Jan Długosz University in Częstochowa
Scientific Issues, Mathematics XV, Częstochowa 2010

ENDOMORPHISM MONOID OF DIAMOND PRODUCT OF TWO COMMON COMPLETE BIPARTITE GRAPHS

Thiradet Jiarasuksakun, Tinnaluk Rutjanisarakul, Worapat Thongjua

Department of Mathematics, Faculty of Science
King Mongkut's University of Technology Thonburi (KMUTT)
126 Pracha-uthit Rd. Bangmod, Thungkru, Bangkok 10140 Thailand
e-mail: thiradet.jia@kmutt.ac.th

Abstract. An endomorphism of a graph $G = (V, E)$ is a mapping $f : V \longrightarrow V$ such that for all $x, y \in V$ if $\{x, y\} \in E$, then $\{f(x), f(y)\} \in E$. Let $End(G)$ be the class of all endomorphisms of graph G. The diamond product of graph $G = (V, E)$ (denoted by $G \diamond G$) is a graph defined by the vertex set $V(G \diamond G) = End(G)$ and the edge set $E(G \diamond G) = \{\{f, g\} \subset End(G) | \{f(x), g(x)\} \in E$ for all $x \in V\}$. Let $K_{m,n}$ be a complete bipartite graph on $m + n$ vertices. This research aims to study the algebraic property of $V(K_{m,n} \diamond K_{m,n}) = End(K_{m,n})$ after we have found that $K_{m,n} \diamond K_{m,n}$ is also a complete bipartite graph on $m^m n^n + n^m m^n$ vertices. The result shows that all of its vertices (endomorphisms) form a noncommutative monoid.

1. Introduction

In the graph theory [2, 5], a graph $G = (V, E)$ consists of a finite nonempty set V of objects called vertices, and a set E of 2-element subsets of V called edges. In this paper we use the following notation and classification of graphs.

- A path denoted P_n is a sequence of $n + 1$ vertices such that from each of its vertices there is an edge to the next vertex in the sequence.
- A cycle denoted C_n consists of n vertices connected in a closed chain.
- A complete graph denoted K_n is a graph on n vertices such that every two distinct vertices of G are adjacent.

- A graph G is called a bipartite graph if $V(G)$ can be partitioned into two subsets U and W, called partite sets, such that every edge of G joins a vertex of U and a vertex of W.
- A complete bipartite graph denoted $K_{m,n}$ is a graph on $m+n$ vertices such that one can partition V into two subsets U and W, where $|U| = m$ and $|W| = n$. Every edge of G joins a vertex of U and a vertex of W as well as every vertex of U is adjacent to every vertex of W.
- A $u - v$ walk in G is a sequence of vertices in G, beginning at u and ending at v such that consecutive vertices in the sequence are adjacent.
- A $u - v$ path in G is a $u - v$ walk in which no vertices are repeated.
- A graph G is called connected if G contains a $u - v$ path for every pair u, v of distinct vertices in G.
- A regular graph is a graph where each vertex has the same number of neighbors, i.e. every vertex has the same degree or valency. A regular graph with vertices of degree k is called a k-regular graph or regular graph of degree k.
- The distance between two vertices u and v in a graph (denoted by $d(u, v)$) is the number of edges in a shortest path connecting them. This is also known as the geodesic distance because it is the length of the graph geodesic between those two vertices. If there is no path connecting the two vertices, i.e. if they belong to different connected components, then conventionally the distance is defined as infinite.
- The diameter of a graph, denoted $\mathrm{diam}(G)$, is the maximum distance between any two vertices in the graph.

