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EXISTENCE AND MULTIPLICITY RESULTS FOR QUASILINEAR EQUATIONS IN THE HEISENBERG GROUP

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Abstract. In this paper we complete the study started in [*Existence of entire solutions for quasilinear equations in the Heisenberg group*, Minimax Theory Appl. 4 (2019)] on entire solutions for a quasilinear equation $(\mathcal{E}_\lambda)$ in $\mathbb{H}^n$, depending on a real parameter λ, which involves a general elliptic operator $\mathbf{A}$ in divergence form and two main nonlinearities. Here, in the so called sublinear case, we prove existence for all $\lambda > 0$ and, for special elliptic operators $\mathbf{A}$, existence of infinitely many solutions $(u_k)_k$.

Keywords: Heisenberg group, entire solutions, critical exponents.

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1. INTRODUCTION

Lately, great attention has been devoted to nonlinear elliptic problems involving critical nonlinearities, in the context of stratified groups. We just refer to [2, 7–9] and to the references therein. Indeed, Geometric Analysis in the Heisenberg group, and more in general in sub–Riemannian manifolds, represents one of the currently most active and exciting areas of mathematics. The main reason lies in the fact that the Heisenberg group $\mathbb{H}^n$, $n = 1, 2, 3, \ldots$, plays an important role in several branches of mathematics, such as representation theory, partial differential equations, number theory, several complex variables and quantum mechanics.

Actually, several recent results have been also established by many authors in the Euclidean elliptic setting. We mention only [11], as well as the references and comments given there, since this paper is an extension of [11] to the Heisenberg setting and also a completion of the study started in [9] on existence of entire solutions for a quasilinear equation $(\mathcal{E}_\lambda)$ in $\mathbb{H}^n$, depending on a real parameter λ, which involves a general elliptic operator $\mathbf{A}$ in divergence form and two main nonlinearities.

More precisely, we consider here as in [9] the one parameter elliptic equation in $\mathbb{H}^n$

$$-\mathrm{div}_H \mathbf{A}(q, D_H u) + a(q)\,|u|^{p-2}\,u = \lambda w(q)\,|u|^{m-2}\,u - h(q)\,|u|^{\mathfrak{m}-2}\,u, \qquad (\mathcal{E}_\lambda)$$

where $\lambda \in \mathbb{R}$ and $\mathbf{A} : \mathbb{H}^n \times \mathbb{H}_H \to \mathbb{H}_H$ admits a potential $\mathscr{A}$, with respect to its second variable ξ, while $\mathbb{H}_H$ is the span generated by $\{X_j, Y_j\}_{j=1}^n$, and

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}$$

are vector fields for any $j = 1, \dots, n$. The horizontal vector field $\mathbf{A}$ satisfies the following assumption $(\mathcal{A}_1)$, required throughout the paper.

$(\mathcal{A}_1)$ The potential $\mathscr{A} = \mathscr{A}(q, \xi)$ is a continuous function in $\mathbb{H}^n \times \mathbb{H}_H$, with continuous derivative with respect to ξ, $\mathbf{A} = \partial_\xi \mathscr{A}(q, \xi)$, and verifies:

(i) $\mathscr{A}(q, 0_H) = 0$ and $\mathscr{A}(q, \xi) = \mathscr{A}(q, -\xi)$ for all $(q, \xi) \in \mathbb{H}^n \times \mathbb{H}_H$;
(ii) $\mathscr{A}(q, \cdot)$ is strictly convex in $\mathbb{H}_H$ for all $q \in \mathbb{H}^n$;
(iii) There exist positive constants c_1, $c_2 > 0$ and an exponent p such that

$$c_1\,|\xi|_H^p \le (\mathbf{A}(q, \xi), \xi)_H, \quad |\mathbf{A}(q, \xi)|_H \le c_2\,|\xi|_H^{p-1} \tag{1.1}$$

for all $(q, \xi) \in \mathbb{H}^n \times \mathbb{H}_H$, where $1 < p < Q$, and $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$, while $(\cdot, \cdot)_H$ is the natural inner product in $\mathbb{H}_H$ and $|\cdot|_H$ the related norm introduced properly in Section 2.

