Free Axisymmetric Flexural Vibrations of Circular Plate with Symmetrically Varying Mechanical Properties Supported on Elastic Foundation

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Abstract

The subject of the paper is a circular plate with clamped edge supported on elastic foundation. Mechanical properties of the plate symmetrically vary in its thickness direction. Free axisymmetric flexural vibration problem of the plate with consideration of the shear effect is analytically studied. Two partial differential equations of motion based on the Hamilton principle are obtained. The system of equations is analytically solved and the fundamental natural frequency of axisymmetric vibration for example plates is derived.

Keywords: circular plate, elastic foundation, shear effect, axisymmetric flexural vibration.

1. Introduction

The circular plates are parts of modern constructions. Nowacki [1] presented the dynamics problems of elastic structures. Ventsel and Krauthammer [2] presented the theoretical foundations of plates and shells. Wang et al [3] investigated the free axisymmetric vibration of transversely isotropic circular plates. Tajeddini et al. [4] described the three-dimensional free vibration of thick isotropic and functionally graded circular and annular plates with variable thickness on Pasternak foundation. Yas and Tahouneh [5] presented the free vibration problems of functionally graded annular plates supported on elastic foundations with consideration of various boundary conditions. Melekzadeh and Farajpour [6] described the axisymmetric free and forced vibrations of circular single- and double-layered nanoplates in an elastic medium. Foyouzat et al. [7] presented an exact solution pertaining to the problem of undamped free vibration of a thin circular plate resting on a Winkler foundation. Ahmad and Khorshidvand [8] analysed free vibration of a circular plate made of a porous materials. Żur [9] presented an analysis and numerical results related to free vibrations of functionally graded circular plates elastically supported on a concentric ring with consideration of the classical plate theory. Magnucki et al. [10] studied analytically and numerically the buckling and free vibrations problems of rectangular plates with symmetrically varying mechanical properties. Magnucki et al. [11] studied analytically and numerically the bending problem of a circular plate with symmetrically thickness-wise varying mechanical properties. Magnucki et al. [12] presented an improved shear deformation theory of bending beams with symmetrically varying mechanical properties in the depth direction. Magnucka et al. [13] studied analytically the bending and buckling problem of a circular plate with symmetrically varying mechanical properties.

The subject of the studies is a circular plate with symmetrically thickness-wise varying mechanical properties of radius *R* and thickness *h* supported on an elastic foundation of the constant-*foundation modulus k* (Fig.1).

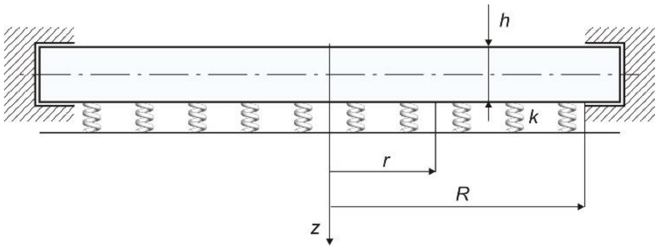

Figure 1. Scheme of the circular plate with clamped edge

2. Analytical model of the plate with consideration of the shear effect

The symmetrical variation of the Young's modulus of the circular plate in its thickness direction is assumed, similarly as in the papers [11], [12] and [13], in the following form

$$E(\zeta) = E_1 f_e(\zeta), \tag{1}$$

where: $f_e(\zeta) = e_0 + (1-e_0)\left(5\zeta^2 - 256\zeta^{10}\right)^{k_e}$ – dimensionless function,

$e_0 = E_0/E_1$ – dimensionless parameter, k_e – exponent (positive real number),

$\zeta = z/h$ – dimensionless coordinate, $E_0 = E(0)$, $E_1 = E(\pm 1/2)$ – Young's modules.

The graphs of the dimensionless function $f_e(\zeta)$ of the example circular plates are shown in Figure 2.

P-1 $e_0=1/3$, $k_e=1$ **P-2** $e_0=1/12$, $k_e=10$ **P-3** $e_0=1/35$, $k_e=40$

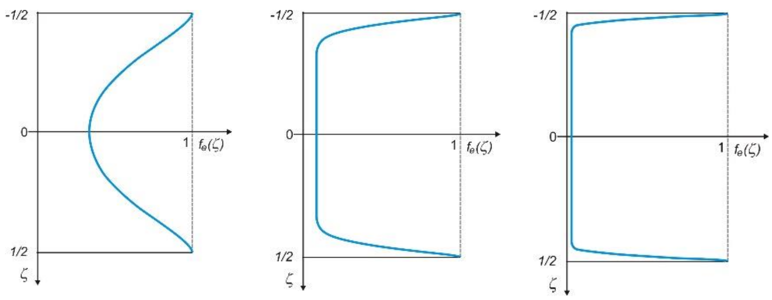

Figure 2. Graphs of the dimensionless function $f_e(\zeta)$ of the example circular plates

The value of the exponent k_e and dimensionless parameter e_0 are decisive for the shapes of symmetrical variability of the Young's modules of the plate (1). In the particular case of $e_0=1$, the modulus of elasticity remains constant i.e. the structure is homogeneous.

