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EXISTENCE OF SOLUTION OF SUB-ELLIPTIC EQUATIONS ON THE HEISENBERG GROUP WITH CRITICAL GROWTH AND DOUBLE SINGULARITIES

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Abstract. For a class of sub-elliptic equations on Heisenberg group $\mathbb{H}^N$ with Hardy type singularity and critical nonlinear growth, we prove the existence of least energy solutions by developing new techniques based on the Nehari constraint. This result extends previous works, e.g., by Han *et al.* [*Hardy-Sobolev type inequalities on the H-type group*, Manuscripta Math. **118** (2005), 235–252].

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1. INTRODUCTION

This paper is concerned with a class of sub-elliptic equations on the Heisenberg group having a nonlinearity with critical nonlinear growth and a singularity of the form

$$-\triangle_{H,p}u - \lambda\frac{|z|^p}{\rho^{2p}}|u|^{p-2}u = \frac{|z|^s}{\rho^{2s}}|u|^{p_*(s)-2}u \quad \text{in} \quad \mathbb{H}^N\backslash\{0\}. \tag{1.1}$$

We begin with some basic definitions and useful results. The Heisenberg group $\mathbb{H}^N$, whose points will be denoted by $\xi = (z,t) = (x,y,t)$, is identified with the Lie group $(\mathbb{R}^{2N+1}, \circ)$ with composition law defined by

$$\xi \circ \xi' = (z + z', t + t' + 2(\langle x', y\rangle - \langle x, y'\rangle)), \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^N$. And for $\xi \in \mathbb{H}^N$, the left translations on $\mathbb{H}^N$ are defined by

$$\tau_\xi : \mathbb{H}^N \to \mathbb{H}^N, \qquad \tau_\xi(\xi') = \xi \circ \xi'.$$

For $\xi, \xi' \in \mathbb{H}^N$, the distance between ξ and ξ' is defined by

$$d(\xi,\xi') = \Big(\big(|x-x'|^2 + |y-y'|^2\big)^2 + (t - t' - 2(\langle x, y'\rangle - \langle x', y\rangle)^2 \Big)^{\frac{1}{4}}.$$

For convenience, the distance of $\xi \in \mathbb{H}^N$ to the origin is denoted by ρ. For $\mu > 0$, a family of dilation on $\mathbb{H}^N$ is defined by

$$\delta_\mu : \mathbb{H}^N \to \mathbb{H}^N, \qquad \delta_\mu(x,y,t) = (\mu x, \mu y, \mu^2 t). \tag{1.3}$$

The homogeneous dimension with respect to this dilation is $Q = 2N+2$. Bases for the corresponding Lie algebra of the Heisenberg group $(\mathbb{R}^{2N+1}, \circ)$ are the left invariant vector fields of the form

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, 2, \ldots, N. \tag{1.4}$$

Denote the horizontal gradient by $\nabla_H = (X_1, \ldots, X_N, Y_1, \ldots, Y_N)$ and write

$$\mathrm{div}_H(\nu_1, \nu_2, \ldots, \nu_{2N}) = \sum_{j=1}^{N} (X_j \nu_j + Y_j \nu_{N+j}).$$

In this way, the sub-Laplacian $\triangle_H$ is expressed by

$$\triangle_H = \mathrm{div}_H(\nabla_H) = \sum_{j=1}^{N} (X_j^2 + Y_j^2).$$

And for $p > 1$, the sub-p-Laplacian $\triangle_{H,p}$ is defined as

$$\triangle_{H,p} u = \mathrm{div}_H(|\nabla_H u|^{p-2} \nabla_H u).$$

The space $\mathcal{D}_0^{1,p}(\mathbb{H}^N)$ is defined as the closure of $C_0^\infty(\mathbb{H}^N)$ under the norm $\|u\|_{\mathcal{D}_0^{1,p}} = \big(\int_{\mathbb{H}^N} |\nabla_H u|^p d\xi\big)^{1/p}$. The Hardy inequality on $\mathcal{D}_0^{1,p}(\mathbb{H}^N)$ is known as

$$\int_{\mathbb{H}^N} |\nabla_H u|^p d\xi \geq \Lambda_p \int_{\mathbb{H}^N} \frac{|z|^p |u|^p}{\rho^{2p}} d\xi,$$

where $\Lambda_p = \left(\frac{Q-p}{p}\right)^p$ is the best constant in the above inequality for $1 < p < Q$, see [12]. For $0 \leq s < p$, the following Hardy-Sobolev inequality

$$M_0 \Bigg(\int_{\mathbb{H}^N} \frac{|z|^s |u|^{p_*(s)}}{\rho^{2s}} d\xi \Bigg)^{\frac{p}{p_*(s)}} \leq \int_{\mathbb{H}^N} |\nabla_H u|^p d\xi, \quad u \in \mathcal{D}_0^{1,p}(\mathbb{H}^N),$$

holds for some positive constant M_0, see [12] for detailed proofs. In here, $p_*(s) = \frac{p(Q-s)}{Q-p}$ is called the critical exponent of the embedding $\mathcal{D}_0^{1,p}(\mathbb{H}^N) \hookrightarrow L^{p_*(s)}(\mathbb{H}^N, \frac{|z|^s}{\rho^{2s}} d\xi)$. On $\mathcal{D}_0^{1,p}(\mathbb{H}^N)$, we define the functionals

$$L(u) = \frac{1}{p} \int\limits_{\mathbb{H}^N} |\nabla_H u|^p d\xi - \frac{\lambda}{p} \int\limits_{\mathbb{H}^N} \frac{|z|^p |u|^p}{\rho^{2p}} d\xi - \frac{1}{p_*(s)} \int\limits_{\mathbb{H}^N} \frac{|z|^s |u|^{p_*(s)}}{\rho^{2s}} d\xi$$

and

$$I(u) = \int\limits_{\mathbb{H}^N} \left(|\nabla_H u|^p d\xi - \lambda \frac{|z|^p |u|^p}{\rho^{2p}} - \frac{|z|^s |u|^{p_*(s)}}{\rho^{2s}} \right) d\xi.$$

Then from the Hardy inequality and the Hardy-Sobolev inequality, one knows that both L and I are well defined. Denote the Nehari set by

$$\mathscr{N} = \left\{ u \in \mathcal{D}_0^{1,p}(\mathbb{H}^N) \backslash \{0\} \colon I(u) = 0 \right\}$$

and define

$$d = \inf \left\{ L(u) \colon u \in \mathscr{N} \right\}. \tag{1.5}$$

Definition 1.1. Let Γ be the set of solutions of (1.1). Namely,

$$\Gamma = \left\{ \phi \in \mathcal{D}_0^{1,p}(\mathbb{H}^N) \colon L'(\phi) = 0 \text{ and } \phi \neq 0 \right\}.$$

Let $\mathcal{G}$ be the set of least energy solutions of (1.1), that is,

$$\mathcal{G} = \left\{ u \in \Gamma \ : \ L(u) \leq L(v) \text{ for any } \ v \in \Gamma \right\}.$$

The following Theorem 1.2 is the main result of the present paper.