Let r be the Heisenberg norm, defined for all $q = (z, t) \in \mathbb{H}^n$ by

$$r(q) = (|z|^4 + t^2)^{1/4}, \quad z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n,\ t \in \mathbb{R},$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^{2n}$. In the entire paper we assume for $(\mathcal{E}_\lambda)$ conditions

$(\mathcal{H}_1)$ (i) $a \in L^\infty_{\mathrm{loc}}(\mathbb{H}^n)$, and $a(q) \ge a_1[1 + r(q)]^{-p}$ for all $q \in \mathbb{H}^n$ and some constant $a_1 \in (0, 1]$.
(ii) The exponents m and $\mathfrak{m}$ are such that $1 < m < \mathfrak{m} < \infty$.
(iii) $0 < h \in L^1_{\mathrm{loc}}(\mathbb{H}^n)$, $0 \le w \in L^1_{\mathrm{loc}}(\mathbb{H}^n)$ and $w \not\equiv 0$.

$(\mathcal{H}_2)$ The coefficients h and w are related by the condition that either

$$\left(\frac{w^{\mathfrak{m}}}{h^m}\right)^{1/(\mathfrak{m}-m)} \in L^1(\mathbb{H}^n), \quad \text{or} \tag{1.2}$$

$$\left(\frac{w^{\mathfrak{m}-1}}{h^{m-1}}\right)^{1/(\mathfrak{m}-m)} \in L^{p^{*\prime}}(\mathbb{H}^n), \quad p^* = \frac{nQ}{n-Q}, \tag{1.3}$$

where $p^{*\prime}$ is the Hölder conjugate of p^*.

For detailed comments, concerning natural conditions under which (1.3) is weaker than (1.2), we refer to the Introduction of [9].

The elliptic equation $(\mathcal{E}_\lambda)$ is quasilinear, with two competing nonlinear terms which combine each other. The presence of a possible critical or supercritical nonlinearity makes the study fairly delicate. In the Euclidean setting several papers are devoted on problems of this kind, we just cite for historical comments given in the papers [1,10,11] and in the references therein. Motivated by these papers, we extend the multiplicity result of [11] from the Euclidean to the Heisenberg setting. However, $(\mathcal{E}_\lambda)$ is more delicate to handle in the Heisenberg setting, since the general framework produces new interesting complications, as shown in [9]. In any case, this paper is strongly based on the preliminary key properties proved in [9].

Equation $(\mathcal{E}_\lambda)$ is variational and the entire (weak) solutions u of $(\mathcal{E}_\lambda)$ are exactly the critical points of the underlying energy functional $\Phi_\lambda : X \to \mathbb{R}$, defined by

$$\Phi_\lambda(u) = \int_{\mathbb{H}^n} \mathscr{A}(q, D_H u)dq + \frac{1}{p}\|u\|_{p,a}^p - \frac{\lambda}{m}\|u\|_{m,w}^m + \frac{1}{\mathfrak{m}}\|u\|_{\mathfrak{m},h}^{\mathfrak{m}}$$

for all $u \in X$, where X is the natural solution space of $(\mathcal{E}_\lambda)$, introduced properly in the next Section 2 together with the involved weighted Lebesgue spaces.

