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Three-phase parabolic inhomogeneities with internal uniform stresses in plane and anti-plane elasticity

X. WANG[1)], P. SCHIAVONE[2)]

[1)] *School of Mechanical and Power Engineering, East China University of Science and Technology, 130 Meilong Road, Shanghai 200237, China, e-mail: xuwang@ecust.edu.cn*

[2)] *Department of Mechanical Engineering, University of Alberta, 10-203 Donadeo Innovation Centre for Engineering, Edmonton, Alberta Canada T6G 1H9, e-mail: p.schiavone@ualberta.ca*

We examine the in-plane and anti-plane stress states inside a parabolic inhomogeneity which is bonded to an infinite matrix through an intermediate coating. The interfaces of the three-phase parabolic inhomogeneity are two confocal parabolas. The corresponding boundary value problems are studied in the physical plane rather than in the image plane. A simple condition is found that ensures that the internal stress state inside the parabolic inhomogeneity is uniform and hydrostatic. Furthermore, this condition is independent of the elastic properties of the coating and the two geometric parameters of the composite: in fact, the condition depends only on the elastic constants of the inhomogeneity and the matrix and the ratio between the two remote principal stresses. Once this condition is met, the mean stress in the coating is constant and the hoop stress on the coating side is also uniform along the entire inhomogeneity-coating interface. The unconditional uniformity of stresses inside a three-phase parabolic inhomogeneity is achieved when the matrix is subjected to uniform remote anti-plane shear stresses. The internal uniform anti-plane shear stresses inside the inhomogeneity are independent of the shear modulus of the coating and the two geometric parameters of the composite.

Key words: three-phase parabolic inhomogeneity, coating, internal uniform stresses, plane elasticity, anti-plane elasticity.

1. Introduction

The question as to whether uniformity of stresses can be achieved inside elastic inhomogeneities has a long history of research and enquiry (see, for example, [1–13]). The majority of these investigations have been focused on inhomogeneities with closed curvilinear contours (or interfaces) such as ellipses and ellipsoids. In a very recent study, Wang and Schiavone [14] have proved the surprising result that for anti-plane elasticity and plane elasticity, the stress field inside a parabolic elastic inhomogeneity bounded by an open interface is still uniform when subjected to a uniform loading at infinity.

In the design of composites, a separate coating of finite thickness is usually intentionally inserted between the internal inhomogeneity and the surrounding matrix with an objective to improve the attachment between the inhomogeneity and the matrix and also to reduce material mismatch induced stress concentrations at the interface (see Ru [15] and the references cited therein). On the other hand, the analysis of the problem of three-phase elastic inhomogeneities offers the fundamental solution for the self-consistent method [16, 17]. Ru [15] obtained a simple condition that ensures that the internal stress state within a three-phase elliptical inhomogeneity with two confocal interfaces in plane deformations is uniform and hydrostatic. Ru *et al.* [18] showed that a three-phase elliptical inhomogeneity with two confocal interfaces under uniform remote anti-plane shear stresses admits an internal uniform stress field. The uniformity of the internal anti-plane stresses within the coated elliptical inhomogeneity is unconditional.

In this paper, we study in detail the in-plane and anti-plane shear deformations of a three-phase parabolic inhomogeneity which is bonded to an infinite matrix through an intermediate coating when the matrix is subjected to uniform remote stresses. The interfaces of the three-phase parabolic inhomogeneity are two confocal parabolic interfaces. The corresponding boundary value problems are studied in the physical plane rather than in the image plane. Under in-plane deformations, a simple condition is derived that ensures that the internal stress state inside the parabolic inhomogeneity is uniform and hydrostatic. Furthermore, this condition is independent of the existence of the intermediate coating and depends only on the elastic constants of the inhomogeneity and the matrix as well as the ratio between the two remote principal stresses. When this condition is met, the internal uniform hydrostatic stress state is simply the remote normal stress perpendicular to the axis of symmetry of the parabolas. Under anti-plane shear deformations, the internal stresses inside the coated parabolic inhomogeneity remain unconditionally uniform. In addition, the internal stress state is independent of the shear modulus of the coating and the two geometric parameters of the composite. A neutral coated parabolic inhomogeneity in anti-plane elasticity is successfully designed using the developed solution. Note that all of the neutral coated inhomogeneities obtained by Milton and Serkov [19], and Jarczyk and Mityushev [20] have only closed curvilinear interfaces.