Definition 1. [1] *A homomorphism of a graph $G = (V, E)$ into a graph $H = (V', E')$ is a mapping $f : V \longrightarrow V'$ which preserves edges: for all $x, y \in V$, if $\{x, y\} \in E$, then $\{f(x), f(y)\} \in E'$. Let $Hom(G, H)$ be the class of all homomorphisms from a graph G into a graph H. In particular, an endomorphism of a graph $G = (V, E)$ is a mapping $f : V \longrightarrow V$ such that for all $x, y \in V$, if $\{x, y\} \in E$, then $\{f(x), f(y)\} \in E$. Let $End(G)$ be the class of all endomorphisms of graph G.*

From this definition, one can easily see that $Hom(G, H)$ may or may not exist. For example, $Hom(P_1, C_3)$ consists of 6 homomorphisms, while $Hom(C_3, P_1)$ is an empty set.

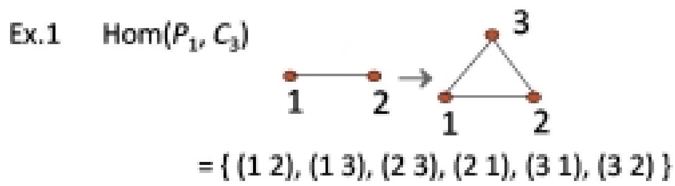

Figure 1: Hom(P_1, C_3).

Definition 2. [1] *The diamond product of a graph* $G = (V, E)$ *and a graph* $H = (V', E')$ *(denoted by* $G \diamond H$*) is a graph defined by the vertex set* $V(G \diamond H) = Hom(G, H)$*, where* $Hom(G, H) \neq \emptyset$*, and the edge set* $E(G \diamond H) = \{\{f, g\} \subset Hom(G, H) | \{f(x), g(x)\} \in E'$ *for all* $x \in V\}$*. In particular, the diamond product of a graph* G *with itself* $(G \diamond G)$ *is defined by the vertex set* $V(G \diamond G) = End(G)$ *and the edge set* $E(G \diamond G) = \{\{f, g\} \subset End(G) | \{f(x), g(x)\} \in E$ *for all* $x \in V\}$*.*

An example of graph $P_1 \diamond C_3$ is shown below.

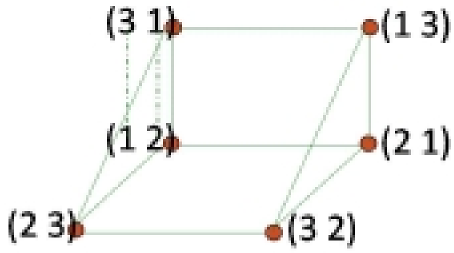

Figure 2: Graph $P_1 \diamond C_3$.

With this definition, there are some interesting results as follows:

Theorem 1. [3] *The graph* $P_m \diamond P_n$ *is connected for all positive integers* m, n *and* $diam(P_m \diamond P_n) = n$*.*

Theorem 2. [3] *Graphs* $P_m \diamond C_n$ *and* $C_n \diamond P_m$ *are connected for all positive integers* m, n*.* $diam(P_m \diamond C_n) \leq m + n$ *and* $diam(C_n \diamond P_m) = n$*.*

Theorem 3. [3] *If* G *is a connected graph, then the graph* $P_m \diamond G$ *is connected for all positive integers* m*, and* $diam(P_m \diamond G) = diam(G) + 2m$*.*

2. Some observations

In this paper, we study the diamond product of two complete bipartite graphs $K_{m,n}$.

- Denote $V(K_{m,n}) = \{1, 2, 3, ..., m, m+1, m+2, ..., m+n\}$, where
  $V_m = \{x \in V(K_{m,n}) \mid x \leq m\}$, and
  $V_n = \{x \in V(K_{m,n}) \mid m+1 \leq x \leq m+n\}$.
  Since $K_{m,n}$ is a complete bipartite graph, each vertex of V_m is adjacent to all vertices of V_n. Every edge joins a vertex of V_m and a vertex of V_n. We can define a function $h : V(K_{m,n}) \to \{0, 1\}$ such that

$$h(x) = \begin{cases} 0 & \text{if } x \in V_m, \\ 1 & \text{if } x \in V_n. \end{cases}$$

  By the definition of a complete bipartite graph, we obtain for all $x, y \in V(K_{m,n})$, $\{x, y\} \in E(K_{m,n})$ if and only if $\mid h(x) - h(y) \mid = 1$.