Theorem 1.2. *If* $1 < p < Q$, $-\infty < \lambda < \Lambda_p$ *and* $0 < s < p$, *then there is* $\phi \in \mathcal{N}$ *such that* $L(\phi) = d$. *Moreover,* ϕ *is a least energy solution of* (1.1).

We recall that a counterpart of (1.1) on $\mathbb{R}^N$ is of the form

$$-\mathrm{div}(|\nabla u|^{p-2} \nabla u) - \frac{\lambda}{|x|^p} |u|^{p-2} u = \frac{1}{|x|^s} |u|^{p^*(s)-2} u, \quad x \text{ in } \mathbb{R}^N \backslash \{0\}, \tag{1.6}$$

with $1 < p < N$, $-\infty < \lambda < ((N-p)/p)^p$, $p^*(s) = p(N-s)/(N-p)$ and $u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N)$. Here $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$ is defined as the closure of $C_0^\infty(\mathbb{R}^N)$ under the norm $\|u\|_{\mathcal{D}_0^{1,p}(\mathbb{R}^N)} := (\int_{\mathbb{R}^N} |\nabla u|^p dx)^{1/p}$. The existence and non-existence and multiplicity of solutions of (1.6) have been studied in the past several years. For instance, in the case of $p = 2$, $\lambda < ((N-2)/2)^2$ and $0 \leq s < 2$, these results can be found in [8, 22]. While for the case of $1 < p < N$, $0 < \lambda < ((N-p)/p)^p$ and $0 \leq s < p$, we refer the interested readers to the papers of [1, 2, 20]. Related results can be found also in [18]. In the setting of the Heisenberg group $\mathbb{H}^N$. Jerison *et al.* [15] firstly proved that

$$-\triangle_H \omega = \omega^{\frac{Q+2}{Q-2}}, \quad \omega > 0, \ \omega \in \mathcal{D}_0^{1,2}(\mathbb{H}^N) \tag{1.7}$$

possesses a solution

$$\omega_0(x,y,t) = \frac{K_0}{(t^2 + (1+|x|^2+|y|^2)^2)^{\frac{Q-2}{4}}},$$

where K_0 is a suitable positive constant. Moreover, every solution to (1.7) takes the form

$$\omega_{\mu\xi} = \mu^{\frac{Q-2}{2}} \omega_0 \circ \delta_\mu \circ \tau_\xi^{-1}.$$

Since the famous paper [15], there are a lot of papers dealing with the semilinear Dirichlet problem on the Heisenberg group. For instance, Citti [9] studies the equation

$$-\triangle_H u + au = u^{\frac{Q+2}{Q-2}} \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.8}$$

where Ω is a smooth bounded domain in $\mathbb{H}^N$. Since (1.8) involves a nonlinearity of critical growth, Citti [9] has proven a representation formula for the Palais-Smale sequence and then proves the existence of positive solutions of (1.8) under suitable conditions for a. Some results of Liouville type for semilinear equations on the Heisenberg group have been studied by Birindelli *et al.* [5, 6]. Uguzzoni [23] has proven a non-existence theorem for a semilinear Dirichlet problem involving critical nonlinearity on the half space of the Heisenberg group. Very recently, Han *et al.* [12] have proven a class of Hardy-Sobolev type inequalities on the H-type group and get the existence of a nontrivial solution of (1.1) in the case of $\lambda = 0$. We also refer the interested readers to [7] for other related results.

The equation (1.1) can be looked as a generalized model on the Heisenberg group and Theorem 1.2 generalizes partially the previous work in [12]. Theorem 1.2 seems to be the first existence result for the problem with double singularity and critical growth. The method of proving Theorem 1.2 is variational. Note that somehow we are facing the double critical case, since, for a bounded domain $\Omega \subset \mathbb{H}^N$ and $0 \in \Omega$, neither $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, \frac{|z|^p}{\rho^{2p}} d\xi)$ nor $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^{p_*(s)}(\Omega, \frac{|z|^s}{\rho^{2s}} d\xi)$ is compact. Hence the standard variational argument can not be used directly. Our idea of proving Theorem 1.2 is based on extending some techniques of the Nehari constraint used in [19]. The detailed proof of Theorem 1.2 will be carried out in Section 2.

Throughout this paper all integrals are taken over $\mathbb{H}^N$ unless stated otherwise. $\langle \cdot, \cdot \rangle_{\mathcal{D}_0^{1,p}}$ denotes the dual product between $\mathcal{D}_0^{1,p}(\mathbb{H}^N)$ and its dual space. The norm in $L^s(\mathbb{H}^N)$ is denoted by $\| \cdot \|_{L^s}$ and we define $E_\lambda(\cdot) = \int_{\mathbb{H}^N} (|\nabla_H \cdot|^p - \lambda \frac{|z|^p |\cdot|^p}{\rho^{2p}}) d\xi$. Positive constants are denoted by C or C_j $(j = 1, 2, \ldots)$, whose values may be different at different places.

2. EXISTENCE OF LEAST ENERGY SOLUTIONS OF (1.1)

This section is devoted to the proof of Theorem 1.2. We will always assume that the assumptions of Theorem 1.2 hold. Keep the notation of $E_\lambda(\cdot)$ in mind. We firstly give several lemmas which will be used in what follows.

Lemma 2.1. *For any $u \in \mathcal{D}_0^{1,p}(\mathbb{H}^N)\backslash\{0\}$, there is a unique $\theta_u > 0$ such that $\theta_u u \in \mathcal{N}$. Moreover, if $I(u) < 0$, then $0 < \theta_u < 1$.*

Proof. Since for $\theta > 0$,

$$L(\theta u) = \frac{\theta^p}{p} E_\lambda(u) - \frac{\theta^{p_*(s)}}{p_*(s)} \int\limits_{\mathbb{H}^N} \frac{|z|^s |u|^{p_*(s)}}{\rho^{2s}} d\xi,$$

one obtains from direct computation that there is a

$$\theta_u = (E_\lambda(u))^{\frac{1}{p_*(s)-p}} \left(\int \frac{|z|^s |u|^{p_*(s)}}{\rho^{2s}} d\xi \right)^{-\frac{1}{p_*(s)-p}}$$

such that $\theta_u u \in \mathcal{N}$. The structure of $L(\theta u)$ implies that this θ_u is unique for each $u \in \mathcal{D}_0^{1,p}(\mathbb{H}^N)\backslash\{0\}$. From the expression

$$I(u) = E_\lambda(u) - \int \frac{|z|^s |u|^{p_*(s)}}{\rho^{2s}} d\xi,$$

we know that if $I(u) < 0$, i.e.,

$$E_\lambda(u) < \int \frac{|z|^s |u|^{p_*(s)}}{\rho^{2s}} d\xi,$$

then $0 < \theta_u < 1$. □

Lemma 2.2. *The set $\mathcal{N}$ is a manifold and there exists $c_0 > 0$ such that for all $u \in \mathcal{N}$, $E_\lambda(u) \geq c_0$.*

