Opuscula Math. 38, no. 3 (2018), 357–377
https://doi.org/10.7494/OpMath.2018.38.3.357 Opuscula Mathematica

FORBIDDEN CONFIGURATIONS FOR HYPOHAMILTONIAN GRAPHS

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Communicated by Adam Paweł Wojda

Abstract. A graph G is called hypohamiltonian if G is not hamiltonian, but $G - x$ is hamiltonian for each vertex x of G. We present a list of 331 forbidden configurations which do not appear in hypohamiltonian graphs.

Keywords: hypohamiltonian graph, forbidden configuration, long cycle.

Mathematics Subject Classification: 05C38, 05C45.

1. INTRODUCTION

Throughout this paper, we consider connected graphs without loops or multiple edges. The used graph terminology is taken from the book [14]. A cycle in a graph G which contains all vertices of G is called *hamiltonian cycle*, and a graph containing such a cycle is called *hamiltonian*. A vertex of degree k is referred to as a k*-vertex*. A *configuration* in a graph $G = (V, E)$ is a pair (H, f) where $H \subseteq G$ is a connected subgraph of G and $f : V(H) \to \mathbb{Z}^+$ is a mapping such that, for each $x \in V(H)$, $f(x) \geq \deg_H(x)$ (less formal, the configuration is a subgraph with specified degrees of its vertices in the supergraph).

One of classical topics in research of hamiltonian graphs is the study of nonhamiltonian graphs whose all vertex-deleted subgraphs are hamiltonian; these graphs are called *hypohamiltonian graphs*. The smallest such graph is the Petersen graph (see [8]) and, by results from [2–4, 11], and [1], n-vertex hypohamiltonian graphs exist for all $n \geq 10, n \notin \{11, 12, 14, 17\}$ (even more, the number of nonisomorphic such graphs grows exponentially with n, see [10]). A lot of work was also done in study of planar hypohamiltonian graphs, concerning mainly their constructions and looking for smallest examples, see [12, 7, 17, 15, 9].

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In this paper, we are interested in forbidden configurations for hypohamiltonian graphs. It is easy to see that a hypohamiltonian graph cannot contain, as a subgraph, a 3-cycle with 3-valent vertex. Surprisingly, to our knowledge, no systematic study of other configurations which cannot appear in hypohamiltonian graphs was done (very recently in [6], it was shown that a hypohamiltonian graph cannot contain an edge common to two 3-cycles and incident with a 4-valent vertex). In order to fill this gap, we present a list of 329 new forbidden configurations for hypohamiltonicity. A part of this list is used in our recent related paper [5] on local structure of planar hypohamiltonian graphs from the point of view of existence of unavoidable configurations (which are still only little explored; along the classical result of Thomassen [13] that each planar hypohamiltonian graph contains a 3-valent vertex, C. Zamfirescu [16] very recently showed that it contains at least four such vertices). Note that our list is not complete as we considered mainly the configurations of small diameter whose central vertex has small degree (the complete list is very likely infinite). The absence of these configurations yields that planar hypohamiltonian graphs are, in certain sense, locally sparse and the upper bound for the number of their edges might be much lower than the general upper bound $3n-6$ for planar graphs (n being the number of vertices). In particular, we believe that the multiplicative coefficient 3 could be decreased; anyway, this cannot be achieved using the presented configuration list as it contains also configurations where vertices of degrees 5 or more are surrounded with triangles only.

2. RESULTS

We start with auxiliary lemma whose instances will be used later in the analysis of long cycle structure in hypohamiltonian graphs:

Lemma 2.1. *Let G be a graph, $v_0 \in V(G)$, and let C be a hamiltonian cycle of $G - v_0$. If C contains*

(a) *an edge v_1v_2, where $v_0v_1, v_0v_2 \in E(G)$, or*

(b) *a subpath $P = [v_1, v_2, v_3, v_4]$, where $v_0v_1, v_0v_3, v_2v_4 \in E(G)$, or*

(c) *two disjoint subpaths $P_1 = [v_1, v_2]$ and $P_2 = [v_3, v_4, v_5]$, where $v_0v_1, v_0v_4, v_2v_4, v_3v_5 \in E(G)$, or*

(d) *two disjoint subpaths $P_1 = [v_1, v_2, v_3]$ and $P_2 = [v_4, v_5]$, where $v_0v_1, v_0v_3, v_2v_4, v_2v_5 \in E(G)$, or*

(e) *two disjoint subpaths $P_1 = [v_1, v_2, v_3, v_4]$ and $P_2 = [v_5, v_6, v_7]$, where v_0v_1, v_0v_3, v_2v_6, $v_4v_6, v_5v_7 \in E(G)$, or*

(f) *two disjoint subpaths $P_1 = [v_1, v_2, v_3, v_4, v_5]$ and $P_2 = [v_6, v_7]$, where v_0v_1, v_0v_3, v_2v_5, $v_4v_6, v_4v_7 \in E(G)$, or*

(g) *two disjoint subpaths* $P_1 = [v_1, v_2]$ *and* $P_2 = [v_3, v_4, v_5, v_6]$*, where* $v_0v_1, v_0v_4, v_2v_5, v_3v_6 \in E(G)$*, or*

(h) *a subpath* $P = [v_1, v_2, v_3, v_4, v_5, v_6, v_7]$*, where* $v_0v_1, v_0v_4, v_2v_7, v_3v_6 \in E(G)$*, or*

(i) *a subpath* $P = [v_1, v_2, v_3, v_4, v_5, v_6, v_7]$*, where* $v_0v_1, v_0v_6, v_2v_5, v_4v_7 \in E(G)$*, or*

(j) *two disjoint subpaths* $P_1 = [v_1, v_2, v_3, v_4, v_5]$ *and* $P_2 = [v_6, v_7]$*, where* v_0v_1, v_0v_5, v_2v_7*,* $v_4v_6 \in E(G)$*,*

then G *is hamiltonian.*

Proof. Let C be a hamiltonian cycle of $G - v_0$. For each of the cases described above, we find a hamiltonian cycle of G:

(a) take $C - [v_1, v_2] + [v_1, v_0, v_2]$ (Figure 1a),

(b) take $C - P + [v_1, v_0, v_3, v_2, v_4]$ (Figure 1b),

(c) take $C - P_1 + [v_1, v_0, v_4, v_2] - P_2 + [v_3, v_5]$ (Figure 1c),

(d) take $C - P_1 + [v_1, v_0, v_3] - P_2 + [v_4, v_2, v_5]$ (Figure 1d),

(e) take $C - P_1 + [v_1, v_0, v_3, v_2, v_6, v_4] - P_2 + [v_5, v_7]$ (Figure 1e),

(f) take $C - P_1 + [v_1, v_0, v_3, v_2, v_5] - P_2 + [v_6, v_4, v_7]$ (Figure 1f),

(g) take $C - P_1 + [v_1, v_0, v_4, v_5, v_2] - P_2 + [v_3, v_6]$ (Figure 1g),

(h) take $C - P + [v_1, v_0, v_4, v_5, v_6, v_3, v_2, v_7]$ (Figure 1h),

(i) take $C - P + [v_1, v_0, v_6, v_5, v_2, v_3, v_4, v_7]$ (Figure 1i),

(j) take $C - P_1 + [v_1, v_0, v_5] - P_2 + [v_6, v_4, v_3, v_2, v_7]$ (Figure 1j).

□

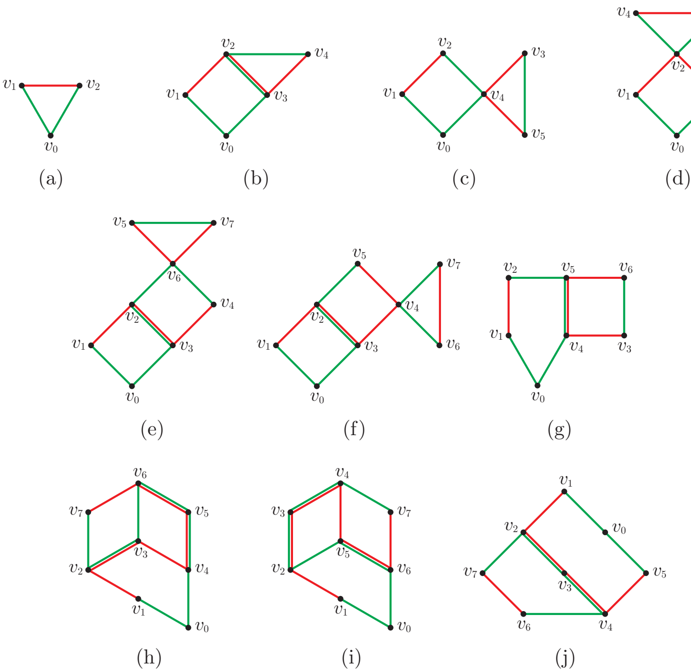


Fig. 1. Parts of hamiltonian cycle in $G - v_0$ (red) and in G (green)

This implies that the above described routings of a cycle which omits exactly one vertex in a hypohamiltonian graph G are not possible.