Finally, we mention that our focus on the parabolic inhomogeneity arises from the very interesting properties recently attributed to parabolic interfaces and barriers (see, for example, the discussion in Obnosov [21] and the references contained therein). For example, Philip [22] has shown that the flow velocity within a parabolic inhomogeneity in a descending unsaturated flow is constant. These remarkable characteristics associated with parabolic barriers led us to ask the question whether analogous results would be available for simple deformations of elastic solids, in particular the 'constant' or 'uniformity' property. Consequently, in the specific case researched here, we consider whether an internal uniform stress distribution is achievable within a three-phase parabolic inhomogeneity. The relevance of such a question to the design of advanced materials lies in the fact that uniform internal stress distributions are often considered as optimal since they eliminate any stress peaks within the inhomogeneity which are well-known to be the primary cause of failure in composite structures.

2. Uniform hydrostatic stress state inside a three-phase parabolic inhomogeneity

We first establish a Cartesian coordinate system $\{x_i\}$ $(i = 1, 2, 3)$. For in-plane deformations of an isotropic elastic material, the three in-plane stresses $(\sigma_{11}, \sigma_{22}, \sigma_{12})$, two in-plane displacements (u_1, u_2) and two stress functions (ϕ_1, ϕ_2) are given in terms of two analytic functions $\varphi(z)$ and $\psi(z)$ of the complex variable $z = x_1 + \mathrm{i}x_2$ as [23]

$$\begin{aligned} &\sigma_{11} + \sigma_{22} = 2[\varphi'(z) + \overline{\varphi'(z)}], \\ &\sigma_{22} - \sigma_{11} + 2\mathrm{i}\sigma_{12} = 2[\bar{z}\varphi''(z) + \psi'(z)], \end{aligned} \tag{2.1}$$

$$\begin{aligned} &2\mu(u_1 + \mathrm{i}u_2) = \kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}, \\ &\phi_1 + \mathrm{i}\phi_2 = \mathrm{i}[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}], \end{aligned} \tag{2.2}$$

where the Kolosov constant $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress, μ and ν $(0 \leq \nu \leq 1/2)$ are the shear modulus and Poisson's ratio, respectively. In addition, the stresses are related to the stress functions through [7]

$$\begin{aligned} \sigma_{11} &= -\phi_{1,2}, \qquad \sigma_{12} = \phi_{1,1}, \\ \sigma_{21} &= -\phi_{2,2}, \qquad \sigma_{22} = \phi_{2,1}. \end{aligned} \tag{2.3}$$

Let t_1,t_2 denote, respectively, the traction components along the x_1- and x_2-directions on a boundary L. If s is the arc-length measured along L such that when facing the direction of increasing s the material is on the left-hand side, it can be shown that [7]:

$$t_1 + \mathrm{i}t_2 = -\frac{d(\phi_1 + \mathrm{i}\phi_2)}{ds}.$$

We consider a three-phase parabolic inhomogeneity with two confocal parabolic interfaces. Let S_1, S_2 and S_3 denote the inhomogeneity, the coating and the matrix, respectively, all of which are perfectly bonded through two confocal

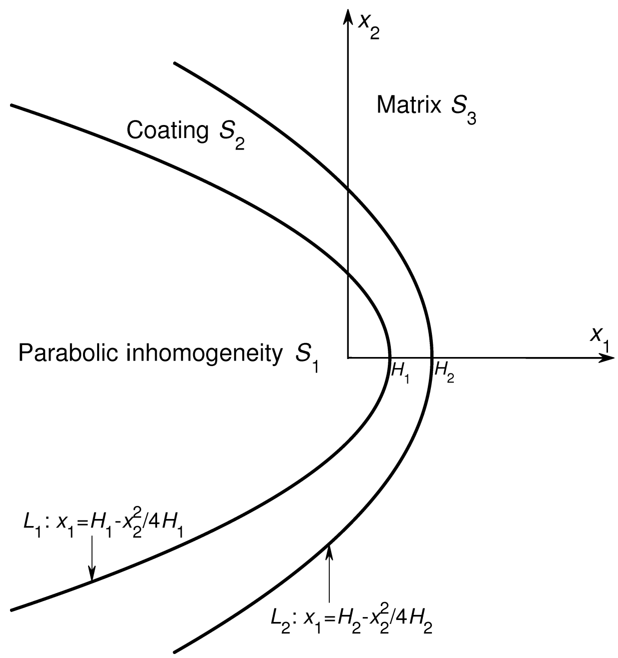

FIG. 1. Three-phase parabolic inhomogeneity with internal uniform stresses.