- Let $f : V(K_{m,n}) \to V(K_{m,n})$ be a homomorphism.
  Then $f \in V(K_{m,n} \diamond K_{m,n})$ if and only if

$$h(f(i)) = \begin{cases} 0 & \text{if } i \in V_m, \\ 1 & \text{if } i \in V_n \end{cases}$$

  or

$$h(f(i)) = \begin{cases} 1 & \text{if } i \in V_m, \\ 0 & \text{if } i \in V_n. \end{cases}$$

  For example, let us take a look at $K_{2,2} \diamond K_{2,2}$.

- We can define a norm as follows:

$$\|f - g\| = \max_{i \in V(K_{m,n})} |h(f(i)) - h(g(i))|.$$

3. Main results

Lemma 1. *For $f, g \in V(K_{m,n} \diamond K_{m,n})$, $\{f, g\} \in E(K_{m,n} \diamond K_{m,n})$ if and only if $\|f - g\| = 1$.*

Proof.
($\Rightarrow$) Let $\{f, g\} \in E(K_{m,n} \diamond K_{m,n})$. We have $\{f(i), g(i)\} \in E(K_{m,n})$ for all $i \in V(K_{m,n})$. Thus $|h(f(i)) - h(g(i))| = 1$ for all $i \in V(K_{m,n})$. This means that $\|f - g\| = 1$.

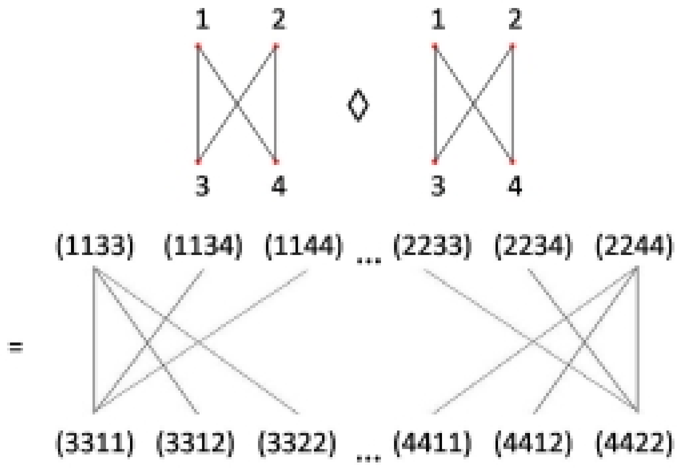

Figure 3: Graph $K_{2,2} \diamond K_{2,2}$

($\Leftarrow$) Let $\|f - g\| = 1$, where $f, g \in V(K_{m,n} \diamond K_{m,n})$. From the definition of norm, $\exists\ i_0 \in V(K_{m,n})$ such that $|h(f(i_0)) - h(g(i_0))| = 1$. Without loss of generality we may assume that $h(f(i_0)) = 0$ and $h(g(i_0)) = 1$.
If $i_0 \in V_m$, then we obtain

$$h(f(i)) = \begin{cases} 0 & \text{if } i \in V_m, \\ 1 & \text{if } i \in V_n \end{cases}$$

and

$$h(g(i)) = \begin{cases} 1 & \text{if } i \in V_m, \\ 0 & \text{if } i \in V_n. \end{cases}$$

So $|h(f(i)) - h(g(i))| = 1$, for all $i \in V(K_{m,n})$.
If $i_0 \in V_n$, then we obtain

$$h(f(i)) = \begin{cases} 1 & \text{if } i \in V_m, \\ 0 & \text{if } i \in V_n \end{cases}$$

and

$$h(g(i)) = \begin{cases} 0 & \text{if } i \in V_m, \\ 1 & \text{if } i \in V_n. \end{cases}$$

So $|h(f(i)) - h(g(i))| = 1$ for all $i \in V(K_{m,n})$. From both cases, we obtain $|h(f(i)) - h(g(i))| = 1$ for all $i \in V(K_{m,n})$. By the definitions of function h and diamond product, $\{f, g\} \in E(K_{m,n} \diamond K_{m,n})$. $\square$