Proof. In the first place, we point out that $\mathcal{N} \neq \emptyset$ follows from the previous lemma. For any $u \in \mathcal{N}$,

$$\begin{aligned} \langle I'(u), u \rangle_{\mathcal{D}_0^{1,p}} &= pE_\lambda(u) - p_*(s) \int \frac{|z|^s |u|^{p_*(s)}}{\rho^{2s}} d\xi = \\ &= (p - p_*(s)) \int \frac{|z|^s |u|^{p_*(s)}}{\rho^{2s}} d\xi < 0, \end{aligned}$$

which implies that $\mathcal{N}$ is a manifold. In the second place, from the Hardy-Sobolev inequality, one obtains that

$$E_\lambda(u) = \int \frac{|z|^s}{\rho^{2s}} |u|^{p_*(s)} d\xi \leq M_0^{-\frac{p_*(s)}{p}} \left(\int |\nabla_H u|^p d\xi \right)^{\frac{p_*(s)}{p}}.$$

If $\lambda \leq 0$, then

$$\int |\nabla_H u|^p d\xi \leq E_\lambda(u).$$

If $0 < \lambda < \Lambda_p$, then

$$\int |\nabla_H u|^p d\xi \leq \left(1 - \frac{\lambda}{\Lambda_p}\right)^{-1} E_\lambda(u).$$

Therefore for $-\infty < \lambda < \Lambda_p$, one obtains that

$$E_\lambda(u) \le M_0^{-\frac{p_*(s)}{p}} \left(\frac{2\Lambda_p - \lambda}{\Lambda_p - \lambda}\right)^{\frac{p_*(s)}{p}} (E_\lambda(u))^{\frac{p_*(s)}{p}}.$$

Therefore we can deduce that there is a $c_0 > 0$ such that $E_\lambda(u) \ge c_0$. □

Lemma 2.3. *If $v \in \mathcal{N}$ and $d = L(v)$, then v is a least energy solution of the equation* (1.1)*.*

Proof. Since v is a minimizer of the minimum d, we obtain from the Lagrange multiplier rule that there is $\theta \in \mathbb{R}$ such that for any $\psi \in \mathcal{D}_0^{1,p}(\mathbb{H}^N)$,

$$\langle L'(v), \psi\rangle_{\mathcal{D}_0^{1,p}} = \theta \langle I'(v), \psi\rangle_{\mathcal{D}_0^{1,p}}.$$

Note that

$$\begin{aligned}\langle I'(v), v\rangle_{\mathcal{D}_0^{1,p}} &= pE_\lambda(u) - p_*(s)\int \frac{|z|^s}{\rho^{2s}}|u|^{p_*(s)} d\xi = \\ &= (p - p_*(s))\int \frac{|z|^s}{\rho^{2s}}|u|^{p_*(s)} d\xi < 0\end{aligned}$$

and $\langle L'(v), v\rangle_{\mathcal{D}_0^{1,p}} = I(v) = 0$. We get that $\theta = 0$. Hence $L'(v) = 0$. According to Definition 1.1, one knows easily that v is a least energy solution of (1.1). □

Lemma 2.4. *If $u \in \mathcal{D}_0^{1,p}(\mathbb{H}^N)$ and $h \in C_0^\infty(\mathbb{H}^N)$, then*

$$\begin{aligned}\int_{\mathbb{H}^N} \frac{|z|^s}{\rho^{2s}}|h|^p|u|^{p_*(s)} d\xi \le M_0^{-1}\left(\frac{2\Lambda_p - \lambda}{\Lambda_p - \lambda}\right) E_\lambda(|h|u)\times \\ \times\left(\int_{supp(h)} \frac{|z|^s}{\rho^{2s}}|u|^{p_*(s)} d\xi\right)^{\frac{p_*(s)-p}{p_*(s)}}.\end{aligned}$$

Proof. Note that for $\lambda < \Lambda_p$, one obtains from the Hardy inequality and an argument similar to those in the proof of Lemma 2.2 that

$$\int |\nabla_H u|^p d\xi \le \left(\frac{2\Lambda_p - \lambda}{\Lambda_p - \lambda}\right) E_\lambda(u). \tag{2.1}$$

Using the Hölder inequality, one can obtain that

$$\begin{aligned}\int \frac{|z|^s}{\rho^{2s}}|u|^{p_*(s)}|h|^p d\xi &\le \left(\int \frac{|z|^s}{\rho^{2s}}|hu|^{p_*(s)} d\xi\right)^{\frac{p}{p_*(s)}} \left(\int_{supp(h)} \frac{|z|^s}{\rho^{2s}}|u|^{p_*(s)} d\xi\right)^{\frac{p_*(s)-p}{p_*(s)}} \le \\ &\le M_0^{-1}\int |\nabla_H(|h|u)|^p d\xi \left(\int_{supp(h)} \frac{|z|^s}{\rho^{2s}}|u|^{p_*(s)} d\xi\right)^{\frac{p_*(s)-p}{p_*(s)}}.\end{aligned}$$

Combining this with (2.1), one deduces the conclusion of Lemma 2.4. □

Lemma 2.5 ([11]). *For any smooth bounded domain $\Omega \subset \mathbb{H}^N$, the inclusion $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact.*

Lemma 2.6. *If $h \in C_0^\infty(\mathbb{H}^N)$, then for any $u \in \mathcal{D}_0^{1,p}(\mathbb{H}^N)$, $|h|^p u \in \mathcal{D}_0^{1,p}(\mathbb{H}^N)$.*

Proof. Since $C_0^\infty(\mathbb{H}^N)$ is dense in $\mathcal{D}_0^{1,p}(\mathbb{H}^N)$, there is a sequence $(\psi_n)_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{H}^N)$ such that $\psi_n \to u$ strongly in $\mathcal{D}_0^{1,p}(\mathbb{H}^N)$. Using Lemma 2.5 and the fact that *supp*(h) is a compact subset of $\mathbb{H}^N$, one can get the conclusion by a direct calculation. □

Lemma 2.7. *For every $u \in \mathcal{N}$, there is $\varepsilon_0 > 0$ and a differentiable functional $\mu(w)$ defined for $w \in \mathcal{D}_0^{1,p}(\mathbb{H}^N)$ with $\|w\|_{\mathcal{D}_0^{1,p}} < \varepsilon_0$ such that $\mu(0) = 1$, $\mu(w)(u - w) \in \mathcal{N}$ and for each $\psi \in \mathcal{D}^{1,p}(\mathbb{H}^N)$,*

$$\langle \mu'(0), \psi \rangle_{\mathcal{D}_0^{1,p}} = \frac{\langle E'_\lambda(u), \psi \rangle_{\mathcal{D}_0^{1,p}} - p_*(s) \int \frac{|z|^s |u|^{p_*(s)-2} u\psi}{\rho^{2s}} d\xi}{(p-1)E_\lambda(u) - (p_*(s) - 1) \int \frac{|z|^s |u|^{p_*(s)}}{\rho^{2s}} d\xi}. \tag{2.2}$$

Proof. For $\mu > 0$ and $w \in \mathcal{D}_0^{1,p}(\mathbb{H}^N)$, we define a function

$$F(\mu, w) = \mu^{p-1} E_\lambda(u - w) - \mu^{p_*(s)-1} \int \frac{|z|^s |u - w|^{p_*(s)}}{\rho^{2s}} d\xi.$$

Since $u \in \mathcal{N}$, we have $F(1, 0) = I(u) = 0$ and

$$F_\mu(1, 0) = (p-1)E_\lambda(u) - (p_*(s) - 1) \int \frac{|z|^s |u - w|^{p_*(s)}}{\rho^{2s}} d\xi = \\ = (p - p_*(s)) \int \frac{|z|^s |u - w|^{p_*(s)}}{\rho^{2s}} d\xi < 0.$$

Applying the implicit function theorem at the point $(1, 0)$, we obtain a $\varepsilon_0 > 0$ and a differentiable functional $\mu \equiv \mu(w)$ defined for $w \in \mathcal{D}_0^{1,p}(\mathbb{H}^N)$ such that $\mu(0) = 1$, $\mu(w)(u - w) \in \mathcal{N}$. Moreover, from $F(\mu(w), w) = 0$, we obtain from direct calculation that (2.2) holds. □

Theorem 2.8. *Under the assumptions of Theorem 1.2, there is $\phi \in \mathcal{N}$ such that $d = L(\phi)$.*

Proof. The proof will be divided into several steps.