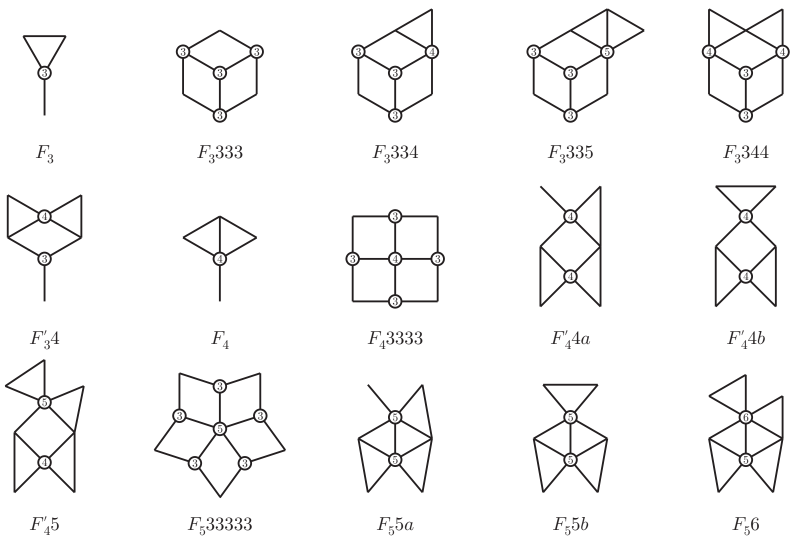


Fig. 2. Particular configurations

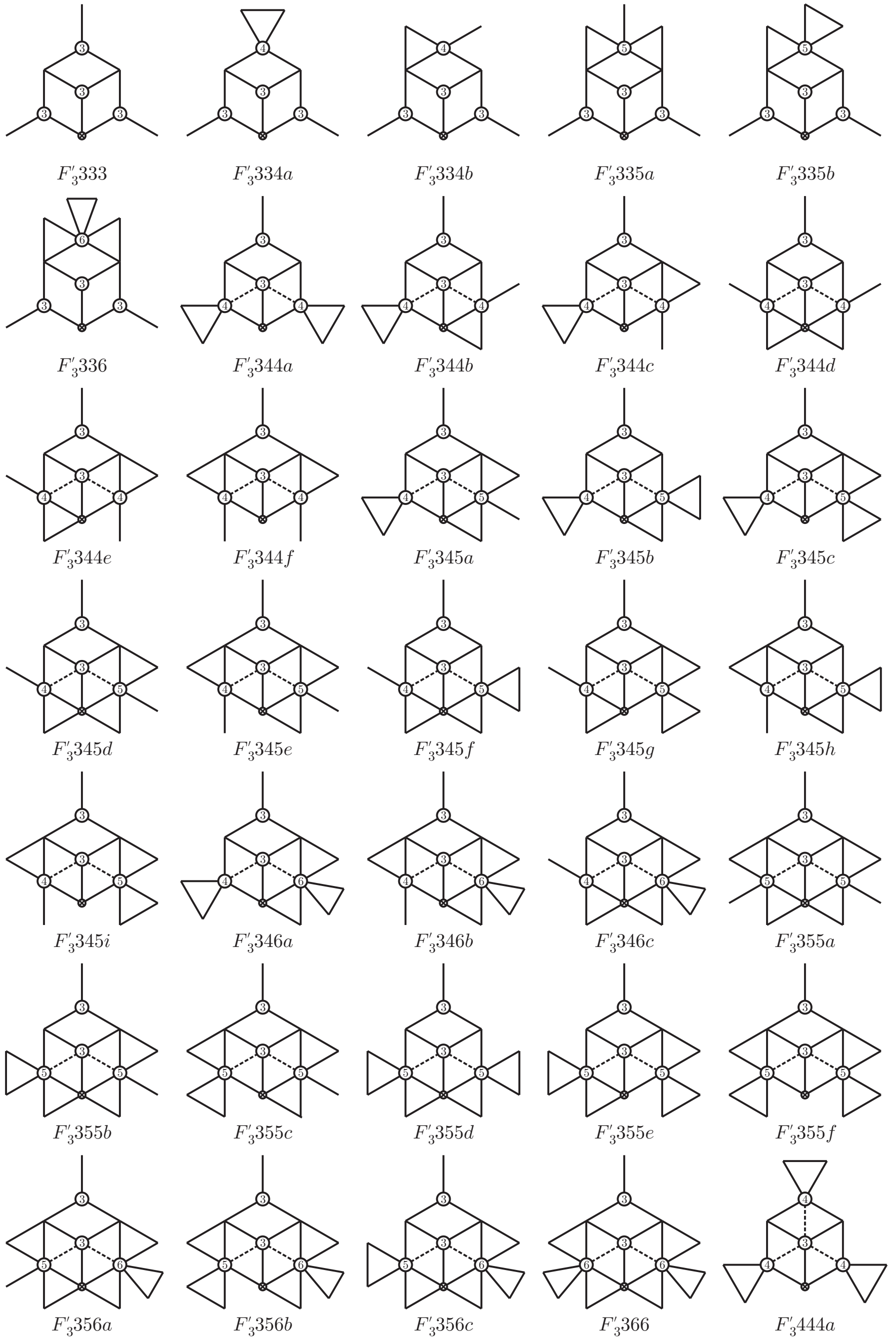


Fig. 3. Configurations with central vertex of degree 3, 5, or 6

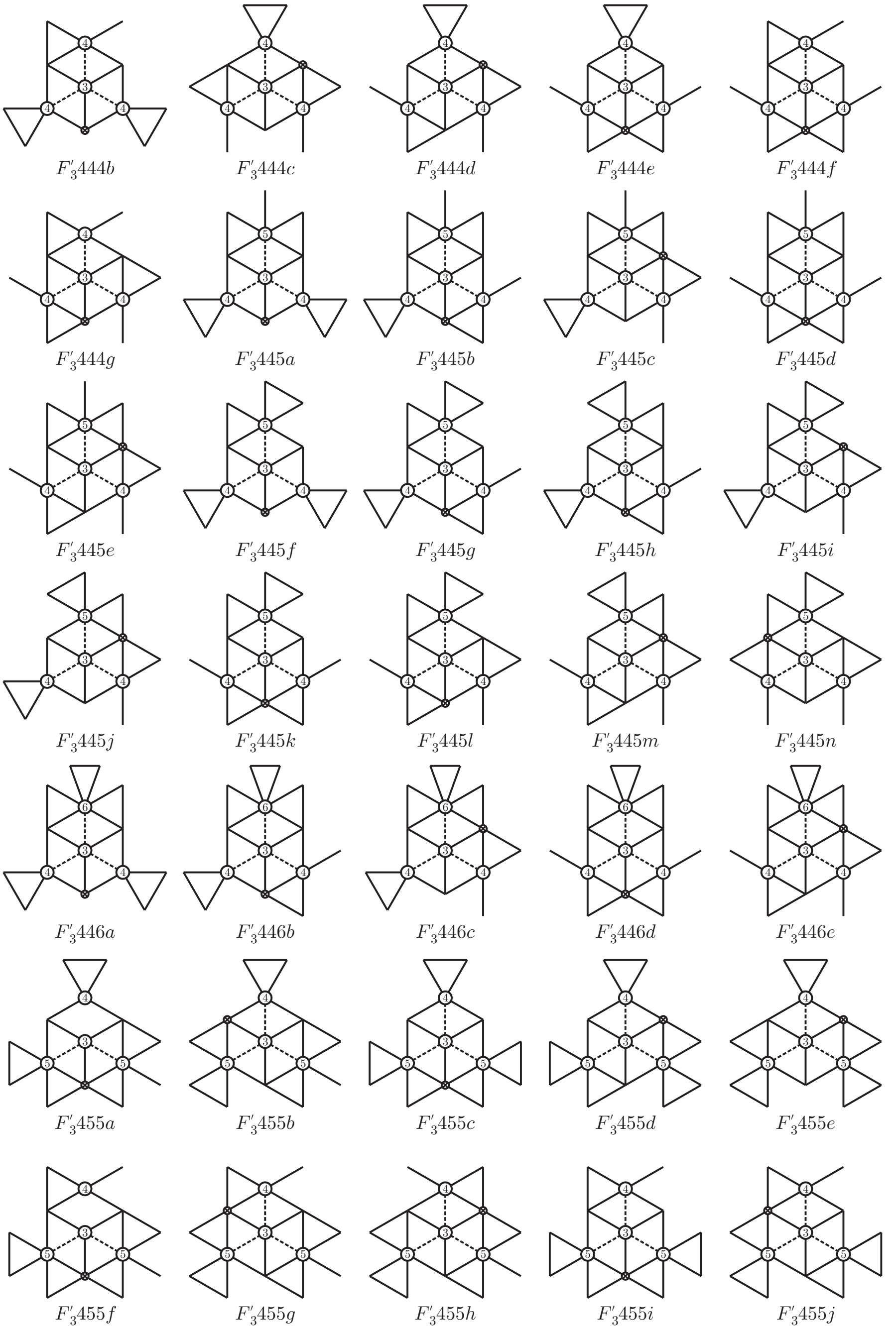


Fig. 4. Configurations with central vertex of degree 3, 5, or 6

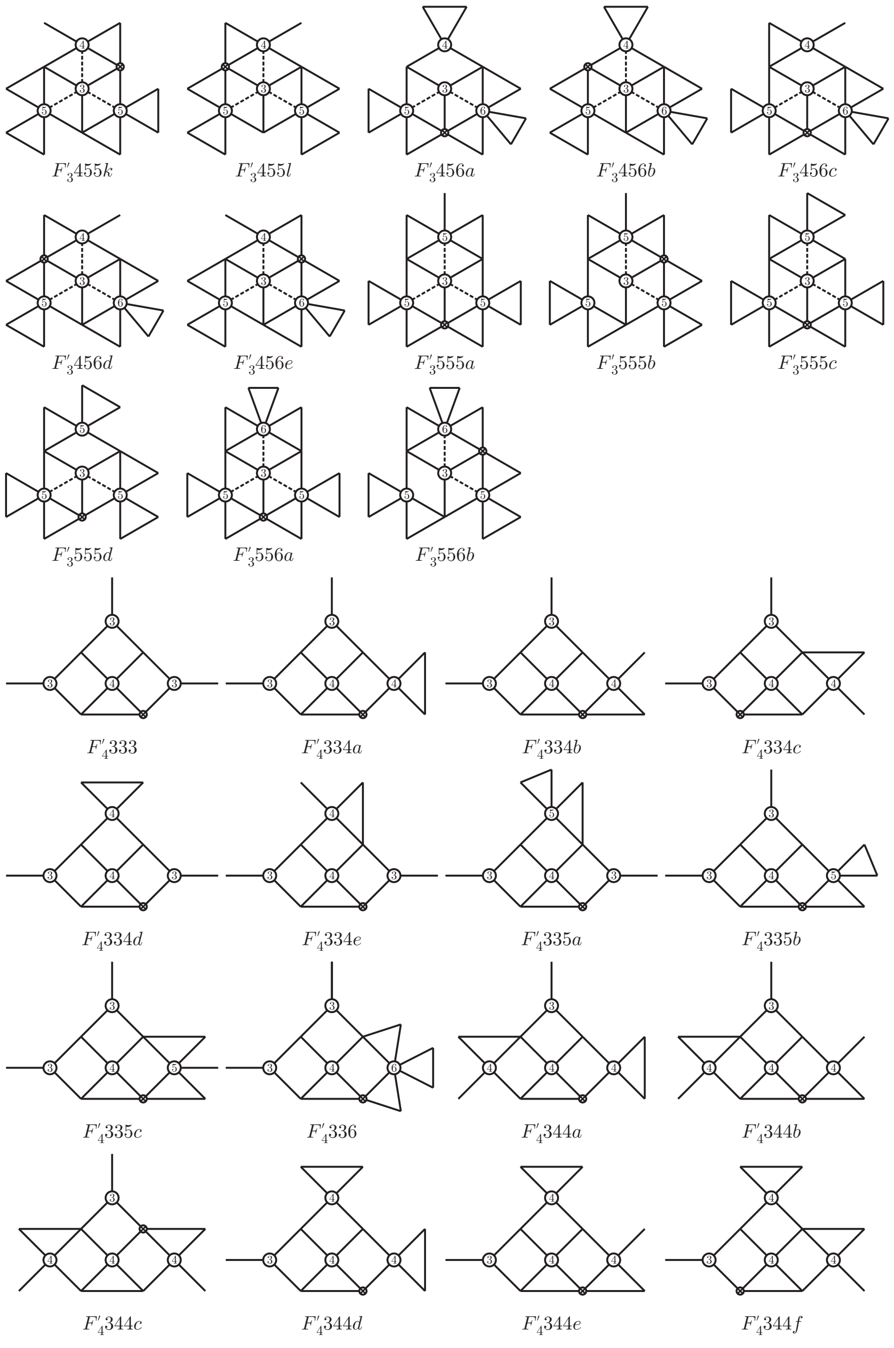


Fig. 5. Configurations with central vertex of degree 3, 5, 6 or 4

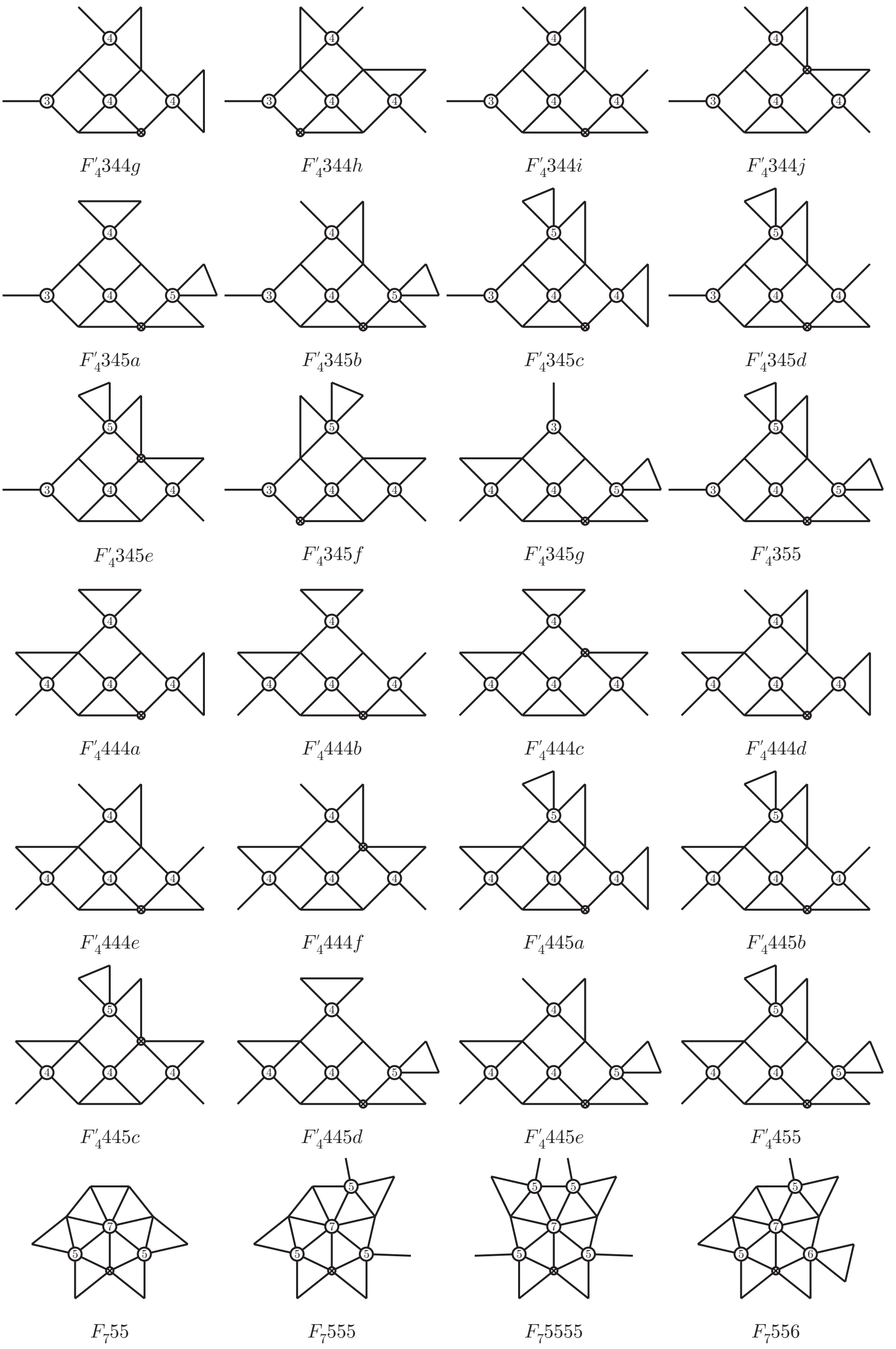


Fig. 6. Configurations with central vertex of degree 4 or 7

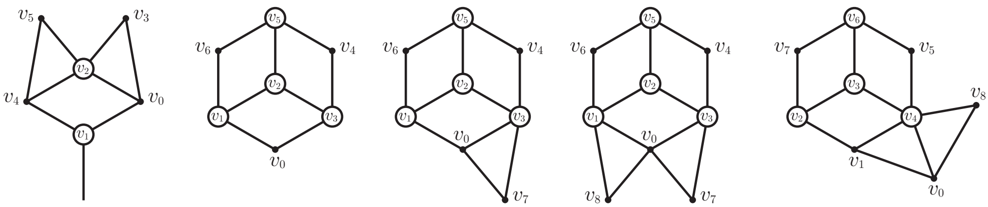
Theorem 2.2. *No hypohamiltonian graph contains any of the forbidden configurations represented on Figures 2–6.*

Proof. Let G be a hypohamiltonian graph.

Configurations F_3 and F_3'

Case 1. Suppose that G contains F_3 (i.e. a triangle $[v_0, v_1, v_2]$ with 3-vertex v_2). Let C be a hamiltonian cycle of $G - v_0$. Since v_2 is a 2-vertex of $G - v_0$, the edge v_1v_2 belongs to C. Thus by Lemma 2.1(a), G is hamiltonian, a contradiction.

Case 2. Suppose that G contains $F_3'4$ (Figure 7). Let C be a hamiltonian cycle of $G-v_0$. By Lemma 2.1(a), C does not contain the edge v_2v_3, thus $P = [v_1, v_4, v_2, v_5]$ is a subpath of C. Hence by Lemma 2.1(b), G is hamiltonian (where $C - P + [v_1, v_0, v_2, v_4, v_5]$ is a hamiltonian cycle of G), a contradiction.



$F_3'4$ F_3333 F_3334 F_3344 F_3335

Fig. 7. Particular configurations with central 3-vertex (Cases 2–4)