Theorem 1.1. *Let $(\mathcal{A}_1)$, $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold and let $m < p$. Then*

(i) $(\mathcal{E}_\lambda)$ *admits at least a nontrivial nonnegative entire solution for any $\lambda > 0$;*
(ii) *if also assumption*
$(\mathcal{A}_2)$ *$\mathscr{A}$ is uniformly convex, i.e. for any $\varepsilon \in (0,1)$ there exists a number $\delta = \delta(\varepsilon) \in (0,1)$ such that either $|\xi - \eta| \le \varepsilon \max\{|\xi|, |\eta|\}$, or*

$$\mathscr{A}(q, (\xi+\eta)/2) \le \frac{1}{2}(1-\delta)[\mathscr{A}(q,\xi) + \mathscr{A}(q,\eta)]$$

for any $q \in \mathbb{H}^n$ and all ξ, $\eta \in \mathbb{H}_H$,
is satisfied, then for all $\lambda > 0$ equation $(\mathcal{E}_\lambda)$ has at least two nontrivial nonnegative entire solutions, and actually $(\mathcal{E}_\lambda)$ possesses infinitely many solutions $(u_k)_k$, whose negative critical values $c_k = \Phi_\lambda(u_k)$ tend to 0 as $k \to \infty$, where Φ_λ is the underlying functional of $(\mathcal{E}_\lambda)$, defined above.

The operator $\mathbf{A}(q,\xi) = \alpha(q)\,|\xi|_H^{p-2}\,\xi$, where $\alpha \in C(\mathbb{H}^n)$ and $0 < \alpha_1 \le \alpha \le \alpha_2$ in $\mathbb{H}^n$ for suitable numbers α_1, $\alpha_2 \in \mathbb{R}^+$, satisfies $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, provided that $1 < p < Q$. This popular prototype is the non–homogeneous horizontal p-Laplacian operator on the Heisenberg group, with continuous coefficient. Therefore, in the subcase $\alpha \equiv 1$, along any $\varphi \in C_c^\infty(\mathbb{H}^n)$

$$\mathrm{div}_H \mathbf{A}(q, D_H\varphi) = \mathrm{div}_H(|D_H\varphi|^{p-2} D_H\varphi) = \Delta_{H,p}\varphi$$

reduces to the so called *horizontal p-Laplacian on the Heisenberg group.* When $p = 2$ it becomes the *Kohn–Spencer Laplacian* operator Δ_H.

Several standard convexity conditions on $\mathscr{A}$ imply $(\mathcal{A}_2)$. This is shown in the Remark of Section 3 of [11]. Concerning the famous condition (1.2) we refer to the historical comments given in [9, 11]. We recall in passing that under very general assumptions Lemma 3.1 of [9] shows that if $(\mathcal{E}_\lambda)$ admits a nontrivial entire solution $u \in X$, then $\lambda > 0$.

The paper is divided into three sections. Section 2 contains some preliminaries and notations and Section 3 is devoted to the proof of Theorem 1.1.

2. PRELIMINARIES

We briefly recall the relevant definitions and notations related to the Heisenberg group functional setting. For a complete treatment, we refer to [3,5,6].

Let $\mathbb{H}^n$ denote the Heisenberg group of dimension $2n+1$, that is the Lie group whose underlying manifold is $\mathbb{R}^{2n+1}$, endowed with the non–commutative group law

$$q \circ q' = \left(z + z', t + t' + 2\sum_{i=1}^{n} (y_i x_i' - x_i y_i') \right)$$

for all q, $q' \in \mathbb{H}^n$, with

$$q = (z,t) = (x_1, \ldots, x_n, y_1, \ldots, y_n, t), \quad q' = (z',t') = (x_1', \ldots, x_n', y_1', \ldots, y_n', t').$$

The Heisenberg group $\mathbb{H}^n$ is the most commutative among the non–commutative Lie groups, and hence gives the greatest opportunity for generalizing the remarkable results stated in Euclidean analysis.

