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The estimation of virial coefficients via the third harmonics measurements

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ABSTRACT

The derivation of the next nonlinear term of the KZK equation is done within original perturbation scheme on the base of the virial expansion for thermic equation of state. We also derived equations for second and third harmonic components in the nearfield of the sound wave generated by a piston transducer. Calculation scheme for a numerical estimation of the integrals for the fundamental and second harmonics is proposed as well as for the averaged third harmonic component. Some results of performed calculations are given for the illustration of the method possibility and comparison with direct finite difference solutions of the KZK equation.

SYMBOLS

- a radius of the circular piston source;
- b dissipation coefficient of the medium;
- c_0 velocity of acoustic wave at infinitesimal amplitude;
- k wave number, $k=\omega/c_0$;
- p acoustic pressure;
- p_0 amplitude of acoustic pressure on the source;
- p' acoustic pressure normalized to p_0, $p' = p/p_0$;
- $|p_n|$ amplitude of n-th harmonic component of acoustic pressure;
- $|p'_n|$ dimensionless (normalized to p_0) amplitude of n-th harmonic component of acoustic pressure;
- P_n complex amplitude of n-th harmonic component of acoustic pressure;
- r radial variable;
- r_0 Rayleigh distance, $r_0 = ka^2/2$;
- z axial variable;
- α_1 linear absorption coefficient, $\alpha_1 = b\omega^2/2\rho_0 c_0^3$;
- ε parameter of nonlinearity, $\varepsilon = 1+B/2A$;
- ρ_0 equilibrium density of the medium;
- σ dimensionless coordinate along acoustic axis, $\sigma = z/r_0$;
- ξ dimensionless coordinate across acoustic axis, $\xi = r/a$;
- τ retarded time $\tau=t-z/c_0$;
- ω angular frequency.

INTRODUCTION

At the paper [1] it was announced that authors derived the additional term to KZK equation. The term is reproduced without derivation details and it is explained that it is originated from the cubic part of the equation of state [2]:

$$P - P_0 = A\left(\frac{\rho-\rho_0}{\rho_0}\right) + \frac{B}{2}\left(\frac{\rho-\rho_0}{\rho_0}\right)^2 + \frac{C}{6}\left(\frac{\rho-\rho_0}{\rho_0}\right)^3 + \ldots \qquad (1)$$

Here we propose the possible way to estimate the constant C of the equation (1) analyzing the measurements of the third harmonics of acoustic nearfield produced by classical circular piston oscillations. The theory we develop includes the derivation of the KZK equation analogue within the next order of accuracy in nonlinear and diffraction parameter [2]. We save and show all the terms that appear in the derivation procedure that suppose equality of the amplitude and configuration parameter. Afterwards we use the temporal Fourier components of the nearfield with slow variable coefficients and introduce nonlinear resonant terms equations in the traditional sense. The study of analytical formulas is performed on the base of results of [3] that give minimal number of integrations taking into account the cylindrical symmetry. The calculations of integrals for first and second harmonics amplitudes is made numerically using the special procedures to simplify formulas in the case of rapidly oscillating integrands [4]. The results are partially reproduced below.

The third harmonics is the main target of our efforts. It is investigated in the framework of the averaging procedure, that is also done in [4] but for the equation of state without a cubic term. The technique allows to take into account the new term of KZK. Namely, this allows us to compare the contributions of the given harmonic components, that correspond to different accuracy in the equation of state.

THEORETICAL MODEL

For a derivation of the KZK analogue with nonlinear terms in the next order of the perturbation theory we follow the pioneering work [5] with the same small parameter μ. We also change the basic equation of state that couple the pressure p and mass density ρ to the third-order expansion (1). The resulting equation has the form:

$$\rho_{\tau z} - \frac{c_0}{2} \cdot \Delta_\perp \rho - \delta \cdot \rho_{\tau\tau\tau} = \varepsilon' \cdot [\rho^2]_{\tau\tau} + + \mu\rho_0 \frac{2B\rho_0 + 2c_0^2 - C\rho_0}{12 c_0^3} [\rho^3]_{\tau\tau}, \quad (2)$$

where $\delta = b/2c_0^3\rho_0$, $\varepsilon' = \varepsilon/2c_0\rho_0$ are the dissipation and nonlinear factors. There are some extra terms in the equation (2) of the second (in ρ) order but we omit them because such terms are small in a high frequency range.