Case 3. Suppose that G contains F_3333, F_3334 or F_3344 (Figure 7). Let C be a hamiltonian cycle of $G - v_0$. By Lemma 2.1(a), C does not contain the edges v_3v_7 (of F_3334 and F_3344) and v_1v_8 (of F_3344), thus $C = [v_1, v_2, v_3, v_4, v_5, v_6]$, a contradiction (note that F_3333 itself is not hypohamiltonian).

Case 4. Suppose that G contains F_3335 (Figure 7). Let C be a hamiltonian cycle of $G - v_0$. By Lemma 2.1(a), C does not contain the edges v_1v_4 and v_4v_8, thus $[v_3, v_4, v_5]$ is a subpath of C. If C contains the edge v_2v_3, then (with respect to 3-vertices v_2, v_3, v_6) $[v_1, v_2, v_3, v_4, v_5, v_6, v_7]$ is a subpath of C, thus G is hamiltonian by Lemma 2.1(h). Otherwise, C contains the edge v_6v_3, thus $[v_1, v_2, v_7, v_6, v_3, v_4, v_5]$ is a subpath of C. Hence G is hamiltonian by Lemma 2.1(i), with contradiction in both cases.

In the sequel, we will analyze 83 configurations $F_3'ijk$ of Figures 3–5. They share the following common features: each of them contains a cycle $[v_0, v_1, v_2, v_6, v_4, v_5]$ and a 3-vertex v_3, which is a common neighbour of v_0, v_2, v_4, whereas v_1, v_5, v_6 have degrees i, j, k $(3 \le i, j, k \le 6)$, respectively, and v_1 (v_5, v_6) is incident with $i - 3$ $(j - 3, k - 3)$ triangles (without common edge).

Case 5. Suppose that G contains a configuration $F_3'ijk$; assume that the indexing of vertices of $F_3'ijk$ (Figure 8) is chosen in the way that v_0 is the special vertex (marked by x-cross in Figures 3–5). Let C be a hamiltonian cycle of $G - v_0$. Clearly, $[v_2, v_3, v_4]$ is a subpath of C, since v_3 is a 2-vertex of $G - v_0$.

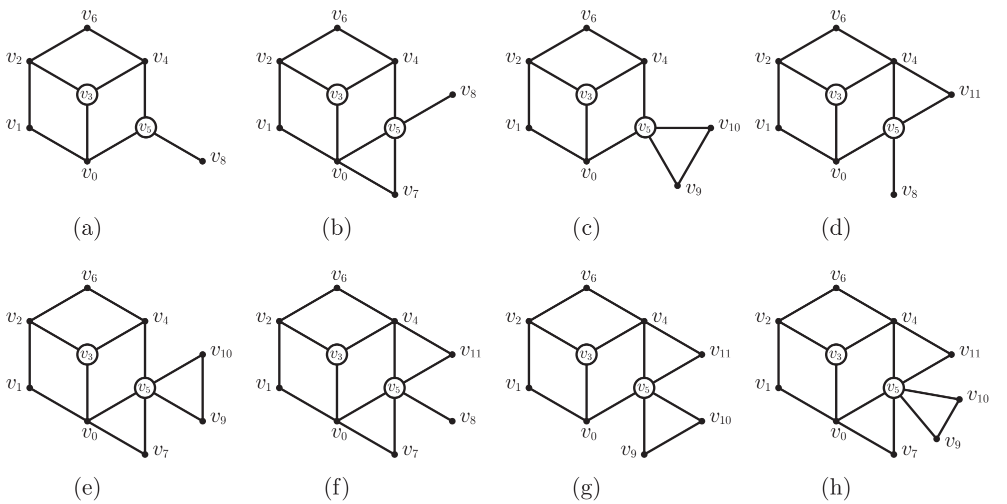