parabolic interfaces L_1 and L_2, as shown in Fig. 1. The inhomogeneity S_1, the coating S_2 and the matrix S_3 occupy the following three regions

$$S_1 : x_1 \leq H_1 - \frac{x_2^2}{4H_1}; \qquad S_2 : H_1 - \frac{x_2^2}{4H_1} \leq x_1 \leq H_2 - \frac{x_2^2}{4H_2}; \qquad S_3 : x_1 \geq H_2 - \frac{x_2^2}{4H_2}, \tag{2.4}$$

where $H_2 > H_1 > 0$, and thus the two confocal parabolic interfaces L_1 and L_2 are described by

$$L_1 : x_1 = H_1 - \frac{x_2^2}{4H_1}; \qquad L_2 : x_1 = H_2 - \frac{x_2^2}{4H_2}. \tag{2.5}$$

We can see that the two parabolic interfaces have a common focus located at the origin of the coordinate system. The two constants H_1 and H_2 are the two geometric parameters of the three-phase composite. In addition, the matrix is subjected to uniform remote in-plane stresses $(\sigma_{11}^{\infty}, \sigma_{22}^{\infty}, \sigma_{12}^{\infty})$. Throughout the paper, the subscripts 1, 2 and 3 are used to identify the respective quantities in S_1, S_2 and S_3.

The boundary value problem for the three-phase parabolic inhomogeneity takes the form

$$\begin{aligned}
&\varphi_2(z)+z\overline{\varphi_2'(z)}+\overline{\psi_2(z)} = \varphi_1(z)+z\overline{\varphi_1'(z)}+\overline{\psi_1(z)}, \\
&\kappa_2\varphi_2(z)-z\overline{\varphi_2'(z)}-\overline{\psi_2(z)} = \Gamma_1\kappa_1\varphi_1(z)-\Gamma_1 z\overline{\varphi_1'(z)}-\Gamma_1\overline{\psi_1(z)}, \quad z\in L_1;
\end{aligned} \tag{2.6a}$$

$$\begin{aligned}
&\varphi_3(z)+z\overline{\varphi_3'(z)}+\overline{\psi_3(z)} = \varphi_2(z)+z\overline{\varphi_2'(z)}+\overline{\psi_2(z)}, \\
&\kappa_3\varphi_3(z)-z\overline{\varphi_3'(z)}-\overline{\psi_3(z)} = \Gamma_2\kappa_2\varphi_2(z)-\Gamma_2 z\overline{\varphi_2'(z)}-\Gamma_2\overline{\psi_2(z)}, \quad z\in L_2;
\end{aligned} \tag{2.6b}$$

$$\begin{aligned}
&\varphi_3(z) \cong \frac{\sigma_{11}^{\infty}+\sigma_{22}^{\infty}}{4}z+O(z^{1/2}), \\
&\psi_3(z) \cong \frac{\sigma_{22}^{\infty}-\sigma_{11}^{\infty}+2\mathrm{i}\sigma_{12}^{\infty}}{2}z+O(z^{1/2}), \quad |z|\to\infty,
\end{aligned} \tag{2.6c}$$

where

$$\Gamma_1 = \frac{\mu_2}{\mu_1}, \qquad \Gamma_2 = \frac{\mu_3}{\mu_2}. \tag{2.7}$$

Equations (2.6a) and (2.6b) describe the continuity conditions of tractions and displacements across the inhomogeneity-coating interface L_1 and the coating-matrix interface L_2, respectively; Eq. (2.6c) gives the asymptotic behaviors of $\varphi_2(z)$ and $\psi_2(z)$ due to the remote loading. The internal stress state inside the parabolic inhomogeneity is uniform and hydrostatic if the two analytic functions $\varphi_1(z)$ and $\psi_1(z)$ defined in the inhomogeneity take the following form

$$\varphi_1(z) = Az, \qquad \psi_1(z) = 0, \qquad z\in S_1, \tag{2.8}$$

where A is a real constant to be determined.