In $\mathbb{H}^n$ the natural origin is denoted by $O = (0,0)$. Define

$$r(q) = r(z,t) = (|z|^4 + t^2)^{1/4} \quad \text{for all } q = (z,t) \in \mathbb{H}^n,$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^{2n}$. The *Korányi norm* is homogeneous of degree 1, with respect to the dilations $\delta_R : (z,t) \mapsto (Rz, R^2 t)$, $R > 0$. Indeed, for all $q = (z,t) \in \mathbb{H}^n$

$$r(\delta_R(q)) = r(Rz, R^2 t) = (|Rz|^4 + R^4 t^2)^{1/4} = R\, r(q).$$

Hence, the *Korányi distance* is

$$d_K(q, q') = r(q^{-1} \circ q') \quad \text{for all } (q,q') \in \mathbb{H}^n \times \mathbb{H}^n,$$

and the *Korányi open ball* of radius R centered at q_0 is

$$B_R(q_0) = \{ q \in \mathbb{H}^n : d_K(q, q_0) < R \}.$$

For simplicity B_R denotes the ball of radius R centered at $q_0 = O$.

The Jacobian determinant of δ_R is R^{2n+2}. The natural number $Q = 2n+2$, which is the so called *homogeneous dimension of* $\mathbb{H}^n$, plays a role analogous to the topological dimension in the Euclidean context.

The Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure on $\mathbb{R}^{2n} \times \mathbb{R}$. It is invariant under left translations and Q–homogeneous with respect to dilations. Hence, as noted in [6], the topological dimension $2n+1$ of $\mathbb{H}^n$ is strictly less than its Hausdorff dimension Q. We denote by $|E|$ the measure of any measurable set $E \subset \mathbb{H}^n$. Then

$$|\delta_R(E)| = R^Q |E|, \qquad d(\delta_R q) = R^Q dq.$$

In particular, if $E = B_R$, then $|B_R| = R^Q w_Q$, where w_Q is the measure of the unit sphere of $\mathbb{H}^n$.

The vector fields for $j = 1, \ldots, n$

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \qquad \frac{\partial}{\partial t},$$

constitute a basis for the real Lie algebra of left–invariant vector fields on $\mathbb{H}^n$. This basis satisfies the Heisenberg canonical commutation relations for position and momentum $[X_j, Y_k] = -4\delta_{jk}\partial/\partial t$, all other commutators being zero. The span of $\{X_j, Y_j\}_{j=1}^n$ is briefly denoted by $\mathbb{H}_H$ and a vector field in $\mathbb{H}_H$ is called *horizontal.*

Let $u \in C^1(\mathbb{H}^n)$ be fixed. The *horizontal gradient* $D_H u$ is

$$D_H u = \sum_{j=1}^n \left[(X_j u)X_j + (Y_j u)Y_j\right],$$

that is, it is an element of $\mathbb{H}_H$. Furthermore, if $f \in C^1(\mathbb{R})$, then $D_H(f \circ u) = f'(u) D_H u$.

The natural inner product in $\mathbb{H}_H$

$$\big(W, Z\big)_H = \sum_{j=1}^n \left(w^j z^j + \widetilde{w}^j \widetilde{z}^j\right)$$

for $W = \{w^j X_j + \widetilde{w}^j Y_j\}_{j=1}^n$ and $Z = \{z^j X_j + \widetilde{z}^j Y_j\}_{j=1}^n$ produces the Hilbertian norm

$$|D_H u|_H = \sqrt{\big(D_H u, D_H u\big)_H}$$

for the horizontal vector field $D_H u$. Moreover, if also $v \in C^1(\mathbb{H}^n)$ then the Cauchy–Schwarz inequality

$$\left|\big(D_H u, D_H v\big)_H\right| \le |D_H u|_H |D_H v|_H$$

continues to be valid.