Fig. 8. Possible neighbourhoods of v_5 in configurations $F_3'ijk$ (Case 5)

Claim 5a. C does not contain the edges v_2v_6 and v_4v_6.

Proof of Claim 5*a*. Suppose that C contains v_4v_6 (the case when C contains v_2v_6 is treated in similar way). Note that in this case C does not contain v_4v_5.

i. If v_5 is a 3-vertex (Figure 8a), then C contains at most one edge incident with v_5, a contradiction.

ii. If v_5 is a 4-vertex, then by Lemma 2.1(a), C does not contain the edge v_5v_7 (Figure 8b), or by Lemma 2.1(c), C does not contain the path $[v_9, v_5, v_{10}]$ (Figure 8c), or else by Lemma 2.1(g), C does not contain the edge v_5v_{11} (Figure 8d). We obtain that C contains at most one edge incident with v_5, a contradiction.

iii. If v_5 is a 5-vertex, then by Lemma 2.1(a) and (c), C contains neither the edge v_5v_7 nor the path $[v_9, v_5, v_{10}]$ (Figure 8e), or by Lemma 2.1(a) and (g), C does not contain the edges v_5v_7 and v_5v_{11} (Figure 8f), or else by Lemma 2.1(c) and (g), C contains neither the path $[v_9, v_5, v_{10}]$ nor the edge v_5v_{11} (Figure 8g); thus C contains at most one edge incident with v_5, a contradiction.

iv. If v_5 is a 6-vertex, then by Lemma 2.1(a), (c), and (g), C contains neither the edge v_5v_7 nor the path $[v_9, v_5, v_{10}]$ nor the edge v_5v_{11} (Figure 8h). Thus C contains at most one edge incident with v_5, a contradiction as well. □

If v_6 is a 3-vertex, then by Claim 5a, C contains at most one edge incident with v_6, a contradiction. In the rest of the Case 5, let v_6 have degree at least 4.