In addition, we can prove the following identities without difficulty

$$\begin{aligned}
&\bar{z}^{1/2} = 2H_1^{1/2} - z^{1/2}, \qquad \text{for } z\in L_1; \\
&\bar{z}^{1/2} = 2H_2^{1/2} - z^{1/2}, \qquad \text{for } z\in L_2.
\end{aligned} \tag{2.9}$$

As a result, the two analytic functions $\varphi_2(z)$ and $\psi_2(z)$ defined in the coating can be determined from Eqs. (2.6a), (2.8) and $(2.9)_1$ as

$$\begin{aligned}
\varphi_2(z) &= \frac{A[\Gamma_1(\kappa_1-1)+2]}{\kappa_2+1}z, \\
\psi_2(z) &= \frac{2A[\Gamma_1(1-\kappa_1)+\kappa_2-1]}{\kappa_2+1}z \\
&\quad - \frac{8AH_1^{1/2}[\Gamma_1(1-\kappa_1)+\kappa_2-1]}{\kappa_2+1}z^{1/2}, \quad z\in S_2.
\end{aligned} \tag{2.10}$$

In a similar manner, the two analytic functions $\varphi_3(z)$ and $\psi_3(z)$ defined in the matrix can be determined from Eqs. (2.6b), (2.10) and $(2.9)_2$ as

$$
\begin{aligned}
\varphi_3(z) &= \frac{A[\Gamma_1\Gamma_2(\kappa_1-1)+2]}{\kappa_3+1}z \\
&\quad+\frac{8A(H_1^{1/2}-H_2^{1/2})(1-\Gamma_2)[\Gamma_1(1-\kappa_1)+\kappa_2-1]}{(\kappa_2+1)(\kappa_3+1)}z^{1/2}, \\
\psi_3(z) &= \frac{2A[\kappa_3-1-\Gamma_1\Gamma_2(\kappa_1-1)]}{\kappa_3+1}z \\
&\quad+\frac{4A\begin{pmatrix} H_1^{1/2}(\Gamma_2+2\kappa_3+1)[\Gamma_1(\kappa_1-1)-\kappa_2+1] \\ +H_2^{1/2}[\Gamma_1\Gamma_2(\kappa_1-1)(2\kappa_2+1)-\Gamma_1(\kappa_1-1)(2\kappa_3+1) \\ +\Gamma_2(\kappa_2-1)+3\kappa_2-4\kappa_3+1] \end{pmatrix}}{(\kappa_2+1)(\kappa_3+1)}z^{1/2} \\
&\quad+\frac{16AH_2(H_2^{1/2}-H_1^{1/2})(1-\Gamma_2)[\Gamma_1(1-\kappa_1)+\kappa_2-1]}{(\kappa_2+1)(\kappa_3+1)}z^{-1/2}, \quad z\in S_3.
\end{aligned} \tag{2.11}
$$

By using Eq. (2.11) to satisfy the remote asymptotic requirements placed on $\varphi_3(z)$ and $\psi_3(z)$ in Eq. (2.6c), we arrive at the following relationships

$$
\begin{aligned}
\frac{A[\Gamma_1\Gamma_2(\kappa_1-1)+2]}{\kappa_3+1} &= \frac{\sigma_{11}^\infty+\sigma_{22}^\infty}{4}, \\
\frac{A[\kappa_3-1-\Gamma_1\Gamma_2(\kappa_1-1)]}{\kappa_3+1} &= \frac{\sigma_{22}^\infty-\sigma_{11}^\infty+2\mathrm{i}\sigma_{12}^\infty}{4}.
\end{aligned} \tag{2.12}
$$

We can see from the second condition in Eq. (2.12) that $\sigma_{12}^\infty = 0$, which implies that the remote principal stresses should act along the x_1 and x_2 axes (in other words, the two remote normal stresses are simply the principal stresses).

It turns out that the necessary and sufficient condition for the existence of the real constant A simultaneously satisfying the two conditions in Eq. (2.12) is:

$$
\frac{\mu_1(\kappa_3-1)-\mu_3(\kappa_1-1)}{2\mu_1+\mu_3(\kappa_1-1)} = \frac{\sigma_{22}^\infty-\sigma_{11}^\infty}{\sigma_{11}^\infty+\sigma_{22}^\infty}, \tag{2.13}
$$

or equivalently

$$
\frac{\sigma_{11}^\infty}{\sigma_{22}^\infty} = \frac{\mu_1(3-\kappa_3)+2\mu_3(\kappa_1-1)}{\mu_1(\kappa_3+1)} > 0, \tag{2.14}
$$

which gives the ratio between the two remote normal stresses for a given set of elastic constants for the inhomogeneity and the matrix. It is interesting to note that the condition in Eq. (2.13) or (2.14) is independent of the elastic properties of the intermediate coating as well as the two geometric parameters H_1 and H_2.