Then the *horizontal divergence* is defined, for horizontal vector fields $W = \{w^j X_j + \widetilde{w}^j Y_j\}_{j=1}^n$ of class $C^1(\mathbb{H}^n, \mathbb{R}^{2n})$, by

$$\operatorname{div}_H W = \sum_{j=1}^n [X_j(w^j) + Y_j(\widetilde{w}^j)].$$

If furthermore $g \in C^1(\mathbb{H}^n)$, then the *Leibnitz formula* holds, namely

$$\operatorname{div}_H(gW) = g \operatorname{div}_H W + \big(D_H g, W\big)_H.$$

The *Kohn–Spencer Laplacian* Δ_H, or equivalently the *horizontal Laplacian in* $\mathbb{H}^n$, of a function u of class $C^2(\mathbb{H}^n)$ is defined by

$$\Delta_H u = \sum_{j=1}^{n}(X_j^2 + Y_j^2)u = \sum_{j=1}^{n}\left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j\frac{\partial^2}{\partial x_j \partial t} - 4x_j\frac{\partial^2}{\partial y_j \partial t}\right)u + 4|z|^2\frac{\partial^2 u}{\partial t^2},$$

and Δ_H is *hypoelliptic* according to the celebrated Theorem 1.1 due to *Hörmander* in [4]. In particular, $\Delta_H u = \mathrm{div}_H D_H u$ for each $u \in C^2(\mathbb{H}^n)$.

A well known generalization of the Kohn–Spencer Laplacian is the *horizontal p-Laplacian* on the Heisenberg group given by

$$\Delta_{H,p}u = \mathrm{div}_H(|D_H u|_H^{p-2}D_H u),$$

which is well defined for all function $u \in C^2(\mathbb{H}^n)$ and $p \in (1,\infty)$. But in this paper $1 < p < Q$.

Throughout the paper we assume that $(\mathcal{A}_1)$, $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$ *are fulfilled*, without further mentioning. Clearly, $(\mathcal{E}_\lambda)$ involves weighted Lebesgue spaces. Hence, denoted by ω a generic weight on $\mathbb{H}^n$ of class $L^1_{\mathrm{loc}}(\mathbb{H}^n)$, for any σ, with $1 < \sigma < \infty$, we define

$$L^\sigma(\mathbb{H}^n,\omega) = \{u : \mathbb{H}^n \to \mathbb{R} \text{ measurable} : \omega^{1/\sigma}|u| \in L^\sigma(\mathbb{H}^n)\},$$

endowed with the norm $\|u\|_{\sigma,\omega} = \|\omega^{1/\sigma}u\|_\sigma$. For the main properties of the weighted spaces $L^p(\mathbb{H}^n,a)$, $L^m(\mathbb{H}^n,w)$ and $L^{\mathfrak{m}}(\mathbb{H}^n,h)$ we refer to Lemma 2.1 of [9] and for a complete general treatment to [5].

The natural solution space, associated to $(\mathcal{E}_\lambda)$, is $X = (X, \|\cdot\|)$, where

$$X = \left\{u \in E : \int_{\mathbb{H}^n} h(q)|u(q)|^{\mathfrak{m}}dq < \infty\right\},$$

and $\|u\| = \|u\|_E + \|u\|_{\mathfrak{m},h}$, while E is the completion of $C_c^\infty(\mathbb{H}^n)$, with respect to the norm $\|u\|_E = \left(\|D_H u\|_p^p + \|u\|_{p,a}^p\right)^{1/p}$, where

$$\|D_H u\|_p = \left(\int_{\mathbb{H}^n}|D_H u(q)|_H^p dq\right)^{1/p}, \quad \|u\|_{p,a} = \left(\int_{\mathbb{H}^n} a(q)|u(q)|^p dq\right)^{1/p}.$$

Clearly E and X are well defined, since $C_c^\infty(\mathbb{H}^n) \subset X \subset E$. Moreover, X is a separable reflexive Banach space, as proved in Lemma 2.2 of [9]. Finally, the key historical condition $(\mathcal{H}_2)$ assures that the embedding

$$X \hookrightarrow\hookrightarrow L^m(\mathbb{H}^n,w)$$

is compact.

A (weak) *entire solution of* $(\mathcal{E}_\lambda)$ is a function $u \in X$ satisfying the identity

$$\int_{\mathbb{H}^n} (\mathbf{A}(q, D_H u), D_H v)_H dq + \int_{\mathbb{H}^n} a(q)\,|u|^{p-2} uv dq$$
$$= \lambda \int_{\mathbb{H}^n} w(q)\,|u|^{m-2} uv dq - \int_{\mathbb{H}^n} h(q)\,|u|^{\mathfrak{m}-2} uv dq$$

for all $v \in X$.