The solution of the equation (2) we shall expand in Fourier series introducing for convenience the amplitude parameter λ:

$$p = \sum_{n=0}^{\infty} p^{(n)} \cdot \lambda^{n+1} \quad (3)$$

Here $p^{(n)} = A^{(n)}(\tau, z) \cdot \exp[i\omega\tau(n+1)]$ +c.c.

The substitution of (3) in (2) gives after transformations:

$$\sum_{n=0}^{\infty} \lambda^{n+1} \cdot Lp^{(n)} =$$
$$-\omega^2 \sum \lambda^{l+m} \left\{ (l+m+2)^2 \left[A^{(l)} \cdot A^{(m)} \cdot e^{j\omega\tau(l+m+2)} + c.c. \right] + \right.$$
$$\left. + (l-m)^2 \left[A^{*(l)} \cdot A^{(m)} \cdot e^{j\omega\tau(l-m)} + c.c. \right] \right\}.$$

The lowest nonlinear resonant terms are indicated in the table:

l	m	$l+m+2$	$l-m$	l	m	$l+m.+2$	$l-m.$
0	0	2	0	1	1	4	0
1	0	3	1	2	0	4	0
0	1	3	-1	0	2	4	-2

Equalizing the resonance terms we go to amplitudes of harmonics. For example the third harmonics may be simpliefied if one introduces new amplitudes as:

$$p^{(2)} = \gamma \exp\left(-9\delta\omega^2 z\right).$$
$$p^{(1)} = \beta \exp\left(-4\delta\omega^2 z\right). \quad (4)$$
$$p^{(0)} = \alpha \exp\left(-\delta\omega^2 z\right).$$

The equation for γ takes the form:

$$6i\omega\gamma_z = c_0 \Delta_\perp \gamma -$$
$$-\left[36\varepsilon' \omega^2 \alpha\beta - 3\omega^2 \rho_0 \frac{C\rho_0 - 2B - c_0^2}{2c_0^3} \gamma^3 \right] \exp\left(4\delta\omega^2 z\right).$$

Below we restrict ourselves only to the standard KZK terms for the illustrations of the possibilities of the method and a comparison with the numerical results and measurements [7]. Integrating by z and over the infinite diameter receiver with normalization by πa^2 gives:

$$\langle \gamma \rangle = \frac{1}{\pi a^2} \int_0^\infty 2\pi r \gamma \, dr =$$
$$= \frac{-12\,\omega\,\varepsilon'}{a^2} \int_0^z e^{4\delta\omega^2 z} \int_0^\infty \alpha\, \beta\, r\, dr\, dz. \quad (5)$$

If we substitute the expressions for α and β from [3] in the equation (5) and integrate it we go to the formula for averaged third pressure harmonics at distance z. The resulting formula for the distances $z \le 1/4$ (where $\exp(4\delta\omega^2 z) \cong 1 + 4\delta\omega^2 z$) may be expressed as linear combination of ε'^2 and $\delta\varepsilon'^2$ with diffraction coefficients that depend only on z. It allows us to simplify calculations of nonlinear and absorption parameters by fitting of measurements to theoretical curves.