Once the condition in Eq. (2.13) or (2.14) is met, the internal uniform hydrostatic stresses are simply given by

$$\sigma_{11} = \sigma_{22} = \sigma_{22}^{\infty}, \quad \sigma_{12} = 0, \quad z \in S_1. \tag{2.15}$$

We can see from Eq. (2.10) that the mean stress is constantly distributed in the coating as

$$\sigma_{11} + \sigma_{22} = \frac{2\sigma_{22}^{\infty}[\Gamma_1(\kappa_1 - 1) + 2]}{\kappa_2 + 1}, \quad z \in S_2. \tag{2.16}$$

In addition, it follows from Eqs. (2.15) and (2.16) that the hoop stress is uniform along the inhomogeneity-coating interface L_1 on the coating side and is given by

$$\sigma_{tt} = \frac{\sigma_{22}^{\infty}[2\Gamma_1(\kappa_1 - 1) + 3 - \kappa_2]}{\kappa_2 + 1}, \quad z \in L_1 \cap S_2. \tag{2.17}$$

By substituting Eqs. (2.10) and (2.11) into Eq. (2.1), we arrive at the stress distributions in the coating and in the surrounding matrix. For convenience and for the sake of brevity, we suppress their specific expressions here noting only that in writing these stress distributions, it is more convenient to introduce a polar coordinate system (r, θ) such that $z = r\exp(i\theta)$. In particular, the mean stress along the coating-matrix interface L_2 on the matrix side is non-uniformly distributed as follows

$$\begin{aligned} \sigma_{11} + \sigma_{22} &= \frac{2\sigma_{22}^{\infty}[\Gamma_1\Gamma_2(\kappa_1 - 1) + 2]}{\kappa_3 + 1} \\ &+ \frac{32\sigma_{22}^{\infty}(H_1^{1/2}H_2^{-1/2} - 1)(1 - \Gamma_2)[\Gamma_1(1 - \kappa_1) + \kappa_2 - 1]}{(\kappa_2 + 1)(\kappa_3 + 1)} g(H_2^{-1}x_2), \\ L_2 : x_1 &= H_2 - \frac{x_2^2}{4H_2}, \end{aligned} \tag{2.18}$$

where

$$g(x) = \frac{1}{x^2 + 4}. \tag{2.19}$$

Apparently, the non-uniformity of the mean stress in Eq. (2.18) comes solely from the even function $g(x)$ of x in Eq. (2.19).

Remark. If $\psi_1(z)$ is assumed to be a linear function of z, the expression of $\varphi_3(z)$ defined in the matrix will contain the term $(2H_2^{1/2} - z^{1/2})^{-1}$, which becomes singular at the point $z = 4H_2$. This observation violates the requirement that $\varphi_3(z)$ should be analytic in the matrix except at the point at infinity. Thus we set $\psi_1(z) = 0$ in Eq. (2.8).

3. Uniform stresses within a three-phase parabolic inhomogeneity in anti-plane shear

In the anti-plane shear deformations of an isotropic elastic material, the two anti-plane shear stress components σ_{31} and σ_{32}, the out-of-plane displacement u_3 and the stress function ϕ_3 can be expressed in terms of a single analytic function $f(z)$ of the complex variable $z = x_1 + \mathrm{i}x_2$ as [7]

$$\sigma_{32} + \mathrm{i}\sigma_{31} = \mu f'(z), \qquad \phi_3 + \mathrm{i}\mu u_3 = \mu f(z), \tag{3.1}$$

and the two stress components can be expressed in terms of the single stress function ϕ_3 as [7]

$$\sigma_{32} = \phi_{3,1}, \qquad \sigma_{31} = -\phi_{3,2}. \tag{3.2}$$

The boundary value problem to be analyzed is similar to that discussed in the previous section except that now the matrix is subjected to uniform remote anti-plane shear stresses $(\sigma_{31}^{\infty}, \sigma_{32}^{\infty})$. The notation used here is identical to that adopted in Section 2. We investigate the existence of a uniform anti-plane stress field inside the parabolic inhomogeneity.