Step 1. Applying the Ekeland variational principle [3] (see also [21]), one has a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{N}$ and u_n satisfies the following properties:

$$L(u_n) < d + \frac{1}{n}, \tag{2.3}$$

$$L(w) \geq L(u_n) - \frac{1}{n}\|w - u_n\| \qquad \text{for any} \quad w \in \mathcal{D}_0^{1,p}(\mathbb{H}^N). \tag{2.4}$$

Since $u_n \in \mathcal{N}$, we know from the Hardy-Sobolev inequality that there are positive constants C_1, C_2 such that

$$C_1 \leq \|u_n\|_{\mathcal{D}_0^{1,p}} \leq C_2.$$

Applying Lemma 2.7 to each u_n, we get a constant ε_{0n} and a differentiable function $\mu_n \equiv \mu_n(w)$ defined for $w \in \mathcal{D}_0^{1,p}(\mathbb{H}^N)$ with $\|w\|_{\mathcal{D}_0^{1,p}} < \varepsilon_{0n}$, such that $\mu_n(0) = 1$ and $\mu_n(w)(u_n - w) \in \mathcal{N}$. By Lemma 2.2, Lemma 2.7 and the fact that $I(u_n) = 0$, one has

$$\|\mu_n'(0)\|_{(\mathcal{D}_0^{1,p}(\mathbb{H}^N))^*} \leq \frac{C_3}{\left|(p-1)E_\lambda(u_n) - (p_*(s)-1)\int \frac{|z|^s|u|^{p_*(s)}}{\rho^{2s}}d\xi\right|} \leq C_4,$$

where $(\mathcal{D}_0^{1,p}(\mathbb{H}^N))^*$ is the dual space of $\mathcal{D}_0^{1,p}(\mathbb{H}^N)$. Next, we prove that $L'(u_n) \to 0$ as $n \to \infty$. Let $0 < \varepsilon < \varepsilon_{0n}$. Set $w_\varepsilon = \varepsilon v$ with $\|v\|_{\mathcal{D}_0^{1,p}} = 1$ and $v_\varepsilon \equiv \mu_n(w_\varepsilon)(u_n - w_\varepsilon)$, then $v_\varepsilon \in \mathcal{N}$. Using Taylor expansion, we obtain that

$$\begin{aligned}
\frac{1}{n}\|u_n - v_\varepsilon\|_{\mathcal{D}_0^{1,p}} &\geq L(u_n) - L(v_\varepsilon) = \\
&= \langle L'(u_n), u_n - v_\varepsilon\rangle_{\mathcal{D}_0^{1,p}} + o(\|u_n - v_\varepsilon\|_{\mathcal{D}_0^{1,p}}) = \\
&= (1-\mu_n(w_\varepsilon))\langle L'(u_n), u_n\rangle_{\mathcal{D}_0^{1,p}} + \varepsilon\mu_n(w_\varepsilon)\langle L'(u_n), v\rangle_{\mathcal{D}_0^{1,p}} + \\
&\quad + o(\|u_n - v_\varepsilon\|_{\mathcal{D}_0^{1,p}}) = \\
&= \varepsilon\mu_n(w_\varepsilon)\langle L'(u_n), v\rangle_{\mathcal{D}_0^{1,p}} + o(\|u_n - v_\varepsilon\|_{\mathcal{D}_0^{1,p}}).
\end{aligned} \tag{2.5}$$

Since

$$\|u_n - v_\varepsilon\|_{\mathcal{D}_0^{1,p}} \leq |1-\mu_n(w_\varepsilon)|\|u_n\|_{\mathcal{D}_0^{1,p}} + C_5\varepsilon|\mu_n(w_\varepsilon)|$$

and $\mu_n(0) = 1$, we obtain that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{\|u_n - v_\varepsilon\|_{\mathcal{D}_0^{1,p}}}{\varepsilon} &\leq C_6\lim_{\varepsilon\to 0}\left(\frac{|\mu_n(w_\varepsilon) - \mu_n(0)|}{\varepsilon} + C_7|\mu_n(w_\varepsilon)|\right) \leq \\
&\leq \left|\langle\mu_n'(0), v\rangle_{\mathcal{D}_0^{1,p}}\right| + C_8 \leq C_9.
\end{aligned}$$

Therefore dividing by ε the inequality (2.5) and passing to the limit as $\varepsilon \to 0$, we deduce from the preceding inequality that there exists a positive constant C_{10} such that

$$\langle L'(u_n), v\rangle_{\mathcal{D}_0^{1,p}} \leq \frac{C_{10}}{n}.$$

Since v is arbitrary, one gets a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{N}$ such that

$$L'(u_n) \to 0, \quad L(u_n) \to d \quad \text{as} \quad n \to \infty.$$

Step 2. From $I(u_n) = 0$ and the Hardy inequality, one deduces from an argument similar to those in the proof of Lemma 2.2 that there is a $c_0 > 0$ such that

$$E_\lambda(u_n) = \int \frac{|z|^s}{\rho^{2s}}|u_n|^{p_*(s)}d\xi \geq c_0. \tag{2.6}$$

Denote $\tilde{c} = M_0(\frac{2\Lambda_p - \lambda}{\Lambda_p - \lambda})$ and choose R_0 such that

$$0 < R_0 < \min\left\{c_0, (2\tilde{c})^{\frac{p_*(s)}{p-p_*(s)}}\right\}$$

and

$$\limsup_{n\to\infty} \int \frac{|z|^s}{\rho^{2s}} |u_n|^{p_*(s)} d\xi > R_0.$$

Let $r_n > 0$ be such that

$$\int\limits_{B(0,r_n)} \frac{|z|^s}{\rho^{2s}} |u_n|^{p_*(s)} d\xi = R_0.$$

Defining

$$\tilde{u}_n(\xi) := r_n^{\frac{Q-p}{p}} u_n(r_n x, r_n y, r_n^2 t),$$

then there holds

$$R_0 = \int\limits_{B(0,1)} \frac{|z|^s}{\rho^{2s}} |\tilde{u}_n|^{p_*(s)} d\xi.$$