The deformation of the straight line normal to the neutral surface of the plate is shown in Figure 3. The upper and lower surfaces of the circular plate are free from shear stresses, therefore, the line of this deformation is perpendicular to these surfaces.

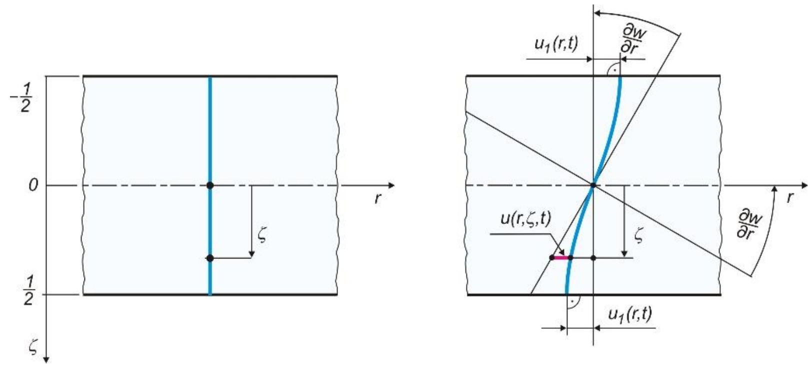

Figure 3. The scheme of deformation of the straight line normal to the neutral surface

The longitudinal displacement compatible with this scheme is as follows

$$u(r,\zeta,t) = -h\left[\zeta\frac{\partial w}{\partial r} - f_d(\zeta)\psi(r,t)\right], \tag{2}$$

where: $\psi(r,t)=u_1(r,t)/h$ – dimensionless function of the displacement.

Taking into account the paper [12] the nonlinear dimensionless function of deformation of the straight line normal to the neutral surface of the plate is assumed in the following form

$$f_d(\zeta) = \frac{1}{C_0}\int\frac{\left(1-4\zeta^2\right)^{k_s}}{f_e(\zeta)}d\zeta\ , \tag{3}$$

where: $C_0 = \int_0^{1/2}\left(1-4\zeta^2\right)^{k_s} / f_e(\zeta)d\zeta$ – dimensionless coefficient, k_s – exponent.

The strains

$$\varepsilon_r(r,\zeta,t) = \frac{\partial u}{\partial r} = -h\left[\zeta\frac{\partial^2 w}{\partial r^2} - f_d(\zeta)\frac{\partial\psi}{\partial r}\right], \tag{4}$$

$$\varepsilon_\varphi(r,\zeta,t) = \frac{u(r,\zeta,t)}{r} = -h\left[\zeta\frac{\partial w}{r\partial r} - f_d(\zeta)\frac{\psi(r,t)}{r}\right], \tag{5}$$

$$\gamma_{rz}(r,\zeta,t) = \frac{\partial w}{\partial r} + \frac{\partial u}{h\partial\zeta} = \frac{df_d(\zeta)}{d\zeta}\psi(r,t). \tag{6}$$

The stresses – the Hooke's law

$$\sigma_r(r,\zeta,t)=\frac{E_1}{1-\nu^2}\left[\varepsilon_r(r,\zeta,t)+\nu\varepsilon_\varphi(r,\zeta,t)\right]f_e(\zeta), \tag{7}$$

$$\sigma_\varphi(r,\zeta,t)=\frac{E_1}{1-\nu^2}\left[\varepsilon_\varphi(r,\zeta,t)+\nu\varepsilon_r(r,\zeta,t)\right]f_e(\zeta), \tag{8}$$

$$\tau_{rz}(r,\zeta,t)=\frac{E_1}{2(1+\nu)}\gamma_{rz}(r,\zeta,t)f_e(\zeta). \tag{9}$$

The elastic strain energy of the plate

$$U_\varepsilon=\pi h\int_0^R\int_{-1/2}^{1/2}\left[\sigma_r(r,\zeta,t)\varepsilon_r(r,\zeta,t)+\sigma_\varphi(r,\zeta,t)\varepsilon_\varphi(r,\zeta,t)+\tau_{rz}(r,\zeta,t)\gamma_{rz}(r,\zeta,t)\right]d\zeta r dr. \tag{10}$$