As already noted in the Introduction, $(\mathcal{E}_\lambda)$ has a variational structure and (weak) entire solutions u of $(\mathcal{E}_\lambda)$ are exactly the critical points of the energy functional $\Phi_\lambda : X \to \mathbb{R}$, defined by

$$\Phi_\lambda(u) = \Phi_{\mathscr{A}}(u) + \Phi_a(u) - \lambda \Phi_w(u) + \Phi_h(u),$$
$$\Phi_{\mathscr{A}}(u) = \int_{\mathbb{H}^n} \mathscr{A}(q, D_H u) dq, \quad \Phi_a(u) = \frac{1}{p}\|u\|_{p,a}^p, \quad \Phi_w(u) = \frac{1}{m}\|u\|_{m,w}^m,$$
$$\Phi_h(u) = \frac{1}{\mathfrak{m}}\|u\|_{\mathfrak{m},h}^{\mathfrak{m}},$$

for all $u \in X$.

3. PROOF OF THEOREM 1.1

Under very general conditions, Lemma 3.1 of [9] asserts that $\lambda > 0$, if $(\mathcal{E}_\lambda)$ admits a nontrivial entire solution $u \in X$. Consequently, *from now on we consider only the case* $\lambda > 0$, without further mentioning.

Before proving Theorem 1.1, we present some useful properties. Since X is a separable and reflexive Banach space thanks to Lemma 2.2. of [9], there exist two sequences $(e_j)_j \subset X$ and $(e_j^*)_j \subset X'$ such that

$$X = \overline{\text{span}}\,\{e_j,\, j = 1, 2, \ldots\}, \qquad X' = \overline{\text{span}}^{w^*}\{e_j^*,\, j = 1, 2, \ldots\},$$

and $< e_i^*, e_j >= \delta_{ij}$, $i,\, j = 1, 2, \ldots$, where $< \cdot, \cdot >$ is the dual pairing between X and its dual space X', while δ_{ij} denotes the Kronecker symbol and $\overline{}^{w^*}$ is the closure of a subset of X' with respect to the weak star topology on X'.

For brevity, we put

$$X_j = \text{span}\,\{e_j\}, \qquad Y_k = \overset{k}{\underset{j=1}{\oplus}} X_j, \qquad Z_k = \overline{\overset{\infty}{\underset{j=k}{\oplus}} X_j}. \tag{3.1}$$

Let us state for completeness a useful result proved in [11].

Lemma 3.1 (Lemma 5.1 of [11]). *Let $\Phi : X \to \mathbb{R}$ be sequentially weakly continuous in X, with $\Phi(0) = 0$. Fix $R > 0$ and put*

$$\beta_k = \sup\,\{\Phi(u)\ :\ \|u\| \le R,\ u \in Z_k\}$$

for all k. Then $\beta_k \to 0$ as $k \to \infty$.

Proof of Theorem 1.1. Fix $\lambda > 0$ and recall that by assumption

$$1 < m < p \quad \text{and} \quad m < \mathfrak{m} \tag{3.2}$$

holds.

(i) The functional Φ_λ is weakly lower semi–continuous in X by Lemma 3.6 of [9] and also coercive in X by Lemma 3.2(i) of [9]. Hence, since X is a reflexive Banach space by Lemma 2.2 of [9], the functional Φ_λ attains its infimum at some $u_1 \in X$ and clearly u_1 is a solution of $(\mathcal{E}_\lambda)$.