Theorem 4. *$K_{m,n} \diamond K_{m,n}$ is a complete bipartite graph on $m^m n^n + n^m m^n$ vertices.*

Proof.
First, let us define $V_m{}^\diamond = \left\{ f \in V(K_{m,n} \diamond K_{m,n}) \mid h(f(i)) = \begin{cases} 0 & \text{if } i \in V_m \\ 1 & \text{if } i \in V_n \end{cases} \right\}$
and $V_n{}^\diamond = \left\{ f \in V(K_{m,n} \diamond K_{m,n}) \mid h(f(i)) = \begin{cases} 1 & \text{if } i \in V_m \\ 0 & \text{if } i \in V_n \end{cases} \right\}$.
Obviously, $V(K_{m,n} \diamond K_{m,n}) = V_m{}^\diamond \cup V_n{}^\diamond$ and $V_m{}^\diamond \cap V_n{}^\diamond = \emptyset$.

To show that the graph of $K_{m,n} \diamond K_{m,n}$ is bipartite, we need to prove that $\{f, g\} \in E(K_{m,n} \diamond K_{m,n})$ if and only if f and g belong to different sets of vertices $V_m{}^\diamond$ and $V_n{}^\diamond$.

($\Rightarrow$) First, let f and g belong to the same set of vertices. Without loss of generality we can assume $f, g \in V_m{}^\diamond$. We have

$$\|f - g\| = \max_{i \in V(K_{m,n})} |h(f(i)) - h(g(i))|.$$

If $i \in V_m$, then $h(f(i)) = 0,\ h(g(i)) = 0$ and

$$\max_{i \in V_m} |h(f(i)) - h(g(i))| = \max |0 - 0| = 0.$$

If $i \in V_n$, then $h(f(i)) = 1,\ h(g(i)) = 1$ and

$$\max_{i \in V_n} |h(f(i)) - h(g(i))| = \max |1 - 1| = 0.$$

Therefore $\|f - g\| = 0$ implies that $\{f, g\} \notin E(K_{m,n} \diamond K_{m,n})$ by Lemma 1. Then we conclude that if f and g belong to the same sets of vertices, there is no edge $\{f, g\}$ in the graph $K_{m,n} \diamond K_{m,n}$.

($\Leftarrow$) Without loss of generality we can take $f \in V_m{}^\diamond$ and $g \in V_n{}^\diamond$. We have

$$\|f - g\| = \max_{i \in V(K_{m,n})} |h(f(i)) - h(g(i))|.$$

If $i \in V_m$, then $h(f(i)) = 0,\ h(g(i)) = 1$ and

$$\max_{i \in V_m} |h(f(i)) - h(g(i))| = \max |0 - 1| = 1.$$

If $i \in V_n$, then $h(f(i)) = 1,\ h(g(i)) = 0$ and

$$\max_{i \in V_n} |h(f(i)) - h(g(i))| = \max |1 - 0| = 1.$$

Then $|h(f(i)) - h(g(i))| = 1$ for all $i \in V(K_{m,n})$, and $\|f - g\| = 1$. Therefore $\{f, g\} \in E(K_{m,n} \diamond K_{m,n})$.

By definition, all the vertices $f \in V_m{}^{\diamond}$ have the same value of $h(f(i))$ for all $i \in V$, and all the vertices $g \in V_n{}^{\diamond}$ have the same value of $h(g(i))$ for all $i \in V$ such that $\|f - g\| = 1$. This means that each vertex of $V_m{}^{\diamond}$ is adjacent to all vertices of $V_n{}^{\diamond}$, making it a complete bipartite graph.