The work of the elastic foundation's reaction

$$W=-2\pi k\int_0^R w^2(r)r dr, \tag{11}$$

where k [N/mm^3] – the constant-*foundation modulus*.
The kinetic energy

$$U_k=\pi C_\rho\rho_1 h\int_0^R\left(\frac{\partial w}{\partial t}\right)^2 r dr, \tag{12}$$

where: t [s] – time, ρ_1 [kg/m^3] – mass density of the upper/lower surfaces of the plate, $C_\rho=\int_{-1/2}^{1/2}\left[\sqrt{e_0}+\left(1-\sqrt{e_0}\right)\left(5\zeta^2-256\zeta^{10}\right)^{k_e}\right]d\zeta$ – dimensionless mass density of the plate.

Thus, based on the Hamilton's principle $\delta\int_{t_1}^{t_2}(U_k-U_\varepsilon+W)dt=0$, two differential equations of motion are obtained in the following form

$$C_\rho\rho_1 hr\frac{\partial^2 w}{\partial t^2}+\frac{E_1h^3}{1-\nu^2}\frac{\partial}{\partial r}\left\{r\Re_w\left[w(r,t),\psi(r,t)\right]\right\}+2krw(r,t)=0, \tag{13}$$

$$\Re_\psi\left[w(r,t),\psi(r,t)\right]+\frac{1}{2}(1-\nu)C_{u2}\frac{\psi(r,t)}{h^2}=0, \tag{14}$$

where: $\Re_w\left[w(r,t),\psi(r,t)\right]=\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(C_{ww}\frac{\partial w}{\partial r}-C_{wu}\psi(r,t)\right)\right]\right\}$,

$\Re_\psi\left[w(r,t),\psi(r,t)\right]=\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(C_{wu}\frac{\partial w}{\partial r}-C_{uu}\psi(r,t)\right)\right]\right\}$, $C_{ww}=\int_{-1/2}^{1/2}\zeta^2 f_e(\zeta)d\zeta$,

$C_{wu}=\int_{-1/2}^{1/2}f_d(\zeta)f_e(\zeta)\zeta d\zeta$, $C_{uu}=\int_{-1/2}^{1/2}f_d^2(\zeta)f_e(\zeta)d\zeta$, $C_{u2}=\int_{-1/2}^{1/2}\left(\frac{df_d}{d\zeta}\right)^2 f_e(\zeta)d\zeta$.

The system of equations of motion (13) and (14) is the basis of analytical studies of the free axisymmetric flexural vibration problem of circular plate with symmetrically thickness-wise varying mechanical properties and with clamped edge and supported on elastic foundation.

3. Analytical study of free axisymmetric flexural vibration of the plate

The system of equations of motion (13) and (14) is approximately solved with assumption of the following two functions

$$w(r,t)=\left[1-\left(\frac{r}{R}\right)^2\right]^2 w_a(t), \quad \psi(r,t)=\left[1-\left(\frac{r}{R}\right)^2\right]\frac{r}{R}\psi_a(t), \tag{15}$$

where: $w_a(t)$, $\psi_a(t)$ – time functions.
These functions satisfy boundary conditions of the clamped edge of the plate: $w(R)=0$, $\partial w/\partial r|_0=0$, $\partial w/\partial r|_R=0$, $\psi(0)=\psi(R)=0$. Substitution of these functions into the eqs (13) and (14) and application of the Galerkin's method, gives two linear algebraic equations

$$C_\rho \rho_1 h \frac{d^2 w_a}{dt^2}+\frac{80}{3}\frac{E_1}{1-\nu^2}\left(\frac{h}{R}\right)^2\left[4C_{ww}\frac{w_a(t)}{R}+C_{uu}\psi_a(t)\right]+2k w_a(t)=0\,, \tag{16}$$

$$4C_{wu}\frac{w_a(t)}{R}+\left[C_{uu}+\frac{1-\nu}{28}\left(\frac{R}{h}\right)^2 C_{u2}\right]\psi_a(t)=0\,. \tag{17}$$

The dimensionless time function from the equation (17) is as follows

$$\psi_a(t)=-4C_{wu}\left[C_{uu}+\frac{1-\nu}{28}\left(\frac{R}{h}\right)^2 C_{u2}\right]^{-1}\frac{w_a(t)}{R}\,. \tag{18}$$