Moreover, by direct calculation, one has

$$\int |\nabla_H \tilde{u}_n|^p d\xi = \int |\nabla_H u_n|^p d\xi, \qquad \int \frac{|z|^p}{\rho^{2s}} |\tilde{u}_n|^p d\xi = \int \frac{|z|^p}{\rho^{2s}} |u_n|^p d\xi$$

and

$$\int \frac{|z|^s}{\rho^{2s}} |\tilde{u}_n|^{p_*(s)} d\xi = \int \frac{|z|^s}{\rho^{2s}} |u_n|^{p_*(s)} d\xi.$$

Therefore the functionals L and I are invariant under the above mentioned transformation, i.e.,

$$L(\tilde{u}_n) = L(u_n) \quad \text{and} \quad I(\tilde{u}_n) = I(u_n).$$

Note that if $h \in \mathcal{D}_0^{1,p}(\mathbb{H}^N)$, then for any $r > 0$, one has

$$\breve{h} := r^{-\frac{Q-p}{p}} h\left(\frac{x}{r}, \frac{y}{r}, \frac{t}{r^2}\right) \in \mathcal{D}_0^{1,p}(\mathbb{H}^N)$$

and $\|\breve{h}\|_{\mathcal{D}_0^{1,p}} = \|h\|_{\mathcal{D}_0^{1,p}}$. Hence for any $\psi \in \mathcal{D}_0^{1,p}(\mathbb{H}^N)$, denoting

$$\breve{\psi}_n := r_n^{-\frac{Q-p}{p}} h\left(\frac{x}{r_n}, \frac{y}{r_n}, \frac{t}{r_n^2}\right),$$

we have that $\breve{\psi}_n \in \mathcal{D}_0^{1,p}(\mathbb{H}^N)$ and $\|\breve{\psi}_n\|_{\mathcal{D}_0^{1,p}} = \|\psi\|_{\mathcal{D}_0^{1,p}}$. Moreover,

$$\begin{aligned}\langle L'(\tilde{u}_n), \psi\rangle_{\mathcal{D}_0^{1,p}} = \int \left(|\nabla_H \tilde{u}_n|^{p-2} \nabla_H \tilde{u}_n \nabla_H \psi - \lambda \frac{|z|^p}{\rho^{2p}} |\tilde{u}_n|^{p-2} \tilde{u}_n \psi\right) d\xi - \\ - \int \frac{|z|^s}{\rho^{2s}} |\tilde{u}_n|^{p_*(s)-2} \tilde{u}_n \psi d\xi = \\ = \langle L'(u_n), \breve{\psi}_n\rangle_{\mathcal{D}_0^{1,p}} = o(1)\|\breve{\psi}_n\|_{\mathcal{D}_0^{1,p}} = o(1)\|\psi\|_{\mathcal{D}_0^{1,p}}.\end{aligned}$$

This proves that $L'(\tilde{u}_n) \to 0$ as $n \to \infty$.

Step 3. Since $(\tilde{u}_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{D}_0^{1,p}(\mathbb{H}^N)$, we may assume that $\tilde{u}_n \rightharpoonup \phi$ weakly in $\mathcal{D}_0^{1,p}(\mathbb{H}^N)$. We claim that $\phi \neq 0$. Arguing by contradiction, we assume that $\phi = 0$. Then, we firstly point out that from Lemma 2.5, $\tilde{u}_n \to \phi$ strongly in $L^p_{loc}(\mathbb{H}^N)$. Secondly, let $h \in C_0^\infty(\mathbb{H}^N)$ and $supp(h) \in B(0,1)$. Using the elementary inequality

$$\left||A+B|^p - |A|^p\right| \leq C_{12}\left(|A|^{p-1}|B| + |B|^p\right), \quad \text{for any } A, B \in \mathbb{R}^{2N},$$

we obtain that

$$\begin{aligned}
&\left|\int \left(|\nabla_H(|h|\tilde{u}_n)|^p - |h\nabla_H \tilde{u}_n|^p\right) d\xi\right| \leq \\
&\leq C\int \left(|h|^{p-1}|\nabla_H \tilde{u}_n|^{p-1}|\nabla_H|h|||\tilde{u}_n| + |\nabla_H|h||^p|\tilde{u}_n|^p\right) d\xi \leq \\
&\leq C \int\limits_{supp(h)} |\nabla_H \tilde{u}_n|^{p-1}|\tilde{u}_n| d\xi + \int\limits_{supp(h)} |\tilde{u}_n|^p d\xi \leq \\
&\leq \left(\int\limits_{supp(h)} |\nabla_H \tilde{u}_n|^p d\xi\right)^{\frac{p-1}{p}} \left(\int\limits_{supp(h)} |\tilde{u}_n|^p d\xi\right)^{\frac{1}{p}} + \\
&\quad + \int\limits_{supp(h)} |\tilde{u}_n|^p d\xi \to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

where we have used the assumption $\phi = 0$, the fact that $\tilde{u}_n \to \phi$ strongly in $L^p_{loc}(\mathbb{H}^N)$ and $(\tilde{u}_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{D}_0^{1,p}(\mathbb{H}^N)$.

Therefore for n large enough, we proved that

$$\int |\nabla_H(|h|\tilde{u}_n)|^p d\xi = \int |h|^p |\nabla_H \tilde{u}_n|^p d\xi + o(1). \tag{2.7}$$

Now using the fact that $\langle L'(\tilde{u}_n), \psi\rangle_{\mathcal{D}_0^{1,p}} = o(1)\|\psi\|_{\mathcal{D}_0^{1,p}}$ and substituting ψ by $|h|^p\tilde{u}_n$, we get that

$$\begin{aligned}
\langle L'(\tilde{u}_n), |h|^p\tilde{u}_n\rangle_{\mathcal{D}_0^{1,p}} = &\int |\nabla_H \tilde{u}_n|^{p-2}\nabla_H \tilde{u}_n(\tilde{u}_n\nabla_H(|h|^p) + |h|^p\nabla_H \tilde{u}_n) d\xi - \\
&- \lambda \int \frac{|z|^p}{\rho^{2p}}|\tilde{u}_n|^p|h|^p d\xi - \int \frac{|z|^s}{\rho^{2s}}|\tilde{u}_n|^{p_*(s)}|h|^p d\xi.
\end{aligned}$$

Combining this with (2.7), Lemma 2.4 and the choice of R_0, we obtain that

$$E_\lambda(|h|\tilde{u}_n) = o(1)\||h|^p\tilde{u}_n\| - \int |\nabla_H \tilde{u}_n|^{p-2}\nabla_H\tilde{u}_n\nabla_H(|h|^p)\tilde{u}_n d\xi + \\ + \int \frac{|z|^s}{\rho^{2s}}|\tilde{u}_n|^{p_*(s)}|h|^p d\xi + o(1) \leq \\ \leq \left(\int\limits_{supp(h)} |\nabla_H\tilde{u}_n|^p d\xi\right)^{\frac{p-1}{p}} \left(\int\limits_{supp(h)} |\tilde{u}_n|^p d\xi\right)^{\frac{1}{p}} + \\ + \tilde{c}E_\lambda(|h|\tilde{u}_n)\left(\int\limits_{supp(h)} \frac{|z|^s}{\rho^{2s}}|\tilde{u}_n|^{p_*(s)}d\xi\right)^{\frac{p_*(s)-p}{p_*(s)}} + o(1) < \\ < \frac{1}{2}E_\lambda(|h|\tilde{u}_n) + o(1),$$