CALCULATION OF INTEGRALS AND COMPARISON OF RESULTS WITH NUMERICAL SOLUTION OF KZK EQUATION

The fundamental harmonic component is expressed as integral of Bessel functions and exponent combination [3]. Introducing dimensionless coordinates $\xi = r/a$ and $\sigma = z/r_0$ we obtain:

$$P_1(\sigma,\xi) = p_0 e^{-\alpha_1 r_0 \sigma} \int_0^{\infty} J_1(\lambda)\, J_0(\lambda\xi)\, \exp\left(\frac{i\,\lambda^2 \sigma}{4}\right) d\lambda,$$

where J_n are nth order Bessel functions. The following integral sum was applied for numerical calculations to account oscillations of the integrand with the biggest scale of J_1:

$$\int_0^{\infty} J_1(\lambda)\, J_0(\lambda\xi)\, \exp\left(\frac{i\,\lambda^2 \sigma}{4}\right) d\lambda \cong$$

$$\cong \int_0^{1} J_1(\lambda)\, J_0(\lambda\xi)\, \exp\left(\frac{i\,\lambda^2 \sigma}{4}\right) d\lambda +$$

$$+ \sum_{l=0}^{L} \int_{1+l\Delta}^{1+(l+1)\Delta} J_1(\lambda)\, J_0(\lambda\xi)\, \exp\left(\frac{i\,\lambda^2 \sigma}{4}\right) d\lambda \cong$$

$$\cong \int_0^{1} \ldots +$$

$$+ \sum_{l=0}^{L} \{J_0(1+l\Delta) - J_0[1+(l+1)\Delta]\} J_0\{[1+(l+0.5)\Delta]\xi\} \times$$

$$\times \exp\left(\frac{i\,[1+(l+0.5)\Delta]^2 \sigma}{4}\right).$$

Results of calculations of the fundamental harmonic component for a circular uniform source in the lossless medium are shown at figure1.