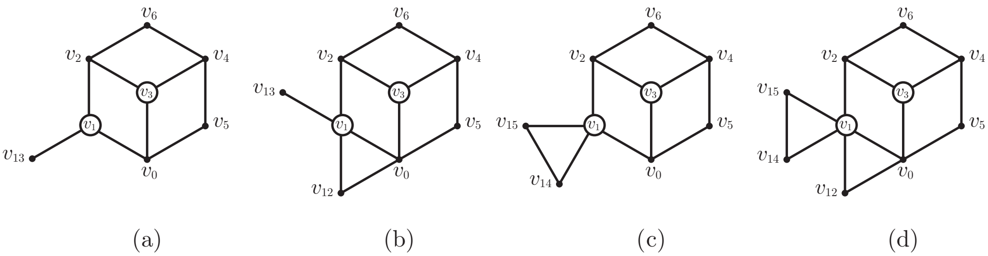


Fig. 9. Possible neighbourhoods of v_1 in configurations $F_3'ijk$ (Case 5)

Claim 5b. If the edge v_1v_2 (v_4v_5) does not belong to any triangle of $F_3'ijk$, then C contains v_1v_2 (v_4v_5).

Proof of Claim 5b. Assume that v_1v_2 does not belong to any triangle (similarly for v_4v_5).

i. If v_1 is a 3-vertex, then C contains the edge v_1v_2 (Figure 9a).

ii. If v_1 is a 4-vertex, then by Lemma 2.1(a), C does not contain the edge v_1v_{12} (Figure 9b), or by Lemma 2.1(c), C does not contain the path $[v_{14}, v_1, v_{15}]$ (Figure 9c). Thus C contains the edge v_1v_2.

iii. If v_1 is a 5-vertex, then by Lemma 2.1(a), C does not contain the edge v_1v_{12} and by Lemma 2.1(c), C does not contain the path $[v_{14}, v_1, v_{15}]$ (Figure 9d). Thus C contains the edge v_1v_2 as well. □

It is easy to check (on Figures 3–5) that (in the case when $\deg(v_6) \geq 4$) at least one of the edges v_1v_2 and v_4v_5 is incident with no triangle of $F_3'ijk$. Suppose v_1v_2 has this property. Then by Claim 5b, $[v_1, v_2, v_3, v_4]$ is a subpath of C.

Claim 5c. C does not contain the edge v_4v_5 and the edge v_4v_6 belongs to a triangle of $F_3'ijk$.

Proof of Claim 5c. Recall that v_6 has degree at least 4.

i. If v_6 is a 4-vertex and the edge v_4v_6 does not belong to a triangle, then by Lemma 2.1(d), C does not contain the edge v_6v_{20} (Figure 10a), or by Lemma 2.1(e), C does not contain the path $[v_{18}, v_6, v_{19}]$ (Figure 10b). Thus C contains at most one edge incident with v_6, a contradiction.

ii. If v_6 is a 4-vertex and the edge v_4v_6 belongs to a triangle (Figure 10c), then C contains the edge v_6v_{16}; thus by Lemma 2.1(d), C does not contain the edge v_4v_5.

iii. If v_6 is a 5-vertex and the edge v_4v_6 does not belong to a triangle (Figure 10d), then by Lemma 2.1(d), C does not contain the edge v_6v_{20} and by Lemma 2.1(e), C does not contain the path $[v_{18}, v_6, v_{19}]$. Thus C contains at most one edge incident with v_6, a contradiction.

iv. If v_6 is a 5-vertex and the edge v_4v_6 belongs to a triangle, then by Lemma 2.1(d), C does not contain the edge v_6v_{20} (Figure 10e), or by Lemma 2.1(e), C does not contain the path $[v_{18}, v_6, v_{19}]$ (Figure 10f). Hence C contains the edge v_6v_{16} and therefore by Lemma 2.1(d), C does not contain the edge v_4v_5.

v. If v_6 is a 6-vertex, then by Lemma 2.1(d), C does not contain the edges v_6v_{20} and by Lemma 2.1(e), C does not contain the path $[v_{18}, v_6, v_{19}]$ (Figure 10g). Hence C contains the edge v_6v_{16} and therefore by Lemma 2.1(d), C does not contain the edge v_4v_5. □

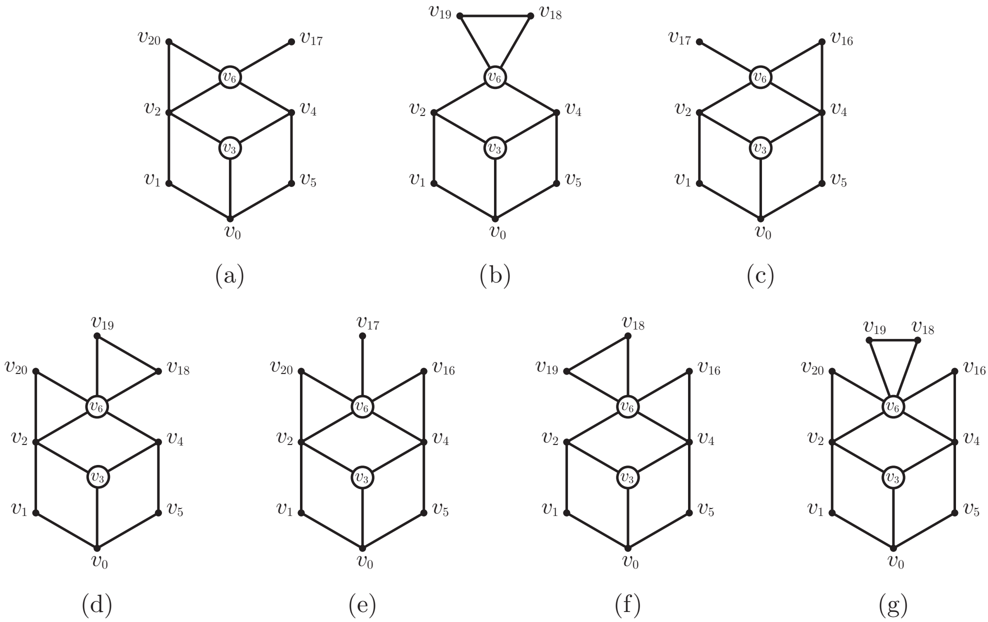


Fig. 10. Possible neighbourhoods of v_6 in configurations F'_3ijk (Case 5)

Now, C contains the path $[v_1, v_2, v_3, v_4]$, C does not contain the edge v_4v_5, and the edge v_4v_6 belongs to a triangle. It is easy to check (on Figures 3–5) that (for v_4v_6 belonging to a triangle and v_1v_2 not belonging to any triangle) the edge v_4v_5 does not belong to any triangle, thus by Claim 5b, C contains the edge v_4v_5, a contradiction.

Configurations F_4 and F'_4

Case 6. Suppose that G contains F_4 (Figure 11). Let C be a hamiltonian cycle of $G - v_0$. By Lemma 2.1(a), C does not contain edges v_1v_2 and v_2v_3, thus C contains at most one edge incident with v_2, a contradiction.

Case 7. Suppose that G contains F'_44a, F'_44b, or F'_45 (Figure 11). Let C be a hamiltonian cycle of $G - v_0$. By Lemma 2.1(a), C contains neither the edge v_1v_3 nor the edge v_5v_6 (of F'_44a and F'_45) and by Lemma 2.1(c), C does not contain the path $[v_7, v_5, v_8]$ (of F'_44b and F'_45). Thus $[v_2, v_3, v_4, v_5]$ is a subpath of C, hence by Lemma 2.1(b), G is hamiltonian, a contradiction.