Substituting this function into the equation (16) one obtains

$$\frac{d^2 w_a}{dt^2}+2\left[\frac{160}{3(1-\nu^2)}\left(C_{ww}-C_s\right)+R\frac{k}{E_1}\left(\frac{R}{h}\right)^3\right]\frac{E_1}{C_\rho \rho_1}\frac{h^2}{R^4}w_a(t)=0\,, \tag{19}$$

where: $C_s=\max\limits_{k_s}\left\{C_{wu}^2\left[C_{uu}+\frac{1-\nu}{28}\left(\frac{R}{h}\right)^2 C_{u2}\right]^{-1}\right\}$ – dimensionless shear coefficient.
The equation (19) is solved with the use of the assumed function

$$w_a(t)=w_a\sin(\omega t), \tag{20}$$

where: w_a – amplitude of the flexural vibration, ω – fundamental natural frequency.
Substituting this function into the equation (19) one obtains the fundamental natural frequency

$$\omega=\sqrt{2}\,\frac{h}{R^2}\sqrt{\left[\frac{160}{3(1-\nu^2)}\left(C_{ww}-C_s\right)+R\frac{k}{E_1}\left(\frac{R}{h}\right)^3\right]\frac{E_1}{C_\rho \rho_1}}\,. \tag{21}$$

Exemplary calculations are carried out for the following data of the plates (Fig.2): material constants E_1=200 GPa, mass density of the upper/lower surfaces of the plate ρ_1=7850 kg/m^3, ν=0.3, radius R=500 mm, thickness h=25 mm, and constant-foundation modulus k=0, 1.0, 2.0, 3.0 N/mm^3.
The results of the calculations are specified in Tables 1–5.

Table 1. Values of the exponent k_s, coefficients C_{ww} and C_s of the example plates

Plates	**P-1**	**P-2**	**P-3**
k_s	0.7293	0.2854	0.1600
C_{ww}	0.066239	0.027627	0.013350
C_s	0.0011326	0.00086118	0.00058309
C_s/C_{ww}	0.01710	0.03117	0.04368

The value of the dimensionless coefficient C_{ww} of flexural rigidity of the plate decreases for consecutive variants of the Young's modulus patterns (Fig.2).

Table 2. Values of the fundamental natural frequency of the plates for k=0

Plates	**P-1**	**P-2**	**P-3**
ω [rad/s]	1616.8	1484.7	1350.6
$\omega/2\pi$ [Hz]	257.3	236.3	215.0

Table 3. Values of the fundamental natural frequency of the plates for k=1.0 N/mm^3

Plates	**P-1**	**P-2**	**P-3**
ω [rad/s]	4039.1	5505.3	7111.9
$\omega/2\pi$ [Hz]	642.8	876.2	1131.9

Table 4. Values of the fundamental natural frequency of the plates for k=2.0 N/mm^3

Plates	**P-1**	**P-2**	**P-3**
ω [rad/s]	5478.6	7641.8	9966.7
$\omega/2\pi$ [Hz]	871.9	1216.2	1586.2

Table 5. Values of the fundamental natural frequency of the plates for k=3.0 N/mm^3

Plates	**P-1**	**P-2**	**P-3**
ω [rad/s]	6611.8	9301.4	12169.2
$\omega/2\pi$ [Hz]	1052.3	1480.4	1936.8

Moreover, in the case of particular, i.e. homogeneous plate, the values of the dimensionless coefficients are as follows: e_0=1, k_e=0, k_s=1, C_{ww}=1/12, C_s=0.000825, C_s/C_{ww}=0.01, C_ρ=1. Thus, the values of fundamental natural frequency of the homogeneous plate are specified in Table 6.

Table 6. Values of the fundamental natural frequency of the homogeneous plates

Found. modulus k [N/m^3]	0	1.0	2.0	3.0
ω [rad/s]	1569.7	3557.4	4779.8	5747.8
$\omega/2\pi$ [Hz]	249.8	566.2	760.7	914.8

While omitting the shear effect (C_s=0) and the elastic foundation (k=0) the fundamental natural frequency (21) of the homogeneous plate takes the following form

$$\omega = \frac{10.33}{R^2}\sqrt{\frac{D}{\rho h}}\,, \tag{22}$$

where: $D = \frac{Eh^3}{12\left(1-\nu^2\right)}$ – flexural rigidity of the plate, E_1=E, ρ_1=ρ – material constants.

This expression is identical with the one presented by Ventsel, and Krauthammer [2].

Taking into account the calculation results specified in Tables 2–6, the influence of the foundation stiffness (foundation modulus k) on the natural frequency of the example plates is graphically presented in Figure 4.