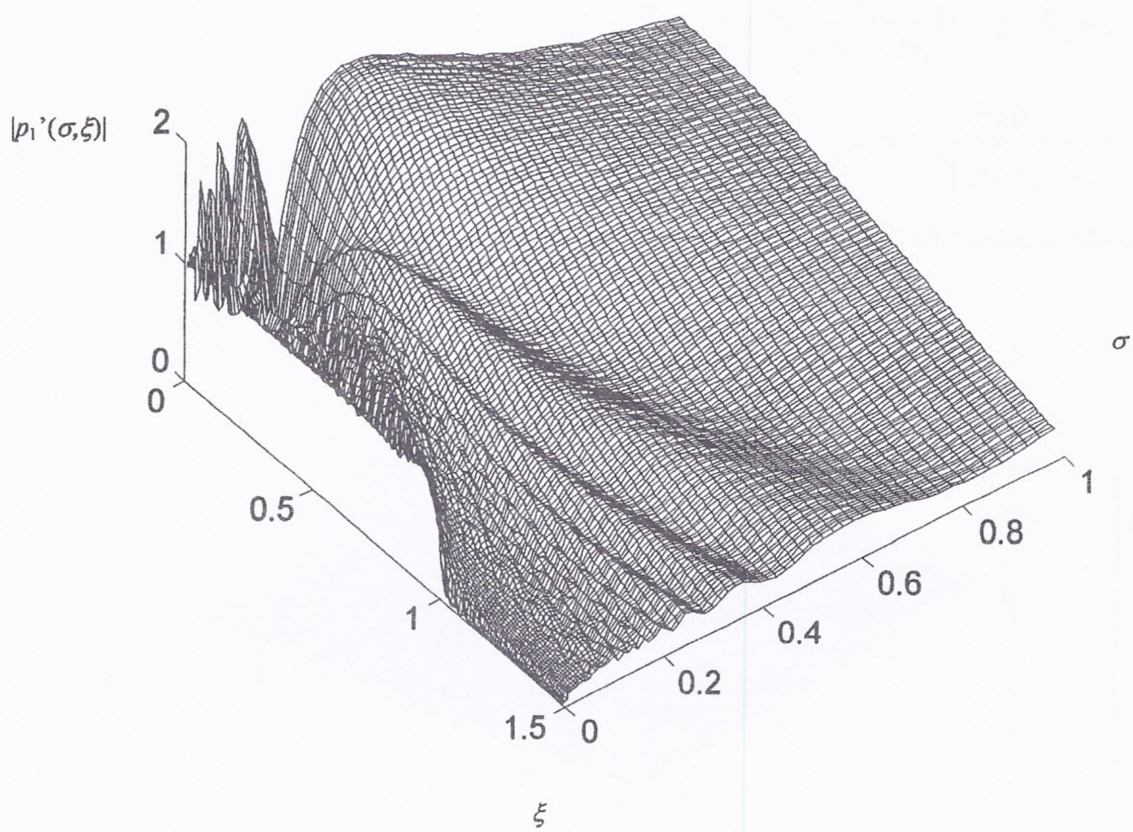

Fig.1. Nearfield pressure amplitudes of the fundamental harmonic component for a circular uniform source in the lossless medium, computed from the solution of the linearized KZK equation.

The second harmonic component is expressed by double integral [3]:

$$P_2(\sigma,\xi) = -\frac{\varepsilon\,(ka)^2}{\rho_0 c_0^2}\, e^{-\alpha_2'\sigma} \int_0^{\sigma} e^{-\frac{2i\xi^2}{\sigma-\sigma'}+\alpha_1 r_0 \sigma'} \frac{d\sigma'}{\sigma-\sigma'} \int_0^{\infty} P_1^2(\sigma,\xi)\, e^{-\frac{2i\xi^2}{\sigma-\sigma'}} J_0\!\left(\frac{4\xi\xi'}{\sigma-\sigma'}\right)\xi'\, d\xi' \;,$$

where the integrand is essentially singular. The calculation of it is performed by more general formula from [6], and division of network integration in parts distinguishing the vicinity of singularity point. Finally:

$$P_2(\sigma_p,\xi_n) \cong -\frac{\varepsilon\,(ka)^2}{\rho_0 c_0^2}\, e^{-\alpha_2 r_0 \sigma} \sum_{k=0}^{p-1} \begin{cases} \dfrac{feg1_{p,n,k}}{\sigma_p-\sigma_k}\,\Delta\sigma_k, & if\ \left|dg1_{p,n,k}\,\Delta\sigma_k\right| \le 0.1, \\[2ex] \dfrac{feg1_{p,n,k}}{\sigma_p-\sigma_k}\sinh\!\left[\dfrac{\Delta\sigma_k}{2}\,dg1_{p,n,k}\right]\dfrac{2}{dg1_{p,n,k}}, & if\ \left|dg1_{p,n,k}\,\Delta\sigma_k\right| > 0.1. \end{cases} \qquad (6)$$

where:

$$feg1_{p,n,k} = e^{-\frac{2i\xi_n^2}{\sigma_p-\sigma_k}+\alpha_1 r_0 \sigma_k} \sum_{l} \begin{cases} feg2_{p,n,k,l}\,\Delta\xi, & if\ \left|dg2_{p,k,l}\cdot\Delta\xi\right| \le 0.1, \\[2ex] feg2_{p,n,k,l}\sinh\!\left[\dfrac{\Delta\xi}{2}\,dg2_{p,k,l}\right]\dfrac{2}{dg2_{p,k,l}}, & if\ \left|dg2_{p,k,l}\cdot\Delta\xi\right| > 0.1. \end{cases}$$

$$feg2_{p,n,k,l} = \left[P_{1k,l}\right]^2 \cdot J_0\!\left(\frac{4\xi_n\xi_l}{\sigma_p-\sigma_k}\right)\cdot \xi_l \exp\!\left[-\frac{2i\xi_l^2}{\sigma_p-\sigma_k}\right],$$

$$dg1_{p,n,k} = -\frac{2i\xi_n^2}{\left(\sigma_p-\sigma_k\right)^2} + \alpha_1 r_0, \qquad dg2_{p,k,l} = -\frac{4i\xi_l}{\sigma_n-\sigma_k}.$$

The results of calculations are presented below in the form of 3D distribution:

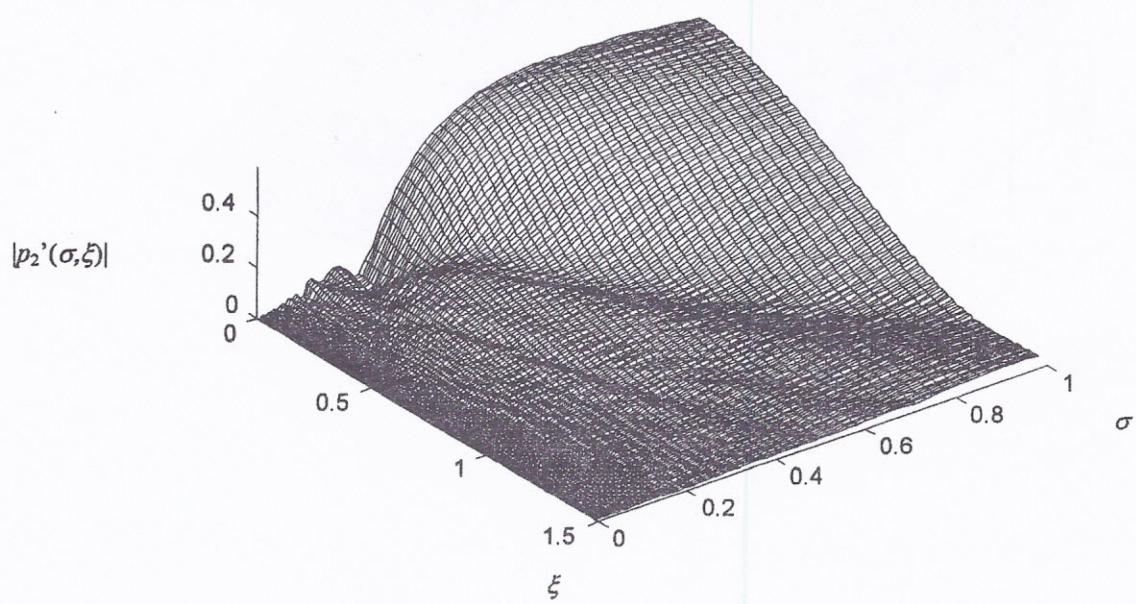

Fig.2. Nearfield pressure amplitudes of the second harmonic component for a circular uniform source ($ka \approx 100$ and $p_0 = 100$ kPa) in the distilled water at $T = 20$ °C, computed using formula (6).

The comparison of calculations using the equation (6) at ξ=0 with the results of numerical integration of the KZK equation is shown at figure 3. As it is seen the explicit analytical solution of the linearized KZK equation for higher source level (p_0 = 100 kPa) and at larger ranges is inconsistent with numerical results obtained using a parabolic approximation. This can be explained by neglecting within a quasilinear approach the extra nonlinear tapering of the amplitude in the vicinity of the beam axis (to transfer energy into higher harmonics).