Now conditions $(\mathcal{A}_1)$(i) and (ii) imply that

$$\mathscr{A}(q,\xi) \le (\mathbf{A}(q,\xi),\xi)_H \quad \text{for all } (q,\xi) \in \mathbb{H}^n \times \mathbb{H}_H. \tag{3.3}$$

Moreover, $(\mathcal{A}_1)$(i) and (iii) give

$$\mathscr{A}(q,\xi) = \int_0^1 \frac{d}{dt}\mathscr{A}(q,t\xi)dt = \int_0^1 \frac{1}{t}(\mathbf{A}(q,t\xi),t\xi)_H dt \ge \frac{c_1}{p}\,|\xi|_H^p\,,$$

which, together with (1.1) and (3.3), yields

$$c\,|\xi|_H^p \le \mathscr{A}(q,\xi) \le (\mathbf{A}(q,\xi),\xi)_H \le c_2\,|\xi|_H^p \tag{3.4}$$

for all $(q,\xi) \in \mathbb{H}^n \times \mathbb{H}_H$.

Fix a nontrivial function $\varphi \in C_c^\infty(\Omega)$. Then, putting $C = \max\{c_2,\,1\}/p$, for all $t > 0$ small enough, we have

$$\begin{aligned}\Phi_\lambda(t\varphi) \le t^p C\left(\int_{\mathbb{H}^n} |D_H\varphi|_H^p\,dq + \int_{\mathbb{H}^n} a(q)\,|\varphi|^p\,dq\right) + \frac{t^{\mathfrak{m}}}{\mathfrak{m}}\int_{\mathbb{H}^n} h(q)\,|\varphi|^{\mathfrak{m}}\,dq \\ -\lambda\frac{t^m}{m}\int_{\mathbb{H}^n} w(q)\,|\varphi|^m\,dq < 0,\end{aligned}$$

since $w > 0$ in Ω and (3.2) is valid.

Thus, $\inf_{u\in X}\Phi_\lambda(u) < 0$. Therefore $(\mathcal{E}_\lambda)$ has a nontrival nonnegative entire solution.

(ii) Since in this case $\mathscr{A}$ is uniformly convex, Theorem 1.1(iii) of [9] asserts that $(\mathcal{E}_\lambda)$ admits at least two nontrivial nonnegative entire solutions in X.

Thanks to the fact that in this case $m < p$, we claim that $(\mathcal{E}_\lambda)$ has a sequence of solutions $(\pm u_k)_k$ such that $\Phi_\lambda(\pm u_k) < 0$ and $\Phi_\lambda(\pm u_k) \to 0$ as $k \to \infty$.

The functional Φ_λ is even in X, since the potential $\mathscr{A}(q,\cdot)$ is even in $\mathbb{H}_H$ for all $q \in \mathbb{H}^n$ by virtue of assumption $(\mathcal{A}_1)$(i). Moreover, Φ_λ is coercive in X by Lemma 3.2(i) of [9] and Φ_λ satisfies the (PS) condition in X by Lemma 3.9 of [9]. Using Definition 5.1 on page 94 of [12], we denote by $\gamma(B)$ the genus of $B \in \mathscr{C}$, where

$$\begin{gathered}\mathscr{C} = \{B \subset X\backslash\{0\} \,:\, B \text{ is compact and } B = -B\}, \\ \mathscr{C}_k = \{B \in \mathscr{C} \,:\, \gamma(B) \ge k\}, \quad c_k = \inf_{B\in\mathscr{C}_k}\sup_{u\in B}\Phi_\lambda(u), \quad k = 1,2,\ldots\end{gathered}$$

Thus
$$-\infty < c_1 \le c_2 \le \ldots \le c_k \le c_{k+1} \le \ldots$$
We assert that $c_k < 0$ for every k.