We know that $K_{m,n} \diamond K_{m,n}$ have two partite sets $V_m{}^{\diamond}$ and $V_n{}^{\diamond}$. From the definition of $V_m{}^{\diamond}$, an endomorphism maps each vertex of V_m into a vertex of V_m giving us m^m choices and maps each vertex of V_n into a vertex of V_n with n^n choices. Thus $|V_m{}^{\diamond}| = m^m n^n$. On the other hand, an endomorphism in $V_n{}^{\diamond}$ maps each vertex of V_m into a vertex of V_n giving us n^m choices and maps each vertex of V_n into a vertex of V_m with m^n choices. Thus $|V_n{}^{\diamond}| = n^m m^n$. Both cases combined, we obtain the number of vertices in the theorem. □

Corollary 1. *$K_{m,n} \diamond K_{m,n}$ is a regular graph if and only if $m = n$.*

Proof.
Since $K_{m,n} \diamond K_{m,n}$ is a complete bipartite graph, we may pick $f \in V_m{}^{\diamond}$ and $g \in V_n{}^{\diamond}$. From Theorem 4, we have the following:

- $\{f, k\} \in E(K_{m,n} \diamond K_{m,n})$ for all $k \in V_n{}^{\diamond}$.
  Thus $\deg(f) = |V_n{}^{\diamond}| = n^m \cdot m^n$ for all $f \in V_m{}^{\diamond}$.
- $\{g, h\} \in E(K_{m,n} \diamond K_{m,n})$ for all $h \in V_m{}^{\diamond}$.
  Thus $\deg(g) = |V_m{}^{\diamond}| = m^m \cdot n^n$ for all $g \in V_n{}^{\diamond}$.

Hence, $K_{m,n} \diamond K_{m,n}$ is a regular graph if and only if $\deg(f) = \deg(g)$, which implies $m = n$. □

Now let us consider the vertex set of $K_{m,n} \diamond K_{m,n}$ with operation of function composition.

Theorem 5. *The vertex (endomorphism) set of $K_{m,n} \diamond K_{m,n}$ with composition form a noncommutative monoid for all positive integers $m, n > 1$.*

Proof.
It is clear that $V(K_{m,n} \diamond K_{m,n})$ is a monoid. To show that in the case when $m, n > 1$, it is noncommutative we can take $f, g \in V(K_{m,n} \diamond K_{m,n})$ such that

$$f(i) = \begin{cases} i & \text{if } i \in V_m, \\ m + 1 & \text{if } i \in V_n, \end{cases}$$

$$g(i) = \begin{cases} m + 2 & \text{if } i \in V_m, \\ i - m & \text{if } i \in V_n. \end{cases}$$

Then we have $(f \circ g)(m) = f(g(m)) = f(m+2) = m+1$. But $(g \circ f)(m) = g(f(m)) = g(m) = m+2$. Thus $f \circ g \neq g \circ f$, making it a noncommutative monoid. $\square$

Remark 1. *This noncommutative monoid is not a group since an endomorphism may not have an inverse. There exists a many-to-one endomorphism such as*

$$f(i) = \begin{cases} 1 & \text{if } \; i \in V_m, \\ m+1 & \text{if } \; i \in V_n. \end{cases}$$

Therefore, this endomorphism set forms only a noncommutative monoid, not a group.

Acknowledgements

The authors would like to thank Prof. Arworn and Ms. Damnernsawad who introduced us to the diamond product of graphs at the seminar at Chiangmai University in July 2008. Prof. Arworn also gave us some useful comments and suggestions.

References

[1] Sr. Arworn, P. Wojtylak. *Connectedness of Diamond Products*, preprint, 2008.

[2] G. Chartrand, P. Zhang. *Introduction to Graph Theory.* McGraw-Hill, 2005.

[3] J. Damnernsawad. *Diamond Product of Paths*, Master Degree Thesis, ChiangMai University, Thailand, 2007.

[4] T. Jiarasuksakun, T. Rutjanisarakul, W. Thongjua. *Diamond Product of Two Common Complete Bipartite Graphs*, Int. Conf. on Algebra and Geometry 2009 (ICAG 2009), Phuket, Thailand, 2009.

[5] D. West. *Introduction to Graph Theory*, 2nd edition. Prentice Hall, 2001.