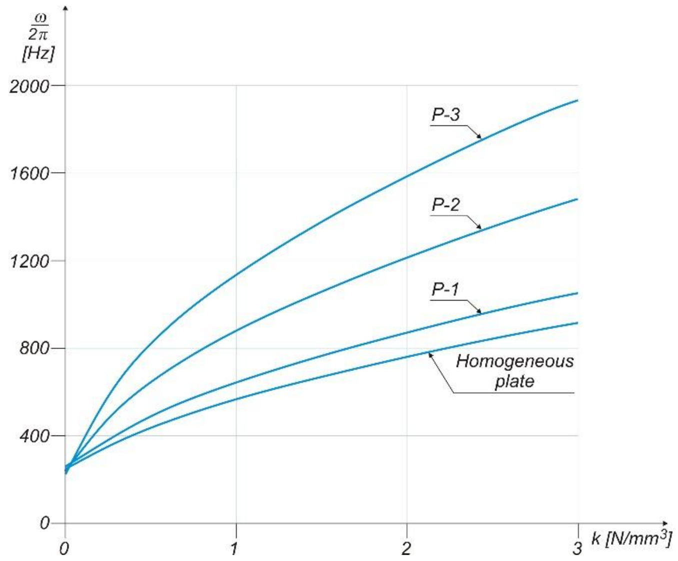

Figure 4. The graph of the natural frequency as a function of the foundation modulus

Thus, the foundation stiffness significantly affects the natural frequency of the plates. The value of the natural frequency grows with the increase of the foundation stiffness.

4. Conclusions

Based on the above analytical studies the following remarks are formulated:

- The presented analytical model of the circular plate is a generalization of the classical theory of plates.
- The symmetrical variation of the Young's modulus of the circular plate in its thickness direction is controlled by the exponent k_e and the dimensionless parameter e_0. The structure is homogeneous for e_0=1, when the modulus of elasticity is constant (E=const.).

- The shear effect (value of the shear coefficient C_s) is small in the axisymmetric flexural vibration problem of example circular plates.
- The influence of the foundation stiffness on the natural frequency of the plates is significant.

References

1. W. Nowacki, *Dynamics of elastic systems*, Chapman & Hall, London, 1963.
2. E. Ventsel, T. Krauthammer, *Thin plates and shells. Theory, Analysis and Applications,* Marcel Dekker, Inc. New York, Basel, 2001.
3. Y. Wang, R. Xu, H. Ding, *Free axisymmetric vibration FGM circular plate*, Appl. Math. Mech., 30 (2009) 1077 – 1082.
4. V. Tajeddini, A. Ohadi, M. Sadighi, *Three-dimensional free vibration of variable thickness thick circular and annular isotropic and functionally graded plates on Pasternak foundation*, Int. J. Mech. Sci. 53 (2011) 300-308.
5. M.H. Yas, V. Tahouneh, *3-D Free vibration analysis of thick functionally graded annular plates on Pasternak elastic foundation via differential quadrature method (DQM),* Acta Mech., 223 (2012) 43 – 62.
6. P. Melekzadeh, A. Farajpour, *Axisymmetric free and forced vibrations of initially stressed circular nanoplates embedded in an elastic medium*, Acta Mech., 223 (2012) 2311 – 2330.
7. M.A. Foyouzat, M. Mofid, J.E. Akin, *Free vibration of thin circular plates resting on an elastic foundation with a variable modulus,* J. Eng. Mech., 142 (2016) Issue 4.
8. E.A. Ahmad, R. Khorshidvand, *Free vibration analysis of saturated porous FG circular plates integrated with piezoelectric actuators via differential quadrature method,* Thin-Walled Struct., 125 (2018) 220 – 233.
9. K.K. Żur, *Quasi-Green's function approach to free vibration analysis of elastically supported functionally graded circular plates*, Comp. Struct., 183 (2018) 600 – 610.
10. K. Magnucki, D. Witkowski, E. Magnucka-Blandzi, *Buckling and free vibrations of rectangular plates with symmetrically varying mechanical properties – Analytical and FEM studies*, Comp. Struct, 220 (2019) 355-361.
11. K. Magnucki, W. Stawecki, J. Lewinski, *Bending of a circular plate with symmetrically thickness-wise varying mechanical properties*, Steel Comp. Struct., 34(6) (2020) 795-802.
12. K. Magnucki, J. Lewinski, E. Magnucka-Blandzi, *An improved shear deformation theory for bending beams with symmetrically varying mechanical properties in the depth direction*, Acta Mech., 231(10) (2020) 4381-4395.
13. E. Magnucka-Blandzi, K. Magnucki, W. Stawecki. *Bending and buckling of a circular plate with symmetrically varying mechanical properties*. Appl. Math. Model., 89(2) (2021) 1198-1205.