As shown in Section 2, the test space $C_c^\infty(\Omega)$ is a subspace of X. Fix $k \in \mathbb{N}$ and choose a k–dimensional linear subspace F_k of $C_c^\infty(\Omega)$. Since all the norms on F_k are equivalent, there exists $\rho_k \in (0,1)$ such that $\varphi \in F_k$ and $\|\varphi\| \le \rho_k$ implies that $\|\varphi\|_\infty \le \delta < 1$. Put
$$S^{(k)}_{\rho_k} = \{u \in F_k \,:\, \|u\| = \rho_k\}.$$
From the compactness of $S^{(k)}_{\rho_k}$ and the fact that $w > 0$ in Ω, for all k there exist constants θ_k, $\eta_k > 0$ such that for all $\varphi \in S^{(k)}_{\rho_k}$
$$\Phi_w(\varphi) = \frac{1}{m}\int\limits_{\mathbb{H}^n} w(q)|\varphi|^m dq \ge \frac{1}{m}\int\limits_{\Omega} w(q)|\varphi|^m dq \ge \theta_k$$
and
$$\Phi_{\mathscr{A}}(\varphi) + \Phi_a(\varphi) + \Phi_h(\varphi) \le C\|\varphi\|_E^p + \Phi_h(\varphi) \le \eta_k.$$
Therefore, for $\varphi \in S^{(k)}_{\rho_k}$ and $t \in (0,1)$
$$\begin{aligned}\Phi_\lambda(t\varphi) &= \Phi_{\mathscr{A}}(t\varphi) + \Phi_a(t\varphi) + \Phi_h(t\varphi) - \lambda\Phi_w(t\varphi)\\ &\le Ct^p\|\varphi\|_E^p + t^{\mathfrak{m}}\Phi_h(\varphi) - \lambda\, t^m\Phi_w(\varphi)\\ &\le \eta_k\big(C\,t^p + t^{\mathfrak{m}}\big) - \lambda\,\theta_k t^m.\end{aligned}$$
Since $1 < m < p$ and $m < \mathfrak{m}$ by (3.2), for all k there exist $t_k \in (0,1)$ and $\varepsilon_k > 0$ so small that for all $\varphi \in S^{(k)}_{\rho_k}$
$$\Phi_\lambda(t_k\varphi) \le -\varepsilon_k < 0, \quad \text{that is } \Phi_\lambda(u) \le -\varepsilon_k < 0$$
for all $u \in S^{(k)}_{t_k\rho_k}$. Finally, $\gamma(S^{(k)}_{t_k\rho_k}) = k$, so that $c_k \le -\varepsilon_k < 0$ for all k and the assertion is proved.

By the genus theory, see for instance Theorem 4.2 and the Remark on page 97 of [12], each c_k is a critical value of Φ_λ. Hence there is a sequence of solutions $(\pm u_k)_k$ such that $\Phi_\lambda(\pm u_k) < 0$. It only remains to show that $c_k \to 0$ as $k \to \infty$.

Since Φ_λ is coercive in X by Lemma 3.2(i) of [9], there exists a constant $R > 0$ such that $\Phi_\lambda(u) > 0$ for all u, with $\|u\| \ge R$. Fix k and let Y_k, Z_k be as in (3.1). Take $B \in \mathscr{C}_k$, so that $\gamma(B) \ge k$. Therefore, according to the properties of genus, $B \cap Z_k \ne \varnothing$. Put
$$\beta_k = \sup\{\lambda\,\Phi_w(u) \,:\, u \in Z_k,\ \|u\| \le R\}.$$
Thus $\beta_k \to 0$ as $k \to \infty$ by Lemma 3.1, since Φ_w is sequentially weakly continuous in X by Lemma 3.5 of [9]. If $u \in Z_k$ and $\|u\| \le R$, then
$$\Phi_\lambda(u) = \Phi_{\mathscr{A}}(u) + \Phi_a(u) + \Phi_h(u) - \lambda\Phi_w(u) \ge -\lambda\Phi_w(u) \ge -\beta_k.$$
Hence, $\sup\limits_{u\in B}\Phi_\lambda(u) \ge -\beta_k$, and so $0 > c_k \ge -\beta_k$. This implies at once that $c_k \to 0$ as $k \to \infty$. $\square$

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