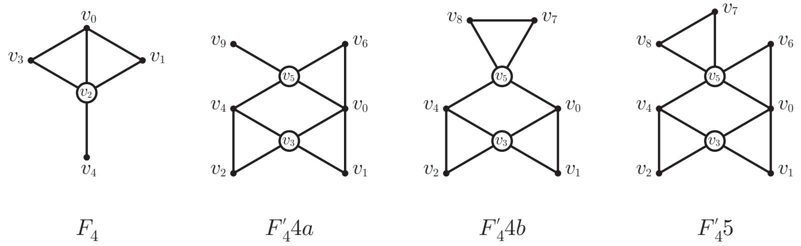


Fig. 11. Particular configurations with central 4-vertex (Cases 6 and 7)

In the sequel, we will analyze 40 configurations F'_4ijk of Figures 5–6. They share the following common features: each of them contains a cycle $[v_0, v_1, v_2, v_3, v_7, v_5, v_6]$ and a 4-vertex v_4, which is a common neighbour of v_0, v_1, v_3, v_5, whereas v_2, v_6, v_7 have degrees i, j, k ($3 \leq i, j, k \leq 6$), respectively, and v_2 (v_6, v_7) is incident with $i-3$ ($j-3, k-3$) triangles (without common edge).

Case 8. Suppose that G contains a configuration F'_4ijk (except of F'_4335c, F'_4336, F'_4344c, F'_4344j, F'_4345e, F'_4444c, F'_4444f, and F'_4445c); assume that the indexing of vertices of F'_4ijk (Figure 12) is chosen in the way that v_0 is the special vertex (marked by x-cross in Figures 5–6). Note that in each of 32 considered configurations F'_4ijk, the edges v_1v_2, v_3v_7, and v_5v_6 do not belong to any triangle of F'_4ijk. Let C be a hamiltonian cycle of $G - v_0$. By Lemma 2.1(a), C does not contain the edge v_1v_4. Thus $[v_3, v_4, v_5]$ is a subpath of C (Figure 12).

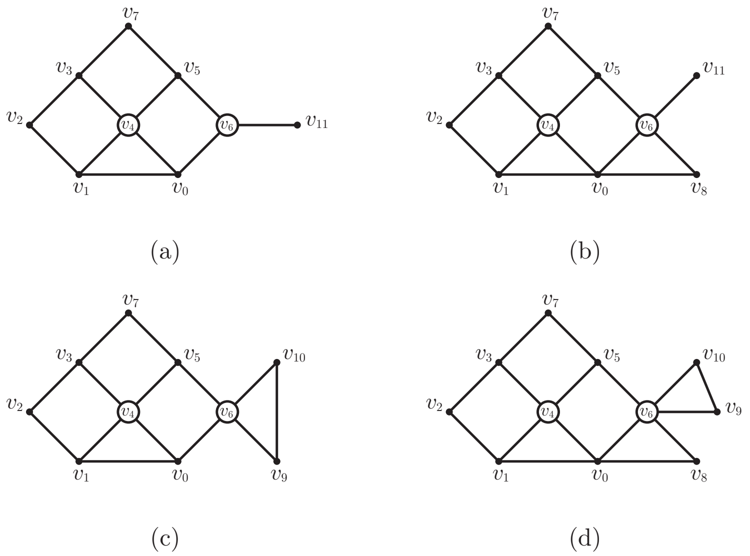


Fig. 12. Possible neighbourhoods of v_6 in configurations F'_4ijk (Case 8)

Claim 8a. C contains the edge v_5v_6.
Proof of Claim 8*a.*

i. If v_6 is a 3-vertex, then it is a 2-vertex of $G - v_0$ (Figure 12a), thus C contains the edge v_5v_6.

ii. If v_6 is a 4-vertex, then by Lemma 2.1(a), C does not contain the edge v_6v_8 (Figure 12b), or by Lemma 2.1(c), C does not contain the path $[v_9, v_6, v_{10}]$ (Figure 12c). Thus C contains the edge v_5v_6.

iii. If v_6 is a 5-vertex, then by Lemma 2.1(a), C does not contain the edge v_6v_8 and by Lemma 2.1(c), C does not contain the path $[v_9, v_6, v_{10}]$ (Figure 12d). Thus C contains the edge v_5v_6 as well. □

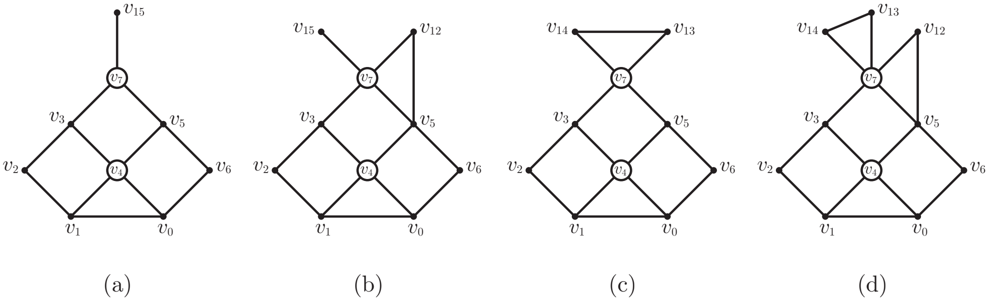


Fig. 13. Possible neighbourhoods of v_7 in configurations $F_4'ijk$ (Case 8)

Now, $P_1 = [v_3, v_4, v_5, v_6]$ is a subpath of C.

Claim 8b. C contains the edge v_3v_7.
Proof of Claim 8*b.*

i. If v_7 is a 3-vertex, then C does not contain the edge v_7v_5 (Figure 13a). Thus C contains the edge v_3v_7.

ii. If v_7 is a 4-vertex, then by Lemma 2.1(d), C does not contain the edge v_7v_{12} (Figure 13b), or by Lemma 2.1(e), C does not contain the path $[v_{13}, v_7, v_{14}]$ (Figure 13c). Thus C contains the edge v_3v_7.

iii. If v_7 is a 5-vertex, then by Lemma 2.1(d), C does not contain the edge v_7v_{12} and by Lemma 2.1(e), C does not contain the path $[v_{13}, v_7, v_{14}]$ (Figure 13d). Thus C contains the edge v_3v_7 as well. □

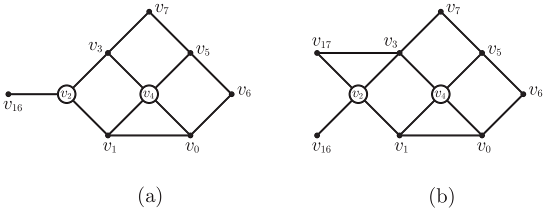


Fig. 14. Possible neighbourhoods of v_2 in configurations $F_4'ijk$ (Case 8)

Now, $P_2 = [v_7, v_3, v_4, v_5, v_6]$ is a subpath of C.

Claim 8c. C contains the edge v_1v_2.
Proof of Claim 8*c.*

i. If v_2 is a 3-vertex, then it is a 2-vertex of $G - P_2$ (Figure 14a), thus the edge v_1v_2 belongs to C.

ii. If v_2 is a 4-vertex (Figure 14b), then by Lemma 2.1(f) C does not contain the edge v_2v_{17}, thus the edge v_1v_2 belongs to C. □

Now, C contains the path $P_2 = [v_7, v_3, v_4, v_5, v_6]$ and the edge v_1v_2, thus by Lemma 2.1(g), G is hamiltonian, a contradiction.

Case 9. Suppose that G contains $F_4'335c$ or $F_4'336$ (Figure 15). Let C be a hamiltonian cycle of $G - v_0$. By Lemma 2.1(a), C does not contain the edges v_1v_4 and v_6v_8. Thus $[v_3, v_4, v_5]$ is a subpath of C. Since v_7 is a 3-vertex, C contains either v_3v_7 or v_5v_7. If C contains v_3v_7, then C does not contain v_2v_3, thus the edge v_1v_2 belongs to C, and by Lemma 2.1(g), G is hamiltonian, a contradiction. If C contains v_5v_7, then by Lemma 2.1(c), C does not contain the path $P = [v_9, v_6, v_{10}]$ (for $F_4'336$), thus C contains the edge v_6v_{11}, and by Lemma 2.1(g), G is hamiltonian, a contradiction as well.