The boundary value problem for the three-phase parabolic inhomogeneity under anti-plane shear deformations has the form

$$\begin{aligned} &f_2(z) + \overline{f_2(z)} = \beta_1 f_1(z) + \beta_1 \overline{f_1(z)},\\ &f_2(z) - \overline{f_2(z)} = f_1(z) - \overline{f_1(z)}, \quad z \in L_1; \end{aligned} \tag{3.3a}$$

$$\begin{aligned} &f_3(z) + \overline{f_3(z)} = \beta_2 f_2(z) + \beta_2 \overline{f_2(z)},\\ &f_3(z) - \overline{f_3(z)} = f_2(z) - \overline{f_2(z)}, \quad z \in L_2; \end{aligned} \tag{3.3b}$$

$$f_3(z) \cong \frac{\sigma_{32}^{\infty} + \mathrm{i}\sigma_{31}^{\infty}}{\mu_3} z + O(z^{1/2}), \quad |z| \to \infty, \tag{3.3c}$$

where

$$\beta_1 = \frac{\mu_1}{\mu_2}, \qquad \beta_2 = \frac{\mu_2}{\mu_3}. \tag{3.4}$$

Equations (3.3a) and (3.3b) describe the continuity conditions of traction and displacement across the two interfaces, respectively whilst Eq. (3.3c) gives the remote asymptotic behavior of $f_3(z)$ due to remote anti-plane shear stresses. The internal stress field inside the parabolic inhomogeneity is uniform if the analytic function $f_1(z)$ defined in the inhomogeneity takes the following form

$$f_1(z) = Cz, \qquad z \in S_1, \tag{3.5}$$

where C is a complex constant to be determined.

By enforcing the conditions in Eqs. (3.3a) and (3.3b) and using the identities in Eq. (2.9) along the two interfaces, we obtain

$$f_2(z) = \frac{C(\beta_1+1)+\bar{C}(\beta_1-1)}{2}z + 2H_1^{1/2}\bar{C}(1-\beta_1)z^{1/2}, \quad z \in S_2, \tag{3.6}$$

$$\begin{aligned} f_3(z) = {} & \frac{C(\beta_1\beta_2+1)+\bar{C}(\beta_1\beta_2-1)}{2}z \\ & + \{H_1^{1/2}(\beta_1-1)[C(\beta_2-1)-\bar{C}(\beta_2+1)] \\ & - H_2^{1/2}(\beta_2-1)[C(\beta_1-1)+\bar{C}(\beta_1+1)]\}z^{1/2}, \quad z \in S_3. \end{aligned} \tag{3.7}$$

Using Eq. (3.7) to satisfy the remote asymptotic behavior of $f_3(z)$ in Eq. (3.3c), we arrive at

$$\frac{C(\beta_1\beta_2+1)+\bar{C}(\beta_1\beta_2-1)}{2} = \frac{\sigma_{32}^{\infty}+\mathrm{i}\sigma_{31}^{\infty}}{\mu_3}, \tag{3.8}$$

from which the complex constant C can be uniquely determined as

$$C = \frac{\sigma_{32}^{\infty}+\mathrm{i}\beta_1\beta_2\sigma_{31}^{\infty}}{\mu_1}. \tag{3.9}$$

Thus the internal uniform stresses inside the parabolic inhomogeneity can be determined as

$$\sigma_{32} = \sigma_{32}^{\infty}, \qquad \sigma_{31} = \beta_1\beta_2\sigma_{31}^{\infty}, \qquad z \in S_1, \tag{3.10}$$

which are, in fact, independent of the shear modulus of the intermediate coating and the two geometric parameters H_1 and H_2. Although the coating does not affect the internal stress field inside the inhomogeneity, it influences the stresses in the matrix (see Eq. (3.7)). By considering this fact, a neutral coated parabolic inhomogeneity ('neutral' in the sense that its introduction does not disturb the original uniform stress field in the surrounding matrix) can be designed as follows:

(i) When $\sigma_{32}^{\infty} \neq 0$ and $\sigma_{31}^{\infty} = 0$, it is deduced from Eq. (3.7) that the two geometric parameters should satisfy the following restriction in order to achieve neutrality

$$\frac{H_1}{H_2} = \frac{\beta_1^2(\beta_2-1)^2}{(\beta_1-1)^2} < 1 \quad \text{when } \mu_1<\mu_3<\mu_2 \text{ or } \mu_1>\mu_3>\mu_2. \tag{3.11}$$