where we have used again the fact that $\tilde{u}_n \to \phi \equiv 0$ strongly in $L^p_{loc}(\mathbb{H}^N)$ and $(\tilde{u}_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{D}^{1,p}_0(\mathbb{H}^N)$. It follows that $E_\lambda(|h|\tilde{u}_n) \to 0$ as $n \to \infty$. Note that for $-\infty < \lambda < \Lambda_p$,

$$\int |\nabla_H(|h|\tilde{u}_n)|^p d\xi \leq \left(\frac{2\Lambda_p - \lambda}{\Lambda_p - \lambda}\right) E_\lambda(|h|\tilde{u}_n),$$

which implies that $\int |\nabla_H(|h|\tilde{u}_n)|^p d\xi \to 0$ as $n \to \infty$. So from the Hardy-Sobolev inequality, we deduce that $\int \frac{|z|^s}{\rho^{2s}}||h|\tilde{u}_n|^{p_*(s)}d\xi \to 0$ as $n \to \infty$. From the choice of h, one obtains that for each $0 < r < 1$,

$$\int\limits_{B(0,r)} \frac{|z|^s}{\rho^{2s}}||h|\tilde{u}_n|^{p_*(s)}d\xi \to 0 \quad \text{as} \quad n \to \infty.$$

Next, choosing a function h such that $\text{supp}(h) \subset B(\xi, \frac{1}{2})$ with $\xi \in \partial B(0,1)$, then similar to the previous computation, one deduces that

$$E_\lambda(|h|\tilde{u}_n) \leq \left(\int\limits_{supp(h)} |\nabla_H\tilde{u}_n|^p d\xi\right)^{\frac{p-1}{p}} \left(\int\limits_{supp(h)} |\tilde{u}_n|^p d\xi\right)^{\frac{1}{p}} + \\ + \tilde{c}E_\lambda(|h|\tilde{u}_n)\left(\int\limits_{supp(h)} \frac{|z|^s}{\rho^{2s}}|\tilde{u}_n|^{p_*(s)}d\xi\right)^{\frac{p_*(s)-p}{p_*(s)}} + o(1) < \frac{1}{2}E_\lambda(|h|\tilde{u}_n) + o(1),$$

where we have used the fact that $p_*(s) < Qp/(Q-p)$ and the locally compact embedding from $\mathcal{D}^{1,p}_0(\Omega) \hookrightarrow L^{p_*(s)}(\Omega)$. Combining this with the previous discussion, one obtains that $\int_{B(0,1)} \frac{|z|^s}{\rho^{2s}} ||h|\tilde{u}_n|^{p_*(s)} d\xi \to 0$ as $n \to \infty$. This is a contradiction. Therefore $\phi \neq 0$.

Step 4. In this step, we will prove that there are some finite points $\{\xi_1, \ldots, \xi_m\}$, such that

$$\tilde{u}_n \to \phi \quad \text{strongly in} \quad \mathcal{D}^{1,p}_{0,loc}(\mathbb{H}^N \backslash \{0, \xi_1, \ldots, \xi_m\}). \tag{2.8}$$

The proof of (2.8) will be divided into several steps. In the first place, from $\tilde{u}_n \rightharpoonup \phi$ weakly in $\mathcal{D}^{1,p}_0(\mathbb{H}^N)$, one has that $\tilde{u}_n \to \phi$ a.e. in $\mathbb{H}^N$. Note that $\nabla_H \tilde{u}_n \in (L^p(\mathbb{H}^N))^{2N}$ and $(|\nabla_H \tilde{u}_n|^{p-2} \nabla_H \tilde{u}_n)_{n \in \mathbb{N}}$ is bounded in $(L^{p'}(\mathbb{H}^N))^{2N}$ with $p' = \frac{p}{p-1}$. We may assume that there is $T \in (L^{p'}(\mathbb{H}^N))^{2N}$ such that

$$|\nabla_H \tilde{u}_n|^{p-2} \nabla_H \tilde{u}_n \rightharpoonup T \quad \text{weakly in} \quad (L^{p'}(\mathbb{H}^N))^{2N}.$$

By letting $n \to \infty$, one gets immediately that

$$\int T \nabla_H \varphi d\xi = \int \left(\frac{\lambda |z|^p |\phi|^{p-2} \phi \varphi}{\rho^{2p}} + \frac{|z|^s |\phi|^{p_*(s)-2} \phi \varphi}{\rho^{2s}} \right) d\xi \tag{2.9}$$

holds for any $\varphi \in \mathcal{D}^{1,p}_0(\mathbb{H}^N)$. Since $(\tilde{u}_n)_{n \in \mathbb{N}}$ is bounded in $\mathcal{D}^{1,p}_0(\mathbb{H}^N)$, the concentration compactness principle [16] (see also a refined version in [12]) implies that there is an at most countable set J such that

$$\begin{cases} (1) & |\nabla_H \tilde{u}_n|^p \rightharpoonup d\alpha \geq |\nabla_H \phi|^p + \sum_{j \in J} \alpha_j \chi_{\xi_j} + \alpha_0 \chi_0, \\ (2) & \frac{|z|^s}{\rho^{2s}} |\tilde{u}_n|^{p_*(s)} \rightharpoonup d\beta = \frac{|z|^s}{\rho^{2s}} |\phi|^{p_*(s)} + \sum_{j \in J} \beta_j \chi_{\xi_j} + \beta_0 \chi_0, \\ (3) & \alpha_j \geq M_0 \beta_j^{p/p_*(s)}, \\ (4) & \frac{|z|^p}{\rho^{2p}} |\tilde{u}_n|^p \rightharpoonup d\gamma = \frac{|z|^p}{\rho^{2p}} |\phi|^p + \gamma_0 \chi_0, \\ (5) & \Lambda_p \gamma_0 \leq \alpha_0, \end{cases} \tag{CCP}$$

where χ_ξ is the Dirac function at ξ. In the second place, we claim that J is a finite set. Indeed, from $\langle L'(\tilde{u}_n), \varphi \rangle_{\mathcal{D}^{1,p}_0} = o(1) \|\varphi\|_{\mathcal{D}^{1,p}_0}$ and choosing $\varphi = \psi \tilde{u}_n$, one gets that

$$\begin{aligned} &\int \Big(|\nabla_H \tilde{u}_n|^p \psi + \tilde{u}_n |\nabla_H \tilde{u}_n|^{p-2} \nabla_H \tilde{u}_n \nabla_H \psi \Big) d\xi = \\ &= \int \Big(\lambda \frac{|z|^p}{\rho^{2p}} |\tilde{u}_n|^p \psi + \frac{|z|^s}{\rho^{2s}} |\tilde{u}_n|^{p_*(s)} \psi \Big) d\xi + o(1). \end{aligned} \tag{2.10}$$

It is now deduced by letting $n \to \infty$ that

$$\int \psi d\alpha + \int \phi T \nabla_H \psi d\xi = \int \psi d\beta + \lambda \int \psi d\gamma. \tag{2.11}$$

On the other hand, substituting φ by $\psi\phi$ in (2.9), we have that

$$\int \Big(\phi T \nabla_H \psi + \psi T \nabla_H \phi \Big) = \int \Big(\lambda \frac{|z|^p}{\rho^{2p}} |\phi|^p \psi + \frac{|z|^s}{\rho^{2s}} |\phi|^{p_*(s)} \psi \Big) d\xi. \tag{2.12}$$

