\
&\qquad = 2b^-(x)M_\alpha f(x).
\end{aligned}$$

In other words, we have a direct estimate for the commutator

$$\begin{aligned}
|[M_\alpha, b]f(x)| &\leqslant |[M_\alpha, b]f(x) - [M_\alpha, |b|]f(x)| + |[M_\alpha, |b|]f(x)| \\
&\leqslant 2b^-(x)M_\alpha f(x) + |[M_\alpha, |b|]f(x)|.
\end{aligned}$$

Since $b \in \mathrm{BMO}(\mathbb{R}^n)$ implies that $|b| \in \mathrm{BMO}(\mathbb{R}^n)$, recalling the assumption $b^- \in L^\infty(\mathbb{R}^n)$ we can combine Theorem 1.2 and Lemma 6.2 to get

$$\|[M_\alpha, b]f\|_\psi \leqslant 2\|b^-\|_\infty \|M_\alpha f\|_\psi + \|[M_\alpha, |b|]f\|_\psi \lesssim \|f\|_\varphi.$$

Next we prove (2) ⇒ (3) and start with Lemma 6.3, which yields

$$\begin{aligned}
\|(b - M_Q b)\chi_Q\|_\psi &\leqslant \|(b - |Q|^{-\frac{\alpha}{n}} M_{\alpha,Q} b)\chi_Q\|_\psi + \|(|Q|^{-\frac{\alpha}{n}} M_{\alpha,Q} b - M_Q b)\chi_Q\|_\psi \\
&\lesssim \|\chi_Q\|_\psi + |Q|^{-\frac{\alpha}{n}} \|(M_{\alpha,Q} b - |Q|^{\frac{\alpha}{n}} M_Q b)\chi_Q\|_\psi.
\end{aligned} \tag{6.5}$$

Letting $x \in Q$ and taking (6.1) and (6.2) into account, we have

$$[M_\alpha, |b|]\chi_Q(x) = M_\alpha(b\chi_Q)(x) - |b(x)|M_\alpha \chi_Q(x) = M_{\alpha,Q} b(x) - |Q|^{\frac{\alpha}{n}}|b(x)|$$

together with $|Q|^{\frac{\alpha}{n}}[M, |b|]\chi_Q(x) = |Q|^{\frac{\alpha}{n}} M_Q b(x) - |Q|^{\frac{\alpha}{n}}|b(x)|$. We subtract the second from the first and get

$$[M_\alpha, |b|]\chi_Q(x) - |Q|^{\frac{\alpha}{n}}[M, |b|]\chi_Q(x) = M_{\alpha,Q} b(x) - |Q|^{\frac{\alpha}{n}} M_Q b(x).$$

This and (6.5) combine to

$$\begin{aligned}
\|(b - M_Q b)\chi_Q\|_\psi &\lesssim \|\chi_Q\|_\psi + |Q|^{-\frac{\alpha}{n}} \|[M_\alpha, |b|]\chi_Q - |Q|^{\frac{\alpha}{n}}[M, |b|]\chi_Q\|_\psi \\
&\leqslant \|\chi_Q\|_\psi + |Q|^{-\frac{\alpha}{n}} \|[M_\alpha, |b|]\chi_Q\|_\psi + \|[M, |b|]\chi_Q\|_\psi.
\end{aligned}$$

As $0 \leqslant |b| \in \mathrm{BMO}(\mathbb{R}^n)$, the commutators are bounded (Lemma 5.1 and Lemma 6.2), and thus

$$\|(b - M_Q b)\chi_Q\|_\psi \lesssim \|\chi_Q\|_\psi + |Q|^{-\frac{\alpha}{n}}\|\chi_Q\|_\varphi + \|\chi_Q\|_\psi.$$

Finally, using (5.1) for the second term on the right-hand side, we have

$$\|(b - M_Q b)\chi_Q\|_\psi \lesssim \|\chi_Q\|_\psi.$$

Then we show (3) ⇒ (1). Hölder's inequality and Lemma 2.7 yield

$$\begin{aligned} \frac{1}{|Q|}\int_Q |b(x) - M_Q b(x)|\,dx &\lesssim \frac{1}{|Q|}\|(b - M_Q b)\chi_Q\|_\psi \|\chi_Q\|_{\psi^*} \\ &\lesssim \frac{1}{|Q|}\|\chi_Q\|_\psi \|\chi_Q\|_{\psi^*} \leqslant C. \end{aligned}$$

Now Lemma 6.1 shows that $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$. □

Now we prove our main theorem regarding commutators of fractional maximal operators. The main difference of Theorem 1.5 to previous proposition is that the norm condition includes fractional maximal operator instead of a standard Hardy–Littlewood maximal operator. Also, it is enough for the norm condition to hold for any weak Φ-function η with structural conditions guaranteeing boundedness of the Hardy–Littlewood maximal operator.

Proof of Theorem 1.5. Equivalence (1) ⇔ (2) was proven in Proposition 6.4. As the implication (4) ⇒ (3) is immediate, we proceed to prove (3) ⇒ (1) and (2) ⇒ (4).