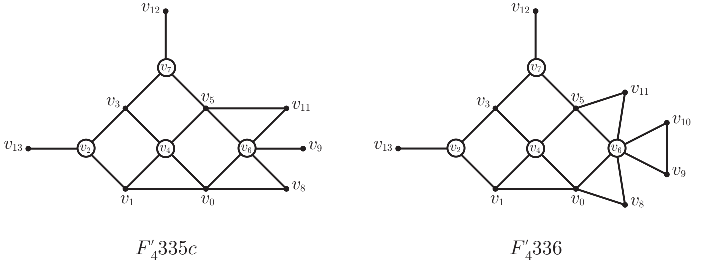


Fig. 15. Particular configurations $F_4'ijk$ (Case 9)

Case 10. Suppose that G contains one of $F_4'344c$, $F_4'344j$, $F_4'345e$, $F_4'444c$, $F_4'444f$, and $F_4'445c$ (Figure 16). Let C be a hamiltonian cycle of $G - v_0$. By Lemma 2.1(a), C does not contain the edge v_1v_9. Thus $[v_2, v_1, v_8]$ is a subpath of C. By Lemma 2.1(b), C does not contain the path $[v_2, v_4, v_3]$, thus C contains the edge v_4v_5.

Claim 10a. C contains the edge v_5v_6.
Proof of Claim 10*a.*

i. If v_6 is a 3-vertex, then it is a 2-vertex of $G - v_0$ ($F_4'344c$), thus C contains the edge v_5v_6.

ii. If v_6 is a 4-vertex, then by Lemma 2.1(a), C does not contain the edge v_6v_{10} ($F_4'344j$, $F_4'444f$), or by Lemma 2.1(c), C does not contain the path $[v_{11}, v_6, v_{12}]$ ($F_4'444c$). Thus C contains the edge v_5v_6.

iii. If v_1 is a 5-vertex, then by Lemma 2.1(a), C does not contain the edge v_6v_{10} and by Lemma 2.1(c), C does not contain the path $[v_{11}, v_6, v_{12}]$ ($F_4'345e$, $F_4'445c$). Thus C contains the edge v_5v_6 as well. □

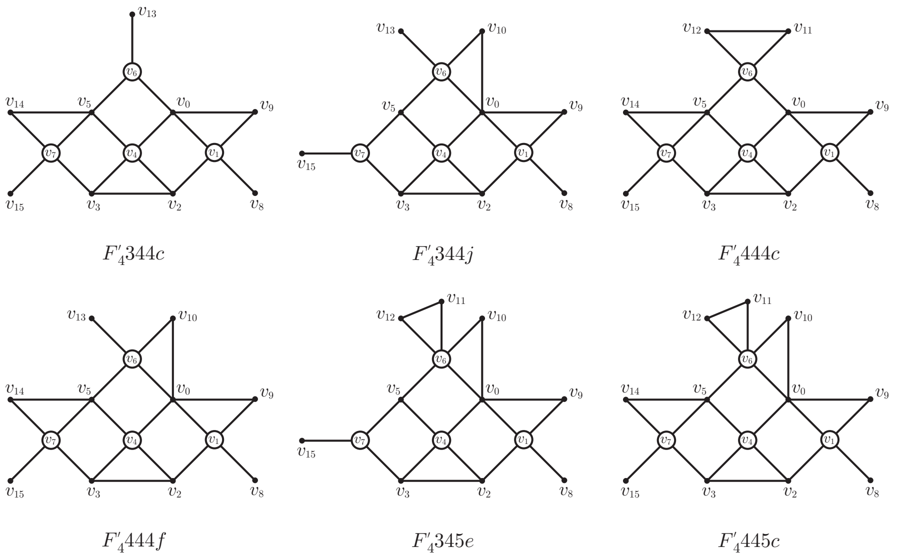


Fig. 16. Particular configurations F'_4ijk (Case 10)

Now, $P_1 = [v_2, v_1, v_8]$ and $P_2 = [v_4, v_5, v_6]$ are subpaths of C.

Claim 10b. C contains the edge v_3v_7.
Proof of Claim 10*b.*

i. If v_7 is a 3-vertex, then it is a 2-vertex of $G - P_2$, thus the edge v_3v_7 belongs to C (concerning the configurations F'_4344j, F'_4345e).

ii. If v_7 is a 4-vertex, then by Lemma 2.1(d), C does not contain the edge v_7v_{14} (concerning the configurations F'_4344c, F'_4444c, F'_4444f, F'_4445c). Thus C contains the edge v_3v_7. □

Now, if C contains the edge v_3v_4, then by Lemma 2.1(g), G is hamiltonian. Otherwise C contains the edge v_2v_4 and by Lemma 2.1(j), G is hamiltonian as well, a contradiction in both cases.

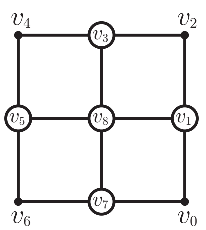


Fig. 17. Configuration F_43333 (Case 11)

Case 11. Suppose that G contains $F_4 3333$ (Figure 17). Let C be a hamiltonian cycle of $G - v_0$. Then the cycle $[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8]$ is a subgraph of C, a contradiction (as the smallest hypohamiltonian graph has 10 vertices).

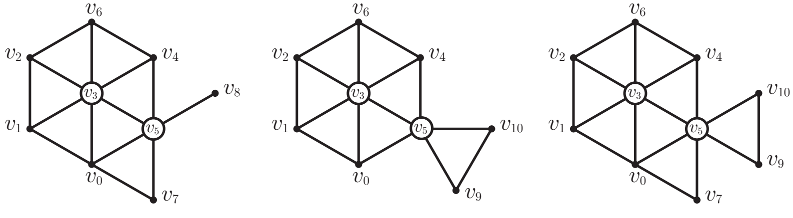
Configurations F_6

In the sequel, we will analyze 42 configurations $F_6 ijk$, each of them results from corresponding $F_3' \ell mn$ (with $\ell = i-1$, $m = j-1$, $n = k-1$) by adding all three dashed edges (Figures 3–5). They share the following common features: each of them contains 6-wheel with the central 6-vertex v_3 and a rim cycle $[v_0, v_1, v_2, v_6, v_4, v_5]$, whereas v_1, v_5, v_6 have degrees i, j, k ($5 \le i, j, k \le 7$), respectively, and v_1 (v_5, v_6) is incident with $i-2$ ($j-2, k-2$) triangles.

Case 12. Suppose that G contains a configuration $F_6 ijk$; assume that the indexing of vertices of $F_6 ijk$ (Figure 18) is chosen in the way that v_0 is the special vertex (marked by x-cross in Figures 3–5). Let C be a hamiltonian cycle of $G - v_0$. Then by Lemma 2.1(a), C does not contain the edges $v_1 v_3$ and $v_3 v_5$ (that is, two of three edges in which $F_6 ijk$ differs from $F_3' \ell mn$). If C does not contain the edge $v_3 v_6$ (the third edge of $F_6 ijk$ not occurring in $F_3' \ell mn$), then the proof is the same as for $F_3' \ell mn$.

Suppose that C contains $v_3 v_6$. Assume first that v_5 is a 5-vertex and $v_0 v_5$ belongs to two triangles of $F_6 ijk$ (Figure 18a; the case when v_1 is a 5-vertex and $v_0 v_1$ belongs to two triangles of $F_6 ijk$ is symmetric). Then, by Lemma 2.1(a), C does not contain the edge $v_5 v_7$, thus C contains the edge $v_4 v_5$. Moreover, if C contains the edge $v_3 v_4$, then by Lemma 2.1(b), G is hamiltonian, otherwise C contains the edge $v_2 v_3$ and by Lemma 2.1(c), G is hamiltonian as well, a contradiction in both cases.

In the remaining configurations $F_6 ijk$ — that is, when v_5 is not a 5-vertex or $v_0 v_5$ belongs to exactly one triangle of $F_6 ijk$ (and, symmetrically, v_1 is not a 5-vertex or $v_0 v_1$ belongs to exactly one triangle of $F_6 ijk$) — it is easy to check (on Figures 3–5) that both edges $v_1 v_2$ and $v_4 v_5$ belong to exactly one triangle of $F_6 ijk$ (i.e. to $[v_1, v_2, v_3]$ or $[v_3, v_4, v_5]$, respectively).



(a) (b) (c)

Fig. 18. Possible neighbourhoods of v_5 in configurations $F_6 ijk$ (Case 12)

Claim 12a. C does not contain the edges v_2v_3 and v_3v_4.
Proof of Claim 5*a*. Suppose that C contains v_3v_4 (if C contains v_2v_3, we argue similarly).