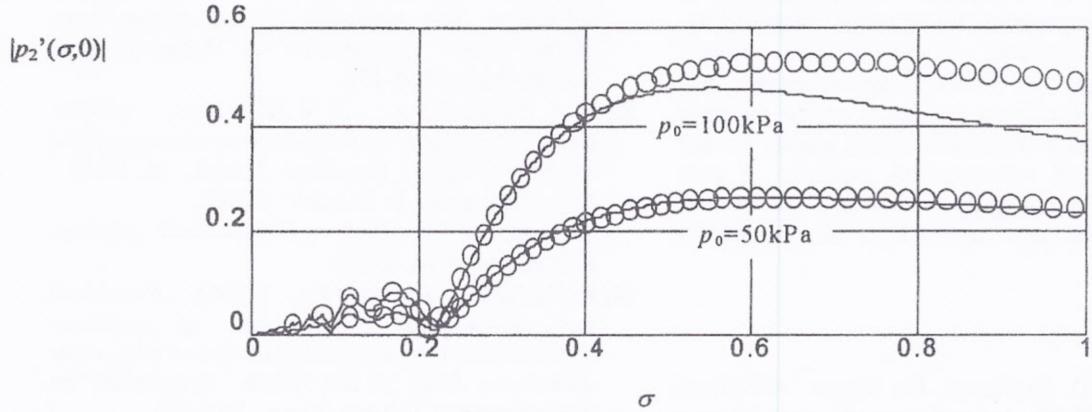

Fig.3. The dependence of the second harmonics amplitudes of the on-axis sound pressure vs. distance for $ka \approx 100$ and p_0 = 50/100 kPa in the distilled water at T = 20 °C (absorption is neglected). Circles show the results of computation according to eq. (6), solid lines are numerical solutions of KZK equation [7].

At least the averaged third harmonic component is esteemed by the simple direct integration formulae:

$$\left\langle P_3(\sigma_p)\right\rangle \cong -\frac{3(ka)^2 \varepsilon}{2\rho_0 c_0^2} \sum_{m=1}^{p} \left(\sum_n P_1(\sigma_m, \xi_n) P_2(\sigma_m, \xi_n) \xi_n \Delta\xi \right) e^{4\delta\omega^2 \sigma_m} \Delta\sigma_m \tag{7}$$

The integration (7) is illustrated by figure 3,

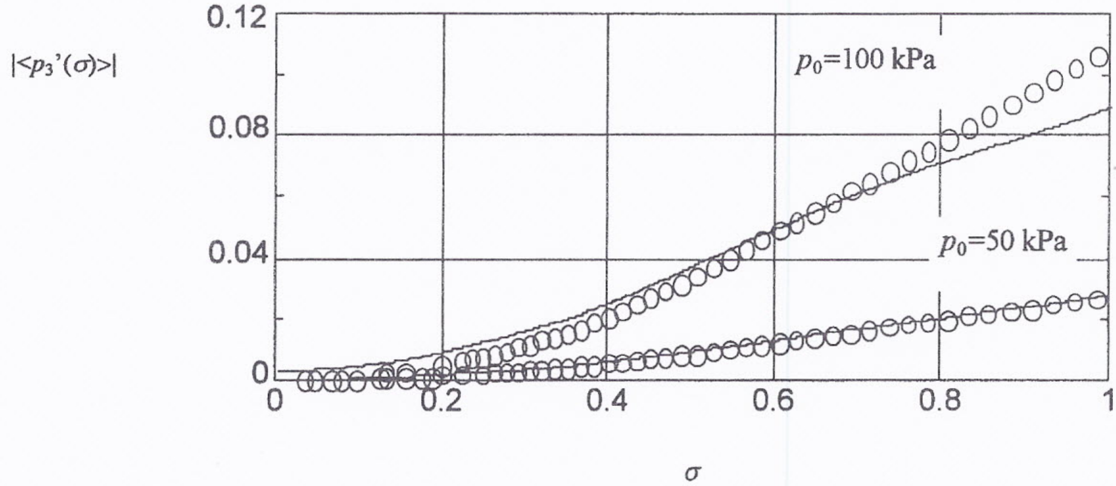

Fig.3. The dependence of the third harmonic component amplitude of the average sound pressure vs. distance between transducers for $ka \approx 100$ and p_0 = 50/100 kPa in the distilled water at T = 20 °C (absorption is neglected). Circles show the results of computation according to eq. (7), solid lines are from averaged numerical solutions of KZK equation [7].

CONCLUSION

The approach we develop is oriented for standard packages firmware PC level. It tends to use combinations of asymptotics and special integral sums to cover difficult parts of integration area. The main profit of resonance harmonics method, we consider here as theoretical formulation, is a possibility of expressing an acoustic nearfield by quadrature components. There is also obvious possible to apply the parabolic approximation and include higher nonlinear constants as for example via state equations. The forthcoming results we see in foundation of measurement methods of new parameters (nonlinearity factors, absorption coefficients) that are involved in the considered scheme.

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