We first assume (3). Choosing any cube Q we get

$$\begin{aligned} \fint_Q |b(x) - M_Q b(x)|\,dx &\leqslant \fint_Q |b(x) - |Q|^{-\alpha/n} M_{\alpha,Q} b(x)|\,dx \\ &\quad + \fint_Q ||Q|^{-\alpha/n} M_{\alpha,Q} b(x) - M_Q b(x)|\,dx =: I_1 + I_2. \end{aligned}$$

We first estimate the term I_1 with Hölder's inequality, Lemma 2.7 and the assumption as follows

$$I_1 \lesssim \tfrac{1}{|Q|}\|(b - |Q|^{-\alpha/n} M_{\alpha,Q} b)\chi_Q\|_\psi \|\chi_Q\|_{\psi^*} \lesssim \tfrac{\|(b - |Q|^{-\alpha/n} M_{\alpha,Q} b)\chi_Q\|_\psi}{\|\chi_Q\|_\psi} \leqslant C.$$

For I_2 let us choose any $x \in Q$. Then (6.1) and (6.2) guarantee us that for $x \in Q$ we have $M\chi_Q(x) = 1$ and $M(b\chi_Q)(x) = M_Q b(x)$ for the maximal function and $M_\alpha \chi_Q(x) = |Q|^{\alpha/n}$ and $M_\alpha(b\chi_Q)(x) = M_{\alpha,Q} b(x)$ for the fractional maximal function. With these observations we estimate the integrand in I_2 as a sum of commutators

$$\begin{aligned} \big||Q|^{-\alpha/n} M_{\alpha,Q} b(x) - M_Q b(x)\big| &\leqslant \Big|Q|^{-\alpha/n}|M_{\alpha,Q} b(x) - |Q|^{\alpha/n}|b(x)|\Big| \\ &\quad + ||b(x)| - M_Q b(x)| \\ &= |Q|^{-\alpha/n}|[M_\alpha, |b|]\chi_Q(x)| + |[M, |b|]\chi_Q(x)|. \end{aligned}$$

This yields

$$I_2 \leqslant \frac{1}{|Q|^{1+\alpha/n}} \int_Q |[M_\alpha, |b|]\chi_Q(x)|\, dx + \frac{1}{|Q|} \int_Q |[M, |b|]\chi_Q(x)|\, dx =: I_2^1 + I_2^2.$$

Now the assumption (3) and a similar argument as in latter part of Lemma 6.3 show that $b \in \mathrm{BMO}(\mathbb{R}^n)$ which further implies that $|b| \in \mathrm{BMO}(\mathbb{R}^n)$. Therefore, we can apply Lemma 6.2 to see that $[M_\alpha, |b|]$ and $[M, |b|]$ are bounded operators. I_2^1 can be shown to be bounded in the following way using Hölder's inequality, the boundedness of the commutator, (5.1) and Lemma 2.7

$$I_2^1 \lesssim \frac{1}{|Q|^{1+\alpha/n}} \|[M_\alpha, |b|]\chi_Q\|_\psi \|\chi_Q\|_{\psi^*} \lesssim \frac{1}{|Q|^{1+\alpha/n}} \|\chi_Q\|_\varphi \|\chi_Q\|_{\psi^*} \leqslant C.$$

In a similar fashion the second term is shown be bounded

$$I_2^2 \lesssim \frac{1}{|Q|} \|[M, |b|]\chi_Q\|_\varphi \|\chi_Q\|_{\varphi^*} \lesssim \frac{1}{|Q|} \|\chi_Q\|_\varphi \|\chi\|_{\varphi^*} \leqslant C.$$

Therefore by Lemma 6.1 we have $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.

Let us then assume (2) and prove (4). Lemma 6.3 shows that (1.2) holds when $\eta = \psi$ satisfying the relation with φ. Let η then be any weak Φ-function such that the maximal operator is bounded from $L^\eta(\mathbb{R}^n) \to L^\eta(\mathbb{R}^n)$. We choose $\psi(x,t) = \eta(x, t^r)$, where $r > \frac{n}{n-\alpha}$ to guarantee that φ satisfies (aInc). Let us denote $\xi(x,t) = \eta(x, t^{r'})$, where r' is the Hölder conjugate of r. Now

$$\eta^{-1}(x,t) = \eta^{-1}(x,t)^{1/r} \eta^{-1}(x,t)^{1/r'} = \psi^{-1}(x,t)\xi^{-1}(x,t)$$

so the generalized Hölder's inequality, Lemma 2.6, is valid with η, ψ and ξ. Applying first it and then (1.2) with $\eta = \psi$ we get

$$\frac{\|(b-|Q|^{-\alpha/n} M_{\alpha,Q} b)\chi_Q\|_\eta}{\|\chi_Q\|_\eta} \lesssim \frac{\|(b-|Q|^{-\alpha/n} M_{\alpha,Q} b)\chi_Q\|_\psi \|\chi_Q\|_\xi}{\|\chi_Q\|_\eta} \lesssim \frac{\|\chi_Q\|_\psi \|\chi_Q\|_\xi}{\|\chi_Q\|_\eta}.$$

From the definitions of ψ and ξ we finally have

$$\frac{\|(b - |Q|^{-\alpha/n} M_{\alpha,Q} b)\chi_Q\|_\eta}{\|\chi_Q\|_\eta} \lesssim \frac{\|\chi_Q\|_\psi \|\chi_Q\|_\xi}{\|\chi_Q\|_\eta} \lesssim \frac{\|\chi_Q\|_\eta^{1/r} \|\chi_Q\|_\eta^{1/r'}}{\|\chi_Q\|_\eta} = 1. \qquad \square$$

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