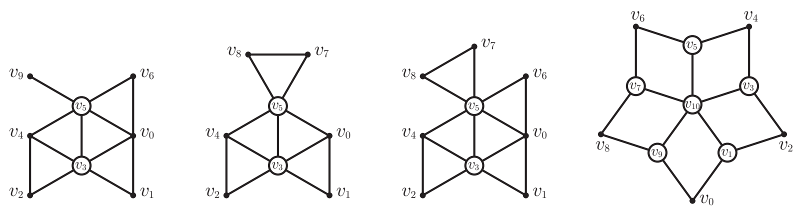
i. If v_5 is a 5-vertex, then by Lemma 2.1(c), C does not contain the path $[v_9, v_5, v_{10}]$ (Figure 18b), thus C contains the edge v_4v_5 and consequently by Lemma 2.1(b), G is hamiltonian, a contradiction.

ii. If v_5 is a 6-vertex, then by Lemma 2.1(a) and (c), C contains neither the edge v_5v_7 nor the path $[v_9, v_5, v_{10}]$ (Figure 18c), thus C contains the edge v_4v_5 and consequently by Lemma 2.1(b), G is hamiltonian, a contradiction as well. □

Finally, C contains at most one edge incident with v_3, a contradiction.

Configurations F_5

Case 13. Suppose that G contains F_55a, F_55b, or F_56 (Figure 19). Let C be a hamiltonian cycle of $G - v_0$. By Lemma 2.1(a), C contains neither the edge v_1v_3 nor the edge v_3v_5 nor else the edge v_5v_6 (of F_55a and F_56) and by Lemma 2.1(c), C does not contain the path $[v_7, v_5, v_8]$ (of F_55b and F_56). Thus $[v_2, v_3, v_4, v_5]$ is a subpath of C, hence by Lemma 2.1(b), G is hamiltonian, a contradiction.



F_55a F_55b F_56 F_533333

Fig. 19. Particular configurations with central 5-vertex (Cases 13 and 14)

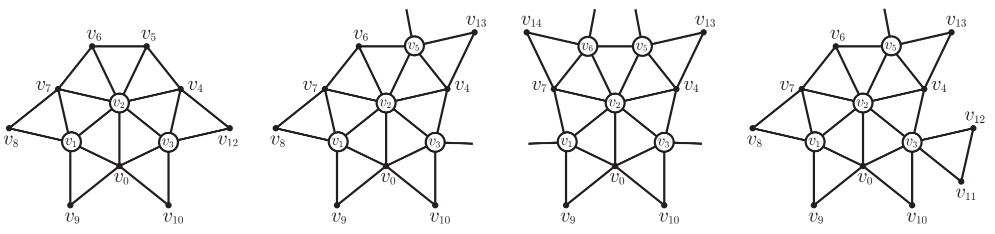
Case 14. Suppose that G contains F_533333 (Figure 19). Let C be a hamiltonian cycle of $G - v_0$. Then the cycle $[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}]$ is a subgraph of C, a contradiction (there is no hypohamiltonian graph on 11 vertices).

In the sequel, we will analyze 147 configurations F_5ijk, each of them results from corresponding $F_3'\ell mn$ (with $\ell = i-1$, $m = j-1$, $n = k$, or $\ell = i-1$, $m = j$, $n = k-1$ or else $\ell = i$, $m = j-1$, $n = k-1$) by adding two dashed edges (Figures 3–5). They share the following common features: each of them contains a cycle $R = [v_0, v_1, v_2, v_6, v_4, v_5]$ and a 5-vertex v_3, which is a common neighbour of five of the six vertices of R, whereas v_1, v_5, v_6 have degrees i, j, k ($3 \le i, j, k \le 7$), respectively, and v_1 (v_5, v_6) is incident with $i-2$ ($j-2, k-2$) triangles or with $i-3$ ($j-3, k-3$) triangles, if v_1 (v_5, v_6) is not adjacent to v_3, respectively.

Case 15. Suppose that G contains a configuration F_5ijk; assume that the indexing of vertices of F_5ijk is chosen in the way that v_0 is the special vertex (marked by x-cross in Figures 3–5). Let C be a hamiltonian cycle of $G - v_0$. In every configuration F_5ijk, either F_5ijk is a subgraph of corresponding F_6pqr (with $p + q + r = i + j + k + 1$, $i \le p \le i + 1$, $j \le q \le j + 1$, $k \le r \le k + 1$) and the proof follows from the proof for F_6pqr, or F_5ijk results from corresponding $F_3'\ell mn$ by adding two edges v_1v_3 and v_3v_5 between neighbours of v_0. Then by Lemma 2.1(a), C does not contain the edges v_1v_3 and v_3v_5 and the proof is the same as for $F_3'\ell mn$.

Configurations F_7

Case 16. Suppose that G contains F_755 (Figure 20). Let C be a hamiltonian cycle of $G - v_0$. By Lemma 2.1(a), C does not contain the edges v_1v_9, v_1v_2, v_2v_3, and v_3v_{10}. Thus $[v_7, v_1, v_8]$ and $[v_4, v_3, v_{12}]$ are subpaths of C. Subsequently, by Lemma 2.1(b), C does not contain the edges v_2v_7 and v_2v_4 and by Lemma 2.1(c), C does not contain the path $[v_5, v_2, v_6]$. Hence C contains at most one edge incident with v_2, a contradiction.



F_755 F_7555 F_75555 F_7556

Fig. 20. Configurations with central 7-vertex (Cases 16 and 17)

Case 17. Suppose that G contains F_7555, F_75555, or F_7556 (Figure 20). Let C be a hamiltonian cycle of $G - v_0$. By Lemma 2.1(a), C does not contain the edges v_1v_9, v_1v_2, v_2v_3, and v_3v_{10}. Thus C contains the edge v_1v_7. Subsequently, by Lemma 2.1(b), C does not contain the path $[v_6, v_2, v_7]$ and by Lemma 2.1(c), C contains neither the path $[v_6, v_2, v_5]$ nor the path $[v_5, v_2, v_4]$.

Claim 17a. C contains the path $[v_6, v_2, v_4]$.
Proof of Claim 17*a.*

i. For F_7555 or F_7556, C contains the path $[v_8, v_1, v_7]$, thus, by Lemma 2.1(b), C does not contain the edge v_2v_7. Hence, C contains the path $[v_6, v_2, v_4]$.

ii. For F_75555, C contains the edge v_1v_7 as well as the edge v_3v_4. Moreover, C does not contain the path $[v_7, v_2, v_4]$, because otherwise C does not contain the edges v_6v_7, v_6v_2, v_5v_2, and v_5v_4, and by Lemma 2.1(d), C does not contain also the edges v_6v_{14} and v_5v_{13}; but then C contains the edge v_5v_6 and, finally, $C - [v_1, v_7, v_2, v_4, v_3] + [v_1, v_0, v_3] - [v_6, v_5] + [v_6, v_7, v_2, v_4, v_5]$ is a hamiltonian cycle of G, a contradiction. Therefore, C contains the path $[v_6, v_2, v_4]$ (or the symmetrical path $[v_5, v_2, v_7]$). $\square$

Claim 17b. C contains the edge v_3v_4.

Proof of Claim 17*b.*

i. It is obvious for the configurations F_7555 and F_75555 (there are only two remaining edges incident with v_3).

ii. For F_7556, by Lemma 2.1(c), C does not contain the path $[v_{11}, v_3, v_{12}]$, thus C contains the edge v_3v_4. □

Now, C contains the path $[v_6, v_2, v_4, v_3]$.

Claim 17c. C contains the edge v_5v_6.

Proof of Claim 17*c.*

By Lemma 2.1(d), C does not contain the edge v_5v_{13}, thus C contains the edge v_5v_6. □

Finally, C contains the path $[v_5, v_6, v_2, v_4, v_3]$ and the edge v_1v_7, hence by Lemma 2.1(g), G is hamiltonian, a contradiction. □

Acknowledgements

The research of I. Fabrici and T. Madaras was partially supported by Science and Technology Assistance Agency under the contract No. APVV-15-0116 and by the Slovak VEGA Grant 1/0368/16. The research of M. Timková was partially supported by Slovak VEGA Grant 1/0389/15 and internal faculty research project no. FEI-2017-39.

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Recieved: October 12, 2017.
Revised: January 26, 2018.
Accepted: February 3, 2018.