Concentrating ψ at ξ_j (here and in the sequel, a function ψ is called "concentrating at ξ_j" if $\psi \in C^1_0(\mathbb{H}^N)$, $\psi(\xi) = 1$ for $|\xi - \xi_j| \leq r$, $\psi(\xi) = 0$ for $|\xi - \xi_j| \geq 2r$,

$|\nabla_H \psi| \leq 4r^{-\frac{2N+1}{Q}}$ and r small), we obtain from (2.11) and (2.12) that $\alpha_j \leq \beta_j$. Combining this with (3) of (CCP) one can deduce that

$$\text{either} \quad \beta_j = 0 \quad \text{or} \quad \beta_j \geq M_0^{\frac{Q-s}{p-s}}. \tag{2.13}$$

Similarly, concentrating ψ at $\xi_0 = 0$ in (2.11) and (2.12), we have that

$$\alpha_0 - \lambda\gamma_0 \leq \beta_0. \tag{2.14}$$

On the other hand, from the Hardy inequality and the Hardy-Sobolev inequality, there is a positive constant M_λ such that for any $u \in \mathcal{D}_0^{1,p}(\mathbb{H}^N)$,

$$\int \Big(|\nabla_H u|^p - \lambda \frac{|z|^p}{\rho^{2p}} |u|^p \Big) d\xi \geq M_\lambda \Big(\int \frac{|z|^s}{\rho^{2s}} |u|^{p_*(s)} d\xi \Big)^{\frac{p}{p_*(s)}}.$$

We have that

$$\int \Big(|\nabla_H (\tilde{u}_n \psi)|^p - \lambda \frac{|z|^p}{\rho^{2p}} |\tilde{u}_n \psi|^p \Big) d\xi \geq M_\lambda \Big(\int \frac{|z|^s}{\rho^{2s}} |\tilde{u}_n \psi|^{p_*(s)} d\xi \Big)^{\frac{p}{p_*(s)}}.$$

Therefore

$$\int \Big(|\tilde{u}_n \nabla_H \psi + \psi \nabla_H \tilde{u}_n|^p \Big) d\xi \geq \lambda \int \frac{|z|^p}{\rho^{2p}} |\tilde{u}_n \psi|^p d\xi + M_\lambda \Big(\int \frac{|z|^s}{\rho^{2s}} |\tilde{u}_n \psi|^{p_*(s)} d\xi \Big)^{\frac{p}{p_*(s)}}.$$

Note that for any $A, B \in \mathbb{R}^{2N}$, there holds

$$\big| |A+B|^p - |A|^p \big| \leq K(|A|^{p-1}|B| + |B|^p).$$

Thus

$$\int \Big(|\psi \nabla_H \tilde{u}_n + \tilde{u}_n \nabla_H \psi|^p - |\psi \nabla_H \tilde{u}_n|^p \Big) d\xi \leq$$
$$\leq K \int \Big(|\psi \nabla_H \tilde{u}_n|^{p-1} |\tilde{u}_n \nabla_H \psi| + |\tilde{u}_n \nabla \psi|^p \Big) d\xi.$$

Now, it is deduced from the Hölder inequality that

$$\int |\psi \nabla_H \tilde{u}_n|^{p-1} |\tilde{u}_n \nabla_H \psi| d\xi \leq$$
$$\leq C_{14} \Big(\int\limits_{r \leq |x| \leq 2r} |\nabla_H \psi|^p |\tilde{u}_n|^p d\xi \Big)^{\frac{1}{p}} \Big(\int\limits_{r \leq |x| \leq 2r} |\nabla_H \tilde{u}_n|^p d\xi \Big)^{\frac{p-1}{p}} \leq$$
$$\leq C_{14} \Big(\int\limits_{r \leq |x| \leq 2r} |\nabla \tilde{u}_n|^p d\xi \Big)^{\frac{p-1}{p}} \times$$
$$\times \Bigg[\Big(\int\limits_{r \leq |x| \leq 2r} |\nabla_H \psi|^Q d\xi \Big)^{\frac{p}{Q}} \Big(\int\limits_{r \leq |x| \leq 2r} |\tilde{u}_n|^{p_*(0)} d\xi \Big)^{\frac{Q-p}{Q}} \Bigg]^{\frac{1}{p}} \leq$$
$$\leq C_{15} \Big(\int\limits_{r \leq |x| \leq 2r} |\nabla_H \tilde{u}_n|^p d\xi \Big)^{\frac{p-1}{p}} \Big(\int\limits_{r \leq |x| \leq 2r} |\tilde{u}_n|^{p_*(0)} \Big)^{\frac{Q-p}{Qp}}.$$

Therefore

$$\lim_{r\to 0}\limsup_{n\to\infty}\int |\psi\nabla_H\tilde{u}_n|^{p-1}|\tilde{u}_n\nabla_H\psi|d\xi=0. \tag{2.15}$$

Similarly, one can prove that

$$\int |\tilde{u}_n\nabla_H\phi|^p\le C_{16}\bigg(\int\limits_{r\le|x|\le 2r}|u_n|^{p_*(0)}\bigg)^{\frac{Q-p}{Qp}}\to 0\quad\text{as}\quad r\to 0.$$

This and (2.15) imply that

$$\lim_{r\to 0}\limsup_{n\to\infty}\int\bigg(|\psi\nabla_H\tilde{u}_n+\tilde{u}_n\nabla_H\psi|^p-|\psi\nabla_H\tilde{u}_n|^p\bigg)d\xi=0$$

and hence

$$\alpha_0-\lambda\gamma_0\ge M_\lambda\beta_0^{p/p_*(s)}. \tag{2.16}$$

Combining (2.16) with (2.14), we obtain that

$$\text{either}\quad \beta_0=0\quad\text{or}\quad \beta_0\ge M_\lambda^{(Q-s)/(p-s)}. \tag{2.17}$$

Therefore J must be a finite set.

In the third place, for any bounded domain $\Omega\subset\mathbb{H}^N$ and $J\cup\{0\}\subset\Omega$, choosing $\psi\in C^1(\bar{\Omega})$ such that $\psi\ge 0$, $\psi(\xi_j)=0$ for $j\in J\cup\{0\}$, we have

$$\begin{aligned}\int\frac{|z|^s}{\rho^{2s}}|\psi\tilde{u}_n|^{p_*(s)}d\xi\to\int\frac{|z|^s}{\rho^{2s}}|\psi\phi|^{p_*(s)}d\xi+\sum_{j\in J}\beta_j\psi^{p_*(s)}(\xi_j)+\beta_0\psi^{p_*(s)}(0)=\\ =\int\frac{|z|^s}{\rho^{2s}}|\psi\phi|^{p_*(s)}d\xi.\end{aligned} \tag{2.18}$$

It follows from the uniform convexity of $L^{p_*(s)}\Big(\mathbb{H}^N;\frac{|z|^s}{\rho^{2s}}d\xi\Big)$ that

$$\psi\tilde{u}_n\to\psi\phi\quad\text{in}\quad L^{p_*(s)}\bigg(\mathbb{H}^N;\frac{|z|^s}{\rho^{2s}}d\xi\bigg). \tag{2.19}$$