(ii) When $\sigma_{31}^{\infty} \neq 0$ and $\sigma_{32}^{\infty} = 0$, Eq. (3.7) again tells us that the two geometric parameters should satisfy the following restriction in order to achieve neutrality

$$\frac{H_1}{H_2} = \frac{(\beta_2-1)^2}{\beta_2^2(\beta_1-1)^2} < 1 \quad \text{when } \mu_1<\mu_3<\mu_2 \text{ or } \mu_1>\mu_3>\mu_2. \tag{3.12}$$

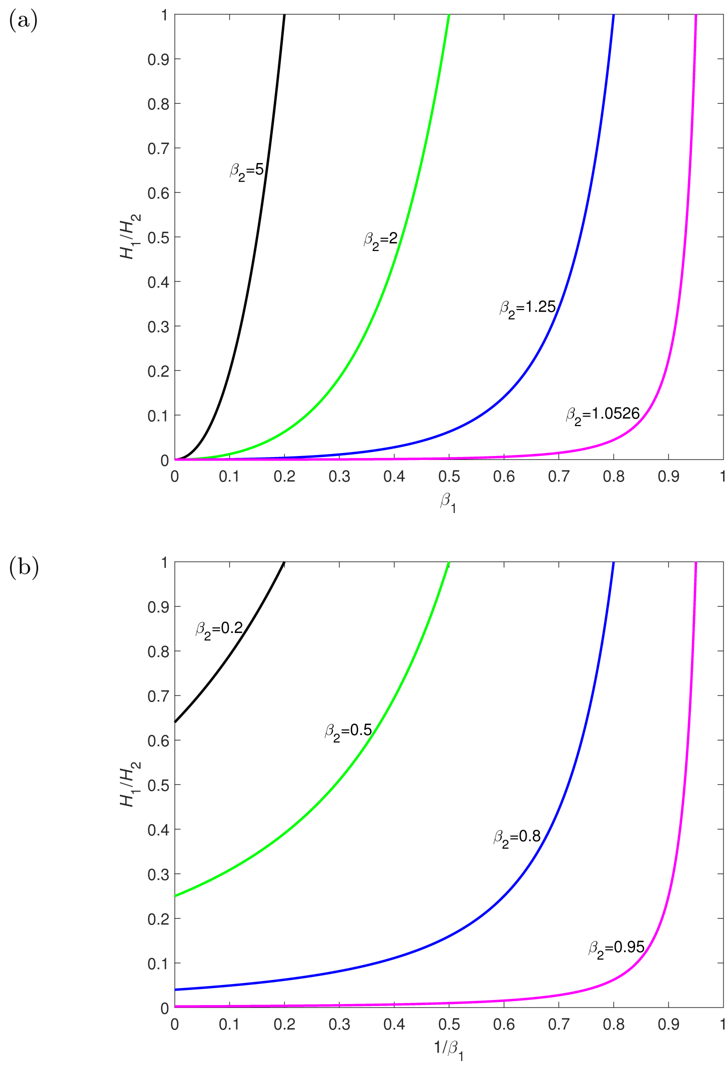

FIG. 2. Variation of H_1/H_2 for different values of β_1 and β_2 for case (i); (a) $\beta_1 < 1/\beta_2 < 1$ (or equivalently $\mu_1 < \mu_3 < \mu_2$), (b) $\beta_1 > 1/\beta_2 > 1$ (or equivalently $\mu_1 > \mu_3 > \mu_2$).

If both σ_{31}^{∞} and σ_{32}^{∞} are non-zero, it is impossible to simultaneously satisfy the two conditions in Eqs. (3.11) and (3.12).

We illustrate in Fig. 2 the variation of H_1/H_2 for different values of β_1 and β_2 for case (i). If we adopt the substitutions $\beta_1 \to 1/\beta_1$ and $\beta_2 \to 1/\beta_2$ in Fig. 2, we arrive at the corresponding result for case (ii). The analysis carried out in this section can be easily extended to the case of a multicoated parabolic inhomogeneity. Interestingly, the internal uniform stress field is independent of the shear moduli of the intermediate multiple coatings and all of the geometric parameters of the composite.

4. Conclusions

We have proved that for plane and anti-plane elastic deformations, the internal stress field inside a coated parabolic inhomogeneity still remains uniform. This uniformity of stresses within the three-phase parabolic inhomogeneity is achieved via the confocal character of the two interfaces. The internal uniform hydrostatic stress state is conditional whereas the internal uniform anti-plane shear stress state is unconditional.

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