Similarly one can obtain that

$$\psi\tilde{u}_n\to\psi\phi\quad\text{in}\quad L^p\bigg(\mathbb{H}^N;\frac{|z|^p}{\rho^{2p}}d\xi\bigg). \tag{2.20}$$

From $\langle L'(\tilde{u}_n),\varphi\rangle_{\mathcal{D}_0^{1,p}}=o(1)\|\varphi\|_{\mathcal{D}_0^{1,p}}$ and choosing $\varphi=\psi(\tilde{u}_n-\phi)$, we obtain that

$$\begin{aligned}&\int\psi(|\nabla_H\tilde{u}_n|^{p-2}\nabla_H\tilde{u}_n-|\nabla_H\phi|^{p-2}\nabla_H\phi)(\nabla_H\tilde{u}_n-\nabla_H\phi)=\\ &=\int\psi\bigg(|\tilde{u}_n|^{p^*(s)-2}\tilde{u}_n+\lambda\frac{|z|^p}{\rho^{2p}}|\tilde{u}_n|^{p-2}\tilde{u}_n\bigg)(\tilde{u}_n-\phi)d\xi+o(1).\end{aligned}$$

It follows that

$$\int \psi \left(|\nabla_H \tilde{u}_n|^{p-2}\nabla_H \tilde{u}_n - |\nabla_H \phi|^{p-2}\nabla_H u\right)(\nabla_H \tilde{u}_n - \nabla_H \phi)\, d\xi \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore an elementary inequality implies (2.8) immediately.

Step 5. In this step, we prove that $I(\phi) = 0$. We will prove that neither $I(\phi) < 0$ nor $I(\phi) > 0$ occurs. Indeed, if $I(\phi) < 0$, we get from Lemma 2.1 that there is a $0 < \theta_\phi < 1$ such that $\theta_\phi \phi \in \mathcal{N}$. Therefore using the Fatou lemma and $I(\tilde{u}_n) = 0$, we get that

$$\begin{aligned} d + o(1) = L(\tilde{u}_n) &= \left(\frac{1}{p} - \frac{1}{p_*(s)}\right)\int \frac{|z|^s |\tilde{u}_n|^{p_*(s)}}{\rho^{2s}} d\xi \geq \\ &\geq \left(\frac{1}{p} - \frac{1}{p_*(s)}\right)\int \frac{|z|^s |\phi|^{p_*(s)}}{\rho^{2s}} d\xi + o(1) = \\ &= \left(\frac{1}{p} - \frac{1}{p_*(s)}\right)\theta_\phi^{-p_*(s)} \int \frac{|z|^s |\theta_\phi \phi|^{p_*(s)}}{\rho^{2s}} d\xi + o(1) = \\ &= \theta_\phi^{-p_*(s)} L(\theta_\phi \phi) + o(1). \end{aligned}$$

It is deduced from $0 < \theta_\phi < 1$ that $d > L(\theta_\phi \phi)$, which is a contradiction because of $\theta_\phi \phi \in \mathcal{N}$.

If $I(\phi) > 0$, then from Step 4 and the Brezis-Lieb lemma [4], one obtains that

$$0 = I(\tilde{u}_n) = I(\phi) + I(v_n) + o(1),$$

where $v_n \equiv \tilde{u}_n - \phi$. In this way $I(\phi) > 0$ implies that

$$\limsup_{n\to\infty} I(v_n) < 0. \tag{2.21}$$

According to Lemma 2.1, we have a sequence θ_{v_n} such that $\theta_{v_n} v_n \in \mathcal{N}$. In order to simplify the notation, we denote $\theta_n := \theta_{v_n}$. Next we claim that $\limsup_{n\to\infty} \theta_n \in (0,1)$. In fact if $\limsup_{n\to\infty} \theta_n = 1$, then there is a subsequence $(\theta_{n_j})_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} \theta_{n_j} = 1$. Thus from $\theta_{n_j} v_{n_j} \in \mathcal{N}$, one deduces that for j large enough, $I(v_{n_j}) = I(\theta_{n_j} v_{n_j}) + o(1) = o(1)$. This contradicts (2.21). Therefore $\limsup_{n\to\infty} \theta_n \in (0,1)$. Since

$$\begin{aligned} d + o(1) = L(\tilde{u}_n) &= \left(\frac{1}{p} - \frac{1}{p_*(s)}\right)\int \frac{|z|^s |\tilde{u}_n|^{p_*(s)}}{\rho^{2s}} d\xi \geq \\ &\geq \left(\frac{1}{p} - \frac{1}{p_*(s)}\right)\int \frac{|z|^s |v_n|^{p_*(s)}}{\rho^{2s}} d\xi + o(1) = \\ &= \left(\frac{1}{p} - \frac{1}{p_*(s)}\right)\theta_n^{-p_*(s)} \int \frac{|z|^s |\theta_n v_n|^{p_*(s)}}{\rho^{2s}} d\xi + o(1) = \\ &= \theta_n^{-p_*(s)} L(\theta_n v_n) + o(1), \end{aligned}$$

one deduces from $\limsup_{n\to\infty} \theta_n \in (0,1)$ that $d > L(\theta_n v_n)$, which is again a contradiction because of $\theta_n v_n \in \mathcal{N}$.

In the sum we proved that $I(\phi) = 0$.

Step 6. Concluding the proof. We claim that

$$\int \left(|\nabla_H v_n|^p - \lambda \frac{|z|^p}{\rho^{2p}} |v_n|^p \right) d\xi \to 0 \quad \text{as} \quad n \to \infty.$$

Arguing by a contradiction. We assume that $\int \left(|\nabla_H v_n|^p - \lambda \frac{|z|^p}{\rho^{2p}} |v_n|^p \right) d\xi \not\to 0$ as $n \to \infty$. Then we have two cases:
(i) if $\int \frac{|z|^s}{\rho^{2s}} |v_n|^{p_*(s)} d\xi \not\to 0$ as $n \to \infty$, we obtain from

$$0 = I(\tilde{u}_n) = I(\phi) + I(v_n) + o(1),$$

$I(\phi) = 0$ and the Brezis-Lieb lemma that

$$d + o(1) = L(\tilde{u}_n) = L(\phi) + L(v_n) + o(1) \geq d + d + o(1),$$

which is a contradiction because of $d > 0$;
(ii) if $\int \frac{|z|^s}{\rho^{2s}} |v_n|^{p_*(s)} d\xi \to 0$ as $n \to \infty$, we have that

$$d + o(1) \quad = L(\tilde{u}_n) = L(\phi) + \frac{1}{p} \int \left(|\nabla_H v_n|^p - \lambda \frac{|z|^p}{\rho^{2p}} |v_n|^p \right) d\xi + o(1) > d,$$

which is also a contradiction. Therefore we deduce that $\tilde{u}_n \to \phi$ in $\mathcal{D}_0^{1,p}(\mathbb{H}^N)$. Thus ϕ is a minimizer of the minimum d. □

Proof of Theorem 1.2. The proof of Theorem 1.2 follows directly from Theorem 2.8 and Lemma 